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Storage capacity of generalized networks

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We discuss the thermodynamic phases and storage capacity of models which are extensions of the Hopfield-Little model of associative memory, generalized to allow for arbitrary polynomial Hamiltonians. The storage capacity increases as an exponential function of the highest power appearing in the polynomial Hamiltonian. We include the effects of replica symmetry breaking in our computations.

The Hopfield-Little model[1] is a network exhibiting associative memory based on the Hamiltonian

$$H=-\frac{1}{2N}\sum_{\mu=1}^{M}\sum_{i\neq j}^{N}S_i\xi_i^\mu S_j\xi_j^\mu \ , \qquad (1)$$

where the S_i are N dynamical variables taking on the values ± 1 and the ξ_i^μ (with $\xi_i^\mu=\pm 1$) are M fixed patterns which are the memories being stored. This Hamiltonian exhibits associative memory because an arbitrary starting configuration S_i in the neighborhood of a given pattern, say ξ_i^1, will, if it is allowed to relax to a local minimum of H, evolve to a final state very close to $S_i=\xi_i^1$ provided that M is not too large. The storage capacity and thermodymamic properties of this model have been studied in detail by Amit, Gutfreund, and Sompolinsky.[2] At $T=0$ they find that for $M<0.14N$ memory patterns can be recovered with an accuracy better than 97%. At higher M or high temperature there is a transition to a spin-glass state in which memory patterns cannot be recovered.

Note that Eq. (1) can be written in the equivalent form

$$H=-\frac{1}{2}\sum_{\mu=1}^{M}\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}S_i\xi_i^\mu\right)^2-1\right] . \qquad (2)$$

In this Rapid Communication we discuss the storage capacity and thermodynamic phases of a generalization of this Hamiltonian to

$$H=-N^{1-p/2}\sum_{\mu=1}^{M}F\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}S_i\xi_i^\mu\right) , \qquad (3)$$

where F is an arbitrary polynomial of degree p,

$$F(x)=\sum_{k=0}^{p}A_k x^k . \qquad (4)$$

There are several reasons for considering this generalization. First, the storage capacity of such a network increases dramatically from the value $M<\alpha_c N$ with $\alpha_c=0.14$ reported by Amit, Gutfreund, and Sompolinsky[2] to $M<\alpha_c(p)N^{p-1}$. (Of course, this dramatic increase in storage capacity is accompanied by an equally dramatic increase in the number and complexity of the couplings between the S_i.) In addition, work on p-spin models of spin glasses[3] analogous to our generalized networks indicates that the structure of the model, in particular the role of replica symmetry breaking and the nature of the spin-glass transition, is quite different for $p>2$ than it is for $p=2$. We find similar effects here. Finally, in the limit of large p we can obtain exact analytic results. Of particular interest is a special case[4] of (3),

$$H=-\frac{1}{\sqrt{2p!N^{p-1}}}\sum_{\mu=1}^{M}\sum_{i_1\neq i_2\neq\cdots\neq i_p}S_{i_1}\xi_{i_1}^\mu S_{i_2}\xi_{i_2}^\mu\cdots S_{i_p}\xi_{i_p}^\mu \ , \qquad (5)$$

which we analyze extensively for $p=3$ and for $p\to\infty$. In this analysis, we compute for the first time the effects of replica symmetry breaking[5] on this model.[6]

We are interested in determining the overlap of a configuration S_i with one of the stored patterns which we arbitrarily take to be ξ_i^1,

$$m=\left\langle\frac{1}{N}\sum_{i=1}^{N}\langle S_i\rangle\xi_i^1\right\rangle_{\text{quench}} . \qquad (6)$$

Here the angle bracket $\langle\cdots\rangle$ represents a thermodynamic average while the angle bracket $\langle\,\rangle_{\text{quench}}$ represents a quenched average over the stored patterns ξ_i^μ. In order to distinguish ξ_i^1 from all the other patterns we apply an external field aligned with ξ_i^1 and then take the limit as this field goes to zero. We perform the quenched average over patterns using the replica method. In the mean-field approximation, valid for large N, the free energy depends on m and on the mean fields,

$$q_{ab} = \left\langle \frac{1}{N} \sum_{i=1}^{N} \langle S_i^a \rangle \langle S_i^b \rangle \right\rangle_{\text{quench}} , \tag{7}$$

and

$$r_{ab} = \frac{1}{MN} \sum_{\mu=2}^{M} \left\langle \sum_{i=1}^{N} \langle S_i^a \rangle \xi_i^\mu \sum_{j=1}^{N} \langle S_j^b \rangle \xi_j^\mu \right\rangle_{\text{quench}} . \tag{8}$$

