The electromagnetic field of a Kerr-Newman source

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The origins of the electromagnetic field of the Kerr-Newman source are a system of currents and of electric surface charges which are distributed over a circular disc of radius a, centered at the origin, and oriented normally to the direction of the angular momentum vector. For a positive total electric charge e, the surface-charge density is negative inside the disc, becoming infinitely negative as the rim of the disc is approached. On the rim, there is a positive charge of infinite density which not only neutralizes the negative charge distributed in the interior of the disc, but also leaves a residue of a positive charge equal to e. Similarly, the currents flow in the negative direction inside the disc, with a current density which becomes infinitely negative at the rim. On the rim there is a positive current of infinite intensity, which generates a magnetic moment compensating the negative magnetic moments distributed in the interior, and leaves a net integrated magnetic moment of magnitude ea, the latter being equal to the dipole component of the total magnetic moment. The gyromagnetic ratio for the total magnetic moment is 2, as it is for the dipole component.

I. INTRODUCTION

The solution of the Einstein-Maxwell equations for a spinning charged mass was given by Kerr[1] and Newman *et al.*[2] In the Kerr-Newman metric, there appears an arbitrary constant a, having the dimension of a length, which is equal to $1/c$ times the angular momentum per unit mass of the source. By analyzing the asymptotic form of the electromagnetic field for large distances from the source, one finds that the Kerr-Newman source possesses a magnetic *dipole* moment μ_D, given by

$$\mu_D = ea \ , \tag{1}$$

where e denotes the electric charge of the source. The angular momentum J of the source, deduced from the asymptotic form of the metric, is

$$J = amc \ , \tag{2}$$

where m denotes the mass of the source. If the total magnetic moment μ were equal to the dipole component μ_D only, then Eqs. (1) and (2) would imply that the gyromagnetic ratio g of the source is

$$g \equiv \frac{\mu}{J} = \frac{e}{mc} \ . \tag{3}$$

Since, classically, the gyromagnetic ratio for a charged mass point moving in a circular orbit is equal to $e/2mc$, it follows that the gyromagnetic ratio given by Eq. (3) is 2, as in Dirac's theory of the electron.[3]

The gyromagnetic ratio of 2 for the Kerr-Newman source is puzzling, considering that it was derived from a purely classical theory. A gyromagnetic ratio of 2 for a classical system raises the question as to whether the gyromagnetic ratio of Dirac's electron is of intrinsic quantum origin.[4] For this reason, as well as for astrophysical applications to the theory of rotating charged stellar models, it is of interest to study the internal electromagnetic structure of the Kerr-Newman source.

Now, in their original paper, Newman *et al.* concluded that the Kerr-Newman source is not a point source, but a ring of mass and charge rotating about the axis of symmetry, the radius of this ring being equal to a. It will be shown below that on the axis of symmetry z, the vertical component of the magnetic field of the Kerr-Newman source is given by

$$H_z = \frac{2ea\,|z|}{(z^2+a^2)^2}, \quad x = y = 0 \ . \tag{4}$$

Consider a line integral of the magnetic field taken on a loop consisting of the z axis extending from $z=-\infty$ to $z=+\infty$, and closing in a circle at infinity. On the circular path, $d\mathbf{s}$ grows like R, whereas $\mathbf{H}$ falls off like R^{-3}, hence the integral $\int \mathbf{H}\cdot d\mathbf{s}$ on the circle vanishes. We therefore have

$$\oint \mathbf{H}\cdot d\mathbf{s} = 2\int_0^\infty H_z dz = 4ea \int_0^\infty \frac{z\,dz}{(z^2+a^2)^2} = \frac{2e}{a} \ . \tag{5}$$

If the magnetic field were due to a current $j(a)$ flowing in a ring of radius a centered on the z axis, and lying in the x-y plane, we would have

$$\oint \mathbf{H}\cdot d\mathbf{s} = 4\pi j(a) = \frac{2e}{a} \ . \tag{6}$$

Since the magnetic moment $\mu(a)$ of such a ring current is given by

$$\mu(a)=\pi a^2 j(a) , \tag{7}$$

we have

$$\frac{2e}{a}=4\pi j(a)=\frac{4\mu(a)}{a^2} , \tag{8}$$

$$\mu(a)=\tfrac{1}{2}ea . \tag{9}$$

The magnetic moment of the proposed ring-current model is, in view of the known distribution of the component H_z of the Kerr-Newman source on the z axis, exactly equal to one-half of the value for the dipole component μ_D of the Kerr-Newman source given in Eq. (1), and the resulting gyromagnetic ratio would be 1, as for a classical system, and not 2.

