Kink-antikink dissociation and annihilation: A collective-coordinate description

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We examine the general kink-antikink (bion) collision process in the driven damped sine-Gordon equation. A collective-coordinate method is used to study several aspects of the bion dissociation and annihilation. Thresholds for the driving term are numerically obtained in both cases. The limitations and reliability of our reducing approach are discussed.

INTRODUCTION

In a recent paper,[1] a collective-coordinate method was used to investigate the sine-Gordon (SG) breather decomposition into a kink-antikink pair under the action of a constant homogeneous force ε. Critical values for this force were analytically obtained and their dependence on the initial phase ϕ of the breather was successfully compared to previous predictions (see Refs. 2 and 3). Yet, limitations occurred in the case $\phi \simeq \pi/2$, due to a broadening of the ε region between decomposition and breatherlike modes. In this case, a number of large-amplitude nonlinear waves are produced, thus making the simple ansatz used in Ref. 1 irrelevant.

In the present paper, we examine the general case of a bion (here we call either a breather or a kink-antikink pair a bion) under the influence of a driving force and damping. As transitions between breather and kink-antikink states are envisaged, we no longer assume the energy parameter k to be constant in time (see Ref. 1). A two-degree-of-freedom system is derived that allows a quantitative dynamical description of various kink-antikink collision processes.

The problem of power balance in a driven damped SG system is considered and we are able to determine the threshold driving term ε_{th} below which a kink-antikink pair annihilates. We compare our results with numerical ones and theoretical predictions presented in Ref. 4. We emphasize that the perturbational method used there leads to results in much better agreement with ours when the velocity dependence of the energy loss is not neglected in the determination of the annihilation threshold.

In order to compare results obtained through a direct numerical integration of the perturbed SG equation with those our reduced system of ordinary differential equations (ODE's) yields, we present a simple equivalence transformation. It relates the collective coordinates with the spatial derivative of the field at points where the latter meets an extremum.

The collective-coordinate transformation we propose here gives a quite reliable dynamical representation of a perturbed bion in terms of interacting particles which are the kink and the antikink with modified shapes.

I. COLLECTIVE-COORDINATE TRANSFORMATION EVEN IN THE NON-HAMILTONIAN CASE

First consider the perturbed SG equation

$$\Phi_{tt} - \Phi_{xx} + \sin\Phi = F[\Phi] \ . \tag{1}$$

For the moment make $F[\Phi]=\varepsilon$. The corresponding Lagrangian density for the field Φ is

$$l = \tfrac{1}{2}\Phi_t^2 - \tfrac{1}{2}\Phi_x^2 - (1-\cos\Phi) + \varepsilon\Phi \ . \tag{2}$$

Now assume that

$$\Phi = \Phi(x, \{a_i(t)\}) \ , \tag{3}$$

where the family of coordinates $\{a_i(t)\}$ includes all the time dependence of the field.

The Lagrange equation for the coordinate a_i reads

$$\frac{\partial L}{\partial a_i} - \frac{d}{dt}\frac{\partial L}{\partial a_{it}} = 0 \ , \tag{4a}$$

where L is the total Lagrangian,

$$L = \int_{-\infty}^{+\infty} l[\Phi(x,\{a_i(t)\})]dx \ . \tag{4b}$$

Hence, the left-hand side of Eq. (4a) also reads

$$\int_{-\infty}^{+\infty} dx \left[\frac{\partial l}{\partial \Phi}\frac{\partial \Phi}{\partial a_i} + \frac{\partial l}{\partial \Phi_x}\frac{\partial \Phi_x}{\partial a_i} + \frac{\partial l}{\partial \Phi_t}\frac{\partial \Phi_t}{\partial a_i} - \frac{\partial}{\partial t}\left[\frac{\partial l}{\partial \Phi_t}\frac{\partial \Phi_t}{\partial a_{it}} \right] \right] \ . \tag{4c}$$

Then, using the following identity:

$$\frac{\partial \Phi_t}{\partial a_{it}} = \frac{\partial \Phi}{\partial a_i} \ , \tag{4d}$$

and provided that

$$\left.\frac{\partial l}{\partial \Phi_x}\frac{\partial \Phi}{\partial a_i}\right|_{x=+\infty} - \left.\frac{\partial l}{\partial \Phi_x}\frac{\partial \Phi}{\partial a_i}\right|_{x=-\infty} = 0 \ , \tag{4e}$$

the expression in formula (4c) becomes

$$\int_{-\infty}^{+\infty} dx \left[\frac{\partial l}{\partial \Phi} - \frac{\partial}{\partial t}\frac{\partial l}{\partial \Phi_t} - \frac{\partial}{\partial x}\frac{\partial l}{\partial \Phi_x} \right] \frac{\partial \Phi}{\partial a_i} . \tag{4f}$$

Therefore, Eq. (4a) is equivalent to

$$\int_{-\infty}^{+\infty} dx \left[\frac{\partial l}{\partial \Phi} - \frac{\partial}{\partial t}\frac{\partial l}{\partial \Phi_t} - \frac{\partial}{\partial x}\frac{\partial l}{\partial \Phi_x} \right] \frac{\partial \Phi}{\partial a_i} = 0 . \tag{5}$$

Note that the expression between large parentheses in formula (4f) is identically zero for any solution of Eq. (1). In fact, when $F[\Phi]=\varepsilon$, for instance, Eq. (5) exactly amounts to projecting the perturbed SG equation (1) [applied to the ansatz function (3)] onto $\partial\Phi/\partial a_i$.