The indices a and b label the n replicas which have been introduced. At the end of the calculation we take the limit $n \to 0$. The case $p=2$ involves a somewhat different calculation and has already been discussed in Ref. 2. Our formulas are valid only for $p > 2$. We begin by considering the replica symmetric ansatz $q_{ab} = q$ and $r_{ab} = r$. In this case, the free energy at temperature $T = 1/\beta$ is given by

$$f = (p-1) A_p m^p + \frac{\alpha\beta}{2} [r(1-q) - G(q)] - \frac{1}{\beta} \int Dz \ln[2\cosh\beta(\sqrt{\alpha r} z + p A_p m^{p-1})] , \tag{9}$$

where we have written $M = \alpha N^{p-1}$ and we use the notation

$$Dz = \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} . \tag{10}$$

The function $G(q)$ in the general case of Eq. (3) is given by

$$G(q) = \sum_{k,k'}^{p} \left[A_k A_{k'} \tfrac{1}{2} [1+(-1)^{k+k'}] \sum_{c=1}^{\min(k,k')} \tfrac{1}{2} [1+(-1)^{k+c}] \frac{k!k'!(k-c-1)!!(k'-c-1)!!}{(k-c)!(k'-c)!c!} (1-q^c) \right] , \tag{11}$$

where by definition

$$n!! = n(n-2)(n-4) \ldots (1), \; n > 0 , \tag{12}$$

and

$$n!! \equiv 1, \; n \le 0 . \tag{13}$$

For a Hamiltonian of the form (5), expression (11) simplifies greatly to

$$G(q) = \tfrac{1}{2} (1-q^p) . \tag{14}$$

Equations (9) and (11) show an interesting feature of the general model (3). Terms involving correlations with the correct memory state, i.e., terms involving m depend only on the leading term $k=p$ in the Hamiltonian (3). However, terms involving the noise q coming from the other memories ξ_i^μ with $\mu \ne 1$, depend on the coefficients A_k for all k. This raises the interesting possibility of choosing the nonleading coefficients A_k for $k < p$ in such a way that the noise is minimized. We will not explore this possibility further here, but instead we focus attention on the simpler case (5) where the free energy per spin is given by (9) and (14).

The free energy (9) with (14) leads to the following equations determining m and q,

$$m = \int Dz \tanh\beta \left[\left(\frac{\alpha p}{2} \right)^{1/2} q^{(p-1)/2} z + \frac{p}{\sqrt{2p!}} m^{p-1} \right] , \tag{15}$$

and

$$q = \int Dz \tanh^2\beta \left[\left(\frac{\alpha p}{2} \right)^{1/2} q^{(p-1)/2} z + \frac{p}{\sqrt{2p!}} m^{p-1} \right] . \tag{16}$$

There are two cases for which (15) and (16) simplify considerably. One is the zero-temperature limit $\beta \to \infty$ for which we find $q=1$ and

$$m = \text{erf} \left[\left(\frac{p}{2\alpha p!} \right)^{1/2} m^{p-1} \right] . \tag{17}$$

The other case is $\alpha \to 0$ which gives $q = m^2$ and

$$m = \tanh \left(\frac{\beta p}{\sqrt{2p!}} m^{p-1} \right) . \tag{18}$$

All results in the replica symmetric case are obtained from Eqs. (15)–(18).

We begin by considering the limit $\alpha \to 0$ described by Eq. (18). For $p=3$ we find that solutions with nonzero $m > 0.807$ exist for $T < 0.504$. For larger T values $m=0$. We can analyze the limit of large p in Eq. (18) explicitly and we find memory states with $m > 1-(2/p)$ for

$$T < \frac{\sqrt{2}p}{\sqrt{p!} e^2 \ln p} . \tag{19}$$

Thus, we see that although the large p models of Eq. (5) allow a large number of memory states they require lower temperatures for memory recovery to take place.