II. GENERAL FEATURES OF THE MAGNETIC FIELD

We find that the magnetic field of the Kerr-Newman source is much more complicated than that of a simple ring current, as is shown in Fig. 1. This figure is drawn on a background of a Cartesian coordinate system (x,y,z) introduced originally by Kerr.[1] For distances much greater than a, the magnetic field resembles closely a dipole field, having the magnetic moment given in Eq. (1). On the disc of radius a lying in the x-y plane, H_z vanishes, in direct contrast to the field in the interior of the ring-current model. The horizontal component of the magnetic field H_λ is of opposite signs on the two sides of the disc. This discontinuity in H_λ gives rise to a current system $j(\lambda)$, flowing in the *negative* direction, with

$$j(\lambda)=\frac{-\lambda}{2\pi(a^2-\lambda^2)^{3/2}}, \quad \lambda<a, \quad j(\lambda)=0, \quad \lambda>a, \quad z=0$$
$$\lambda=(x^2+y^2)^{1/2} , \tag{10}$$

where $j(\lambda)d\lambda$ denotes the current flowing in a circular strip of the disc lying between the radii λ and $\lambda+d\lambda$. The total magnetic moment $M(\lambda)$ generated by the currents flowing inside a disc of radius λ is given by

$$M(\lambda)=\frac{ea}{2}\left[-\frac{a}{(a^2-\lambda^2)^{1/2}}+2-\frac{1}{a}(a^2-\lambda^2)^{1/2}\right], \quad \lambda<a \tag{11}$$

showing that $M(\lambda)$ becomes negatively infinite as the rim is approached. The magnetic moment $\overline{M}$ generated by the currents flowing between $\lambda=a-\epsilon$ and $\lambda=a^+$ approaches the value

$$\overline{M}=+\frac{ea^2}{2(a^2-\lambda^2)^{1/2}}, \quad \epsilon\to 0 . \tag{12}$$

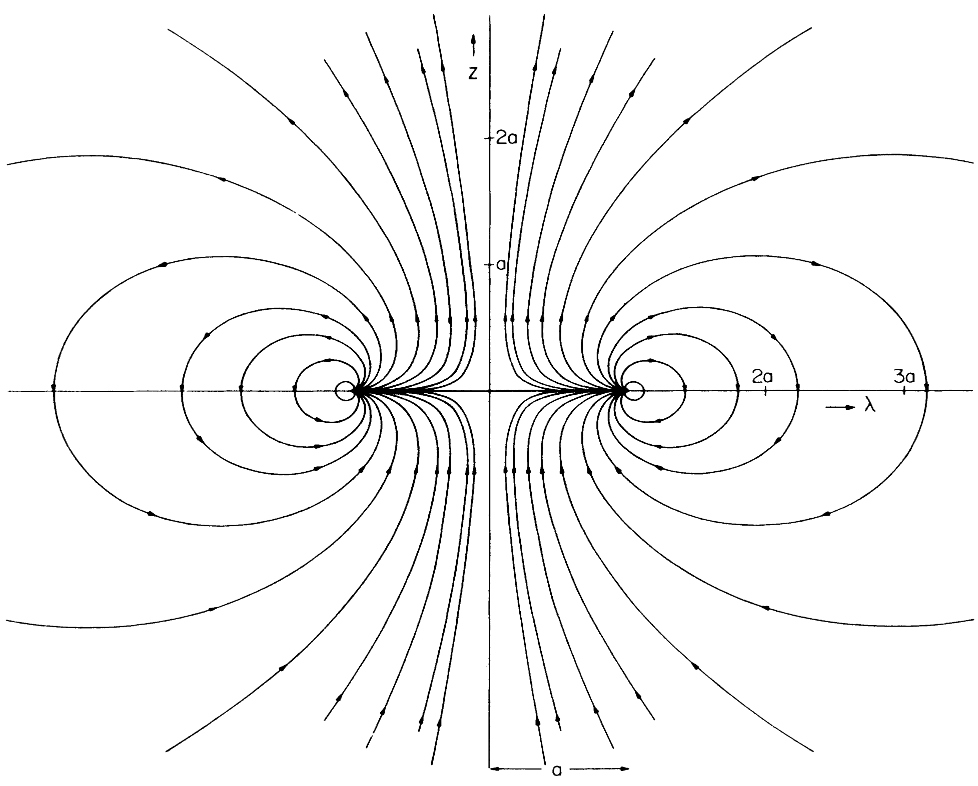

FIG. 1. The magnetic field **H** of a Kerr-Newman source in the λ-z plane. $\lambda=(x^2+y^2)^{1/2}$.

Hence, the total magnetic moment μ is

$$\mu = M(\lambda) + \overline{M} \to ea, \quad \lambda \to a^+ , \tag{13}$$

as given in Eq. (1), for the dipole component μ_D.

Equation (12) was derived from the relation

$$\int_{\lambda}^{a^+} j(\lambda) d\lambda = \frac{e}{2\pi(a^2-\lambda^2)^{1/2}} , \tag{14}$$

showing that there exists a positive current of infinite intensity flowing in the rim of the disc which more than compensates the effect of the interior negative currents represented by Eq. (10).

For $z=0$ and $\lambda > a$, we have

$$H_z = -\frac{ea}{(\lambda^2-a^2)^{3/2}}, \quad z=0, \quad \lambda > a . \tag{15}$$

Hence, for $\lambda = a + \epsilon$, H_z grows like $\epsilon^{-3/2}$ as the rim is approached, compared to the ϵ^{-1} variation near a current wire, and the ϵ^{-3} variation near a magnetic dipole.

III. GENERAL FEATURES OF THE ELECTRIC FIELD

The electric field of the Kerr-Newman source is shown in Fig. 2. The arrows were drawn so as to fit the asymptotic Coulomb field for a positive charge, with the directions in the interior then following by continuity. We note that a positive test charge is attracted towards the disc, showing that the disc is laden with a negative electric surface charge of density $\sigma(\lambda)$, say, per unit area. The value of $\sigma(\lambda)$ follows from the discontinuity in the vertical component E_z of the electric field on the disc.

We have

$$E_z^+ = -\frac{ea}{(a^2-\lambda^2)^{3/2}}, \quad z=0^+, \quad \lambda < a \tag{16}$$

$$E_z^- = +\frac{ea}{(a^2-\lambda^2)^{3/2}}, \quad z=0^-, \quad \lambda < a . \tag{17}$$

Hence

$$\sigma(\lambda) = \frac{1}{4\pi}(E_z^+ - E_z^-) = -\frac{ea}{2\pi(a^2-\lambda^2)^{3/2}} , \tag{18}$$

$$\sigma(\lambda) = 0, \quad \lambda > a . \tag{19}$$

The total electric charge $Q(\lambda)$ contained inside a disc of radius λ is given by

$$Q(\lambda) = 2\pi \int_0^{\lambda} \sigma(\lambda)\lambda\, d\lambda = e\left[1 - \frac{a}{(a^2-\lambda^2)^{1/2}}\right], \quad \lambda < a . \tag{20}$$

$Q(\lambda)$ is everywhere negative, and becomes infinitely negative as $\lambda \to a$. On the other hand, the total charge $\overline{Q}$ contained in a circular strip, taken between λ and $\lambda = a^+$, is equal to