If we take the zero-temperature limit described by Eq. (17), we find for $p=3$ that memory recovery with $m > 0.838$ occurs for $\alpha \le \alpha_c = 0.126$. In Fig. 1 the dashed curve shows the percentage of errors, $(1-m)/2$, made in recovering a particular memory configuration as a function of α, again for $p=3$ and zero temperature. For $\alpha > \alpha_c = 0.126$ the error rate goes to 50% since there is no longer any correlation between states of the system and the memory states. (It should be stressed that the mean-field calculation we are doing is only sensitive to states that are stable in the thermodynamic limit. In any simulation, we expect that there will be correlations between long-lived metastable states and the memory patterns even for $\alpha > \alpha_c$. This "remnant" magnetization is known to occur for $p=2$.) In the limit of large p we find that at zero temperature

$$m > 1 - \frac{1}{p} \left(\frac{1}{\pi \ln p} \right)^{1/2} , \tag{20}$$

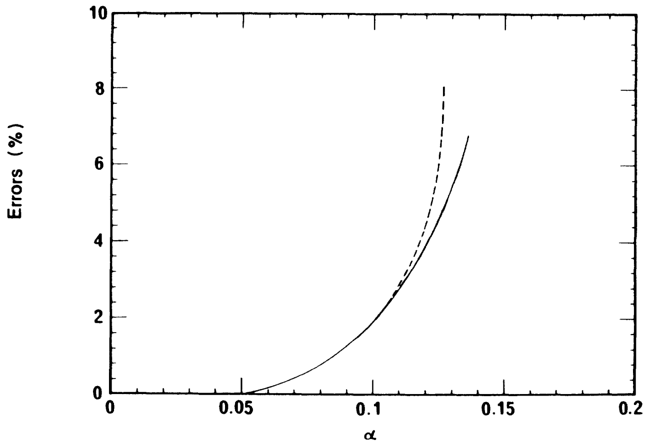

FIG. 1. The error rate $(1-m)/2$ as a function of α. The dashed curve is the replica symmetric result, while the solid curve includes the effects of replica symmetry breaking.

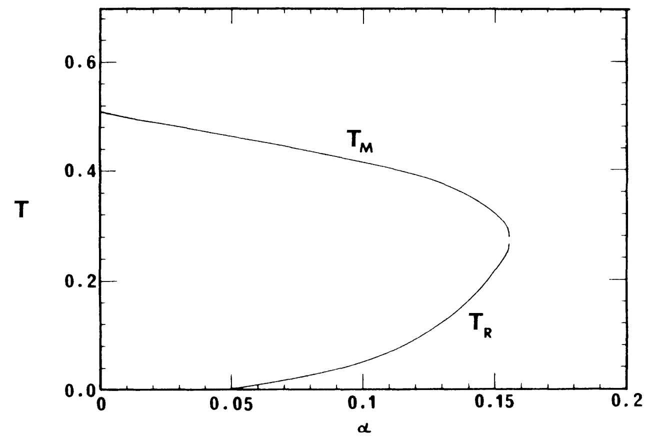

FIG. 2. The phase diagram for the $p=3$ model. Below the line labeled T_M there are metastable and stable states correlated with the stored patterns and above the line is a spin-glass phase. Below the line marked T_R the replica symmetric ansatz is unstable.

provided that

$$\alpha \leq \alpha_c = \frac{p}{2p!\ln p} . \tag{21}$$

Although α goes to zero for large p the total number of memory states allowed is large,

$$M = \alpha N^{p-1} \to \frac{e}{2\ln p}\left[\frac{1}{2\pi p}\right]^{1/2}\left[\frac{eN}{p}\right]^{p-1} . \tag{22}$$

At $T=0$ the energy of a state with memory overlap m is

$$E = -\frac{1}{\sqrt{2p!}}m^p - \left[\frac{\alpha p}{\pi}\right]^{1/2}\exp\left[-\frac{pm^{2(p-1)}}{\alpha 2p!}\right] . \tag{23}$$

For $p=3$, the memory recovery state $m \neq 0$ has a lower energy than the spin-glass state with $m=0$ provided that $\alpha < 0.089$. When p is very large the recovery state has lower energy for $\alpha < \pi/(2p!p)$.