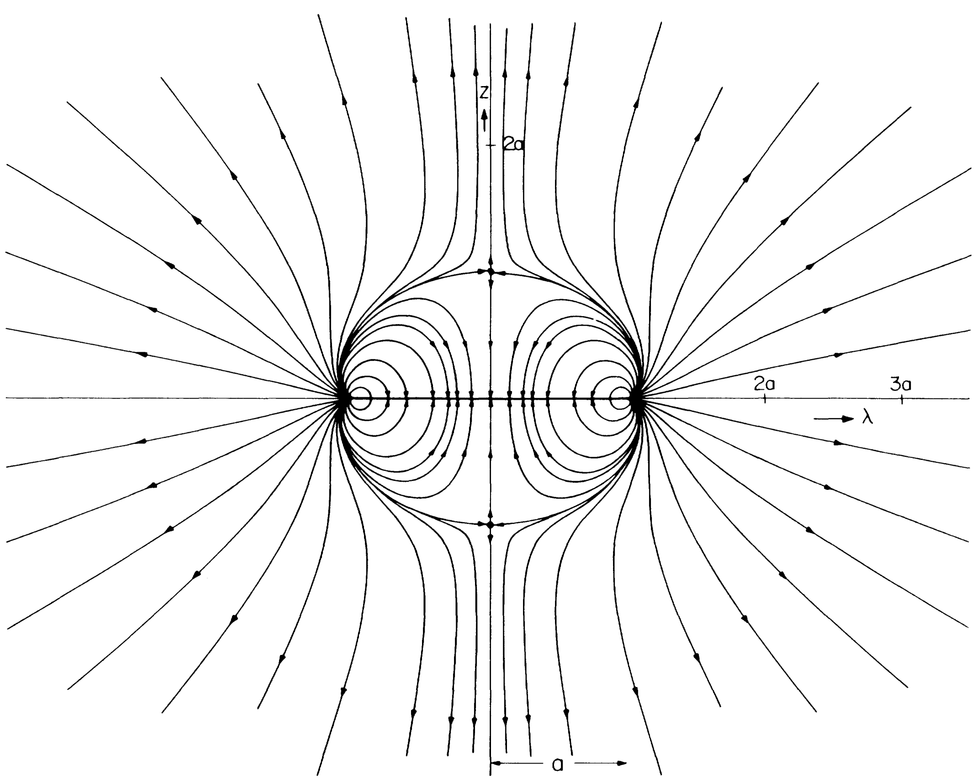

FIG. 2. The electric field $\mathbf{E}$ of a Kerr-Newman source in the λ-z plane. $\lambda = (x^2+y^2)^{1/2}$.

$$\bar{Q} = +\frac{ea}{(a^2-\lambda^2)^{1/2}} , \tag{21}$$

yielding a finite value for the total charge:

$$Q(\lambda)+\bar{Q}=e . \tag{22}$$

As in the case of the magnetic field, we have

$$E_\lambda = \frac{ea}{(a^2-\lambda^2)^{3/2}}, \quad z=0, \quad \lambda > a . \tag{23}$$

The electric field grows like $\epsilon^{-3/2}$ for $\lambda = a+\epsilon$, compared to the ϵ^{-1} variation near a charged wire, and the ϵ^{-2} variation near an electric point source.

Note that on the z axis, there are two saddle points at $z=\pm a$. There are no sources at these points because $|\mathbf{E}|$ vanishes there.

IV. THEORY

In Finkelstein's formulation of the Kerr-Newman solution,[4,5] the electromagnetic field can be represented as the gradient of a potential:

$$\mathbf{E}+i\mathbf{H} = -e\nabla(\Omega^{-1}) , \tag{24}$$

where

$$\Omega = [x^2+y^2+(z-ia)^2]^{1/2}$$
$$=(R^2-a^2-2iaz)^{1/2}, \quad R=(x^2+y^2+z^2)^{1/2} . \tag{25}$$

For large values of R, the potential e/Ω has the asymptotic expansion

$$\frac{e}{\Omega} \to \frac{e}{R} + \frac{ea^2}{2R^3} + \frac{ieaz}{R^3} + \cdots . \tag{26}$$

Since the potential of a magnetic dipole of magnetic moment μ_D is equal to $(\mu_D z/R^3)$, we identify μ_D with ea, as given in Eq. (1).

As a first step, we have to make the function Ω single valued in the complex z plane, since it has branch points on the imaginary axis at z_1 and z_2, given by

$$z_1 = i(a+\lambda), \quad z_2 = i(a-\lambda), \quad \lambda = (x^2+y^2)^{1/2} . \tag{27}$$

We make cuts on the imaginary z axis from z_1 up to $z=+i\infty$, and from z_2 down to $z=-i\infty$, as is shown in Fig. 3. It is seen that for $\lambda > a$ [Fig. 3(a)] the real z axis is not crossed by the branch lines. Hence, Ω and the electromagnetic field are analytic outside the disc. On the other hand, for $\lambda < a$ [Fig. 3(b)] the lower branch line crosses the real z axis. It follows that for $\lambda < a$, the potential Ω changes sign at $z=0$:

$$\Omega^- = -\Omega^+, \quad z=0, \quad \lambda < a , \tag{28}$$

where Ω^- denotes the value of the radical in Eq. (25) for $z=0^-$, and Ω^+ for $z=0^+$. It will be shown below that this feature of a sign reversal has the important physical consequence that the sources of the electromagnetic field are confined to the interior and to the rim of a disc of radius a, centered at the origin, and lying in the x-y plane.