The number of errors which is made in recovering any given stored pattern is $(1-m)N/2$. For the values of α we have been considering, this number is infinite for $N \to \infty$ despite the fact that the error rate shown in Fig. 1 is quite small. If we demand that the total number of errors is finite as $N \to \infty$, we must require the more stringent bound

$$\alpha < \frac{p}{2p!\ln N} . \tag{24}$$

Next we consider the system with finite values of both the parameters T and α. We plot the full phase diagram for $p=3$ in Fig. 2. The area under the upper curve marked T_M in Fig. 2 represents the region in this parameter space where there exist stable or metastable states correlated with the stored memory patterns. Above this curve is a spin-glass phase and then at high temperature an ordinary paramagnetic phase. As shown in Fig. 1, the transition from the spin-glass state to states with nonzero m involves a discontinuous jump in the value of m.

Up to now we have assumed that the symmetry between the replicas we have introduced is unbroken. We can check to see if this assumption is valid by examining the stability of fluctuations around the replica symmetric solutions we have found. In order for these solutions to be stable, we find that we must have

$$\frac{2q^{2-p}}{p(p-1)} - \alpha\beta^2\int Dz\cosh^{-4}\beta\left[\left[\frac{\alpha p}{2}\right]^{1/2}q^{(p-1)/2}z + \frac{p}{\sqrt{2p!}}m^{p-1}\right] > 0 . \tag{25}$$

This is true for the area above the lower curve marked T_R in Fig. 2. Below this curve the effects of replica symmetry breaking must be included. Comparing Fig. 2 with the results of Ref. 2, we see that replica symmetry breaking has a larger effect for $p>2$ than it did for $p=2$. The importance of replica symmetry breaking at zero temperature can be seen by noting that the upper curve in Fig. 2, if extrapolated smoothly down to $T=0$ (where it is no longer valid due to the instability of the replica symmetric ansatz), would suggest a value of α_c greater than 0.15, whereas the replica symmetric calculation discussed above gave $\alpha_c = 0.126$. Clearly, replica symmetry-breaking effects are fairly important, especially at $T=0$. In view of this fact, we must include replica symmetry breaking in our calculations.

We will evaluate here the effects of including a single level of replica symmetry breaking[5] on the $p=3$ model at zero temperature. In this approximation the free energy becomes a function of six variables r_1, r_0, q_1, q_0, x, and m and for the Hamiltonian (5) it is

$$f=\frac{p-1}{\sqrt{2p!}}m^p+\frac{\alpha\beta}{2}\{r_1-\tfrac{1}{2}[1-xq_0^2-(1-x)q_1^2]-xr_0q_0-(1-x)r_1q_1\}$$
$$-\frac{1}{\beta x}\int Dz_0\ln\left[2^x\int Dz_1\cosh^x\beta\left(\sqrt{\alpha r_0}z_0+[\alpha(r_1-r_0)]^{1/2}z_1+\frac{p}{\sqrt{2p!}}m^{p-1}\right)\right] . \quad (26)$$

Variation with respect to q_1 and q_0 gives trivially $r_1=pq_1^{p-1}/2$ and $r_0=pq_0^{p-1}/2$. At $T=0$, $q_1=1$ and it is convenient to replace the variable x with $\beta_c=x\beta$.[3] Expression (26) must then be maximized with respect to β_c and q_0 and minimized with respect to m to determine the values of these parameters. The results for $p=3$ are indicated by the solid curve in Fig. 1. Note that the memory recovery properties of the model were underestimated by our earlier replica symmetric calculation (the dashed curve) and that, as we expected, the value of α_c has increased. We find memory recovery states with $m>0.872$ provided that $\alpha\leq\alpha_c=0.135\pm0.001$.

The calculation we have performed included only the first level of replica symmetry breaking, but interestingly, a preliminary investigation indicates that there is a small region of the phase diagram where it is exact. However, at zero temperature the single replica symmetry-breaking scheme we have considered is not stable. It seems likely that higher orders of replica symmetry breaking will continue to increase the value of α_c.

The powerful technique of introducing replicas for performing quenched averages along with the replica symmetry-breaking ansatz of Parisi[5] allows us to consider in detail the properties of models as complicated as those of Eq. (3). Comparing with the results of Amit, Gutfreund, and Sompolinsky[2] for $p=2$, we see that introducing further nonlinearity into the Hopfield-Little model[1] greatly enhances its ability to store memories at the expense of slightly increasing the recovery error rate. A preliminary analysis indicates that the results for large p given here for the replica symmetric ansatz remain exact when replica symmetry breaking is introduced. This and further issues will be examined more thoroughly in a future publication.[7]

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