On the Riemann sheet, as cut in Fig. 3, we have, at $z=0$,

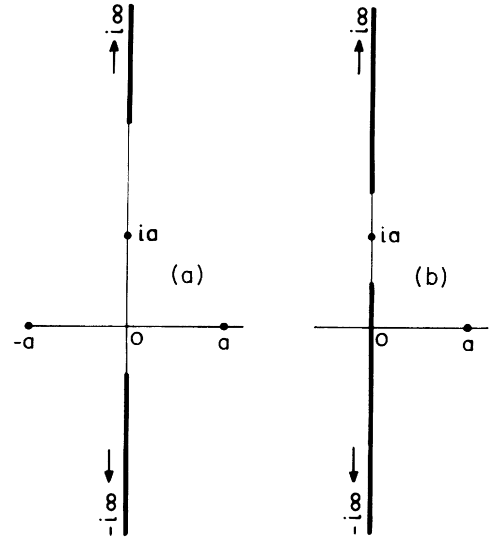

FIG. 3. Branch-line cuts in the complex z plane. (a) $\lambda=(3/2)a$, (b) $\lambda=(1/2)a$.

$$\Omega^+ = [\lambda^2+(-ia)^2]^{1/2} = -i(a^2-\lambda^2)^{1/2} ,$$
$$\Omega^- = +i(a^2-\lambda^2)^{1/2}, \quad z=0, \quad \lambda < a , \tag{29}$$

$$\Omega^+ = (\lambda^2-a^2)^{1/2} = \Omega^-, \quad z=0, \quad \lambda > a . \tag{30}$$

At $\lambda=0$, we take

$$\Omega^+ = +(z-ia), \quad \Omega^- = -(z-ia), \quad \lambda=0 , \tag{31}$$

this assignment following from the requirement of continuity at $\lambda=0$ with the expressions given in Eq. (29), as $\lambda \to 0$.

V. THE MAGNETIC FIELD

It follows from Eq. (24) that the vertical component of the magnetic field H_z is given by

$$H_z = -\mathrm{Im}\frac{\partial}{\partial z}\left[\frac{e}{\Omega}\right] = +e\,\mathrm{Im}\frac{(z-ia)}{\Omega^3} , \tag{32}$$

where Im denotes the imaginary part. At $z=0$,

$$H_z^+ = e\,\mathrm{Im}\frac{(-ia)}{(-i)^3(a^2-\lambda^2)^{3/2}} = 0 = H_z^-, \quad z=0, \quad \lambda < a . \tag{33}$$

By (30) and (32), we have

$$H_z^+ = -\frac{ea}{(\lambda^2-a^2)^{3/2}} = H_z^-, \quad z=0, \quad \lambda > a . \tag{34}$$

Now

$$H_\lambda = -e\,\mathrm{Im}\frac{\partial}{\partial\lambda}(\Omega^{-1}) = e\,\mathrm{Im}\frac{\lambda}{\Omega^3} . \tag{35}$$

Hence,

$$H_\lambda^+ = -\frac{e\lambda}{(a^2-\lambda^2)^{3/2}}, \quad z=0^+, \quad \lambda < a \tag{36}$$

$$H_\lambda^- = +\frac{ea}{(a^2-\lambda^2)^{3/2}}, \quad z=0^-, \quad \lambda<a , \tag{37}$$

$$H_\lambda = 0, \quad z=0, \quad \lambda>a . \tag{38}$$

At $\lambda=0$, we get from Eqs. (31) and (32)

$$H_z = \frac{2ea\,|z|}{(z^2+a^2)^2}, \quad \lambda=0 . \tag{39}$$

We now study the structure of the sources giving rise to the magnetic field in the Kerr-Newman source. It is clear from Fig. 1 that the sources are located inside the disc and on its rim, since the horizontal component of the magnetic field H_λ suffers there a reversal as the surface of the disc is crossed, and since the field is continuous outside the disc. By taking a line integral L_1 of the magnetic field on a loop in the λ-z plane starting at λ^+, going to $(\lambda^+ + d\lambda^+)$, and then returning to λ^- on the negative side of the disc, we get

$$L_1 = 2H_\lambda^+ d\lambda = -\frac{2e\lambda\, d\lambda}{(a^2-\lambda^2)^{3/2}} = 4\pi j(\lambda) d\lambda = \frac{4\mu(\lambda) d\lambda}{\lambda^2} . \tag{40}$$

Here, $j(\lambda)d\lambda$ denotes the current flowing between λ and $(\lambda+d\lambda)$, and $\mu(\lambda)d\lambda$ the magnetic moment generated by this current. We see that this magnetic moment is *negative*, and is given by

$$\mu(\lambda) = -\frac{ea\lambda^3}{2(a^2-\lambda^2)^{3/2}} . \tag{41}$$

The total magnetic moment $M(\lambda)$ of the currents flowing inside a disc of radius λ is

$$M(\lambda) = \int_0^\lambda \mu(\lambda) d\lambda = -\frac{ea}{2}\left[\frac{a}{(a^2-\lambda^2)^{1/2}} - 2 + \frac{1}{a}(a^2-\lambda^2)^{1/2}\right], \quad \lambda<a . \tag{42}$$

Clearly, $M(\lambda)$ becomes negatively infinite when λ approaches a from the inside.

As $\lambda \to a$, the lower branch line in Fig. 3 touches the real z axis and the singularity z_2 of Ω lands on the real z axis. In order to ascertain the contribution to the magnetic moment originating from the vicinity of the rim of the disc, we evaluate a line integral L_2 of the magnetic field, starting at λ^+, going up and then down to cross the λ axis at $\lambda>a$, and returning to λ^- from below. We have

$$L_2 = \oint \mathbf{H}\cdot d\mathbf{s}_2 = -\mathrm{Im} \oint \frac{\partial}{\partial s}\left[\frac{e}{\Omega}\right] ds = -\mathrm{Im}\left[\frac{e}{\Omega^-} - \frac{e}{\Omega^+}\right] = \frac{2e}{(a^2-\lambda^2)^{1/2}} , \tag{43}$$

by (29). It follows that

$$\frac{2e}{(a^2-\lambda^2)^{1/2}} = 4\pi \int_\lambda^{a^+} j(\lambda) d\lambda = \frac{4}{a^2}\int_\lambda^{a^+}\left[\frac{a^2}{\lambda^2}\right]\mu(\lambda) d\lambda \simeq \frac{4}{a^2}\int_{\lambda-\epsilon}^{a^+} \mu(\lambda) d\lambda = \frac{4}{a^2}\overline{M} , \tag{44}$$

to within a relative error of order ϵ/a. Hence the contribution $\overline{M}$ from sources close to the rim becomes positively infinite as

$$\overline{M} = +\frac{ea^2}{2(a^2-\lambda^2)^{1/2}} , \tag{45}$$

giving

$$\mu \simeq M(\lambda) + \overline{M} = \frac{ea}{2}\left[2 - \frac{1}{a}(a^2-\lambda^2)^{1/2}\right] \to ea, \quad \lambda \to a^+ . \tag{46}$$

The total magnetic moment μ of the Kerr-Newman source is therefore equal to the dipole component μ_D given in Eq. (1), and the gyromagnetic ratio for the total moment is equal to 2.

VI. THE ELECTRIC FIELD

We have from Eq. (24)

$$E_z = -\mathrm{Re}\frac{\partial}{\partial z}\left[\frac{e}{\Omega}\right] = +e\,\mathrm{Re}\frac{(z-ia)}{\Omega^3} , \tag{47}$$

$$E_\lambda = -\mathrm{Re}\frac{\partial}{\partial \lambda}\left[\frac{e}{\Omega}\right] = +e\,\mathrm{Re}\frac{\lambda}{\Omega^3} , \tag{48}$$

where Re denotes the real part. Using Eq. (29) in Eq. (47), we confirm relations (16) to (20). From (30) it follows that

$$E_z = 0, \quad z=0, \quad \lambda>a . \tag{49}$$

We also find from (47) and (31) that

$$E_z = \frac{e(z^2-a^2)}{(z^2+a^2)^2}, \quad z>0, \quad \lambda=0 , \tag{50}$$

$$E_z = -\frac{e(z^2-a^2)}{(z^2+a^2)^2}, \quad z<0, \quad \lambda=0 , \tag{51}$$

in line with the arrows drawn in Fig. 2. Similarly, we find that

$$E_\lambda = \frac{e\lambda}{(\lambda^2-a^2)^{3/2}}, \quad z=0, \quad \lambda>a , \tag{52}$$

$$E_\lambda = 0, \quad z=0, \quad \lambda<a . \tag{53}$$

With the integrated charge $Q(\lambda)$, which is contained within a disc of radius λ given in Eq. (20), we now proceed to evaluate the remaining charge contained between λ and a^+. One way of doing this is to use Gauss's theorem by evaluating the electric flux normal to the surface of a torus, centered on the rim of the disc, and having a circular cross section of radius τ ($<a$). On this circle we have

$$z=\tau\sin\psi,\quad \lambda=a-\tau\cos\psi\ , \tag{54}$$

with $\psi=0$ at λ^+ and $\psi=2\pi$ at λ^-, on the two sides of the disc.

We have

$$\Omega=(\tau^2-2a\tau e^{i\psi})^{1/2}\ . \tag{55}$$

For the component E_τ normal to the surface of the torus we have

$$E_\tau=-\mathrm{Re}\frac{\partial}{\partial\tau}\left[\frac{e}{\Omega}\right]=+e\,\mathrm{Re}\frac{(\tau-ae^{i\psi})}{\Omega^3}\ . \tag{56}$$

By Gauss's theorem, the charge $Q(\tau)$ contained inside the torus of cross-sectional radius τ is given by

$$\begin{aligned}Q(\tau)&=\frac{1}{4\pi}\int_0^{2\pi}E_\tau 2\pi\lambda\tau\,d\psi\\&=\tfrac{1}{2}e\tau\,\mathrm{Re}\int_0^{2\pi}(a-\tau\cos\psi)\frac{(\tau-ae^{i\psi})}{\Omega^3}\\&=\frac{e\tau}{2}\mathrm{Re}\left|\frac{ia}{\tau\Omega}-\frac{\sin\psi}{\Omega}\right|_{\psi=0}^{\psi=2\pi}=\frac{e}{2}\mathrm{Re}\left[\frac{ia}{\Omega^-}-\frac{ia}{\Omega^+}\right]\\&=+\frac{ea}{(a^2-\lambda^2)^{1/2}}\ .\end{aligned} \tag{57}$$

By Eq. (20), the total charge Q is

$$Q=Q(\lambda)+Q(\tau)=e\ . \tag{58}$$

VII. MAPPING THE ELECTROMAGNETIC FIELD

Instead of integrating the differential equation

$$\frac{dz}{d\lambda}=\frac{E_z}{E_\lambda} \tag{59}$$

for the electric lines of force, and the equation

$$\frac{dz}{d\lambda}=\frac{H_z}{H_\lambda} \tag{60}$$

for the magnetic lines of force, we found it preferable to use instead the electromagnetic stream function Ψ, based on the potential of Eq. (24), namely,

$$\Psi=-\frac{(z-ia)}{\lambda\Omega}\ , \tag{61}$$

from which the fields can be derived:

$$E_z=\frac{1}{\lambda}\mathrm{Re}\frac{\partial}{\partial\lambda}(\lambda\Psi),\quad E_\lambda=-\mathrm{Re}\frac{\partial\Psi}{\partial z}\ , \tag{62}$$

$$H_z=\frac{1}{\lambda}\mathrm{Im}\frac{\partial}{\partial\lambda}(\lambda\Psi),\quad H_\lambda=-\mathrm{Im}\frac{\partial\Psi}{\partial z}\ . \tag{63}$$

A solution of Eq. (59), giving the electric lines of force, is

$$-\mathrm{Re}(\lambda\Psi)=\mathrm{Re}\frac{(z-ia)}{\Omega}=\mathrm{const}\ , \tag{64}$$

as becomes evident on taking the differential of $(\lambda\Psi)$, and using (62). Similarly, a solution of Eq. (60) for the magnetic lines of force is

$$\mathrm{Im}(\lambda\Psi)=-\mathrm{Im}\frac{(z-ia)}{\Omega}=\mathrm{const}\ . \tag{65}$$

We now introduce coordinates (r,θ) (Kerr,[1] Finkelstein,[5]) defined so that

$$\begin{aligned}&\Omega=[\lambda^2+(z-ia)^2]^{1/2}=r-ia\cos\theta\ ,\\&\lambda=(x^2+y^2)^{1/2}\ .\end{aligned} \tag{66}$$

From this definition, it follows that

$$\lambda=(a^2+r^2)^{1/2}\sin\theta,\quad z=r\cos\theta\ , \tag{67}$$

and that r is related to $R=(x^2+y^2+z^2)^{1/2}$ through the equation

$$r^4-(R^2-a^2)r^2-a^2z^2=0\ . \tag{68}$$

For large values of R, $r\rightarrow R$. In the (x,y,z) coordinate system used in previous sections, the lines $r=$const are confocal ellipsoids, and the lines $\theta=$const are hyperboloids of one sheet.

Using Eq. (66), we find that

$$\mathrm{Re}\frac{(z-ia)}{\Omega}=\cos\theta+\frac{\cos\theta\sin^2\theta}{u^2+\cos^2\theta}=B\ , \tag{69}$$

where

$$u=\frac{r}{a}\ . \tag{70}$$

Hence, the electric lines of force are determined parametrically from

$$u=\left[\frac{\cos\theta(1-B\cos\theta)}{B-\cos\theta}\right]^{1/2}=u(\theta)\ . \tag{71}$$

The (λ,z) coordinates of an electric line characterized by a given value of the parameter B are determined from

$$\lambda=a(1+u^2)^{1/2}\sin\theta,\quad z=au\cos\theta\ . \tag{72}$$

Similarly, one finds for the magnetic lines

$$-\mathrm{Im}\frac{(z-ia)}{\Omega}=\frac{u\sin^2\theta}{u^2+\cos^2\theta}=\frac{1}{A}\ , \tag{73}$$

$$u=A\sin^2\theta\pm(A^2\sin^4\theta-\cos^2\theta)^{1/2}\ , \tag{74}$$

where the coordinates (λ,z) of a magnetic line labeled by a given value of the constant A are determined, again, from Eq. (72).

The electric line of force for $B=1$ in Eq. (71) is the one passing through the saddle points in Fig. 2 at $z=\pm a$. It is not quite a circle, its equation being given by

$$R^2=a^2(1+z^{2/3}-z^{4/3})\ . \tag{75}$$

It should be noted that the coordinates (r,θ), introduced in Eqs. (66)–(68), are identical with the spherical coordinates in which Chandrasekhar[6] succeeded in separating Dirac's equation in Kerr geometry. The point $r=0$, $\theta=\pi/2$, corresponds to the circle $\lambda=a$, which is the rim of the disc.

As for the image of the disc in the (r,θ) plane, we note from Eq. (66) that when $z=0$,

$$\Omega=\mp i(a^2-\lambda^2)^{1/2}=r-ia\cos\theta \ , \tag{76}$$

where the minus sign applies to the side of the disc at $z=0^+$, and the plus sign to the 0^- side. If r is real it must vanish on the disc, and then

$$\lambda=a\sin\theta \ , \tag{77}$$

with $0<\theta<\pi/2$ on the positive side, and $\pi/2<\theta<\pi$ on the negative side. If $r=(u+iv)$, then the vanishing of the imaginary part of $(r-ia\cos\theta)^2$ requires that $u(v-a\cos\theta)$ vanish. The root $v=a\cos\theta$ does not yield points lying in the interior of the disc. The root $u=0$ gives

$$\lambda=[a^2-(v-a\cos\theta)^2]^{1/2} \ . \tag{78}$$

Hence, r can take on a purely imaginary value iv, with $|v|\le 2a$. When $\theta=\pi/2$, Eq. (78) yields $\lambda=(a^2-v^2)^{1/2}$, $v\le a$.

The image of the disc in the (r,θ) plane is either

$$r=0, \quad \theta=\sin^{-1}\left[\frac{\lambda}{a}\right] , \tag{79}$$

or

$$r=iv \ ,$$
$$\theta=\cos^{-1}\left[\frac{r\pm(a^2-\lambda^2)^{1/2}}{a}\right] , \tag{80}$$
$$|v|\le|a\pm(a^2-\lambda^2)^{1/2}| \ .$$

VIII. DISCUSSION

In the analysis of the electromagnetic field of the Kerr-Newman source, only two physical parameters enter, namely, the charge e, and the length a, but not the mass m, nor the angular momentum J. Only three of these four parameters can be assigned arbitrary values, because of the relation $J=cma$. Our results should be of interest in the study of the electromagnetic field in rotating charged stellar models, including the theory of black holes.

We were led to this investigation in the course of testing a proposed model of the atomic nucleus[7] taken as a Kerr-Newman source, under the assumption that the angular momentum of the source is equal to the intrinsic spin angular momentum of the nucleus.

The parameter a in this model of the nucleus is of the order of the compton wavelength of the nucleus ($\hbar/mc$). Now that we know that the Kerr-Newman source is not pointlike, but has a physical extension over a disc of the order of the Compton wavelength of the nucleus, with a complicated distribution of currents and surface charges, the fact that it has a gyromagnetic ratio of 2 is less striking.

Note added in proof. In an illuminating paper, W. Israel [W. Israel, Phys. Rev. D **2**, 641 (1970)] discusses the electromagnetic field of the Kerr-Newman metric. Under the assumption that the sources are limited to the disk of radius a, he derives relations (10) and (18) for the current distribution and the charge distribution over the disk.

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