In a non-Hamiltonian case, such as $F[\Phi]=\varepsilon-\alpha\Phi_t$, one is naturally led to extend Eq. (5) by projecting Eq. (1) onto the "mode" $\partial\Phi/\partial a_i$. This mode carries the change of the ansatz function Φ due to a small variation of the coordinate a_i.

We recall the following algebraic identity (see Ref. 1):

$$4\tan^{-1}\exp[k(x+y)]+4\tan^{-1}\exp[-k(x-y)]-2\pi = 4\tan^{-1}\left[\frac{\sinh ky}{\cosh kx}\right] , \tag{6}$$

which expresses the sum of a kink and an antikink profile as a bion profile. In this formula, y (respectively, $-y$) is the relative position of the antikink component (respectively, of the kink component) with respect to the center of symmetry of the whole profile. The number k characterizes the shape of both components. For the sake of simplicity, we shall replace the product ky by Y in the forthcoming calculations.

Now consider the ansatz function

$$\Phi(x,Y(t),k(t))=4\tan^{-1}\left[\frac{\sinh Y(t)}{\cosh[k(t)x]}\right] , \tag{7}$$

and substitute this ansatz for the field in Eq. (1) with $F[\Phi]=\varepsilon-\alpha\Phi_t$.

Then, by a mere projection of Eq. (1) onto $\partial\Phi/\partial Y$ and $\partial\Phi/\partial k$, one gets (after some lengthy calculations) the following reduced system of coupled nonlinear ODE's:

$$\begin{aligned} P &= \frac{\partial \Lambda}{\partial Y_t} , \\ Q &= \frac{\partial \Lambda}{\partial k_t} , \\ P_t+\alpha P &= \frac{\partial \Lambda}{\partial Y} , \\ Q_t+\alpha Q &= \frac{\partial \Lambda}{\partial k} , \end{aligned} \tag{8a}$$

where

$$\Lambda = 8\frac{Y_t^2}{k}\left[1+\frac{2Y}{\sinh 2Y}\right] - 16YY_t\frac{k_t}{k^2} + \frac{2}{3}\frac{k_t^2}{k^3}\left[(\pi^2+4Y^2)\left[1+\frac{2Y}{\sinh 2Y}\right]+8Y^2\right] - 8k\left[1-\frac{2Y}{\sinh 2Y}\right] - \frac{8}{k}\tanh^2 Y\left[1+\frac{2Y}{\sinh 2Y}\right] + \frac{4\pi}{k}Y . \tag{8b}$$

Though the choice of Y was advisable to derive the above system, we shall return to the more "physical" coordinate y in the interpretation of further results.

II. A SIMPLE NUMERICAL DIAGNOSIS

For the type of perturbation we consider here (i.e., $F[\Phi]=\varepsilon-\alpha\Phi_t$), the solution Φ of Eq. (1) remains symmetrical about its center of mass as long as $\Phi(x=+\infty)=\Phi(x=-\infty)$. This is actually the case if one starts with a bion as an initial condition. Let us denote x_0 the position of the center of mass. Ascribing the role of a new degree of freedom to x_0 would amount to adding the following extra term in the expression of Λ [see formula (8b)]:

$$8kx_{0t}^2\left[1-\frac{2Y}{\sinh 2Y}\right] . \tag{8c}$$

We would also have two additional equations in system (8a), namely,

$$R=\frac{\partial \Lambda}{\partial x_{0t}}$$

and (8d)

$$R_t+\alpha R=\frac{\partial \Lambda}{\partial x_0}=0 .$$

Therefore, we decide to set x_0 and x_{0t} both equal to zero at $t=0$ since the dynamical behavior of x_0 is obviously of no interest from a physical point of view. From those considerations we shall only be interested in the part of the solution which is on the left-hand side of the center position $x=0$.

Then two quantities are easily obtained through a handy post-treatment of the results issued from a numerical simulation of Eq. (1). Namely, those of the extremum value of the x derivative of the field and the value of its corresponding position on the negative x axis.

Concerning the ansatz function in formula (7), those quantities are also readily obtained and read

$$\Phi_{X\,\max}=\Phi_x\,|_{x=X}=2k\tanh Y , \tag{9a}$$

with

$$X=-k^{-1}\mathrm{arg}\sinh(\cosh Y) . \tag{9b}$$

Thus we have provided ourselves with a practical way of diagnosing the validity of system (8a),(8b) by comparing the respective determinations of $\Phi_{X\,\max}$ and X resulting either from the complete system governed by Eq. (1), or from the reduced-model described by the system with a finite number of degrees of freedom (8a),(8b). The collective-coordinate description we propose here will then be relevant as long as both determinations coincide in a physically acceptable way.

III. BION DISSOCIATION

Let us start with the problem of the breather decomposition we began to study in Ref. 1. In this chapter we assume that we have no damping (i.e., $\alpha=0$). As in Ref. 1, the initial conditions are those given by an exact breather taken at $t=0$. We recall the form of a breather in its center-of-mass reference frame:

$$\Phi_B = 4\tan^{-1}\left[\frac{(1-\omega_B^2)^{1/2}}{\omega_B}\,\frac{\sin(\omega_B t+\phi)}{\cosh[(1-\omega_B^2)^{1/2}x]}\right]. \tag{10}$$

Equating Φ_B and the ansatz given in formula (7) at $t=0$, we get

$$\begin{aligned} &k(t=0)=(1-\omega_B^2)^{1/2}, \\ &k_t(t=0)=0, \\ &\sinh[Y(t=0)]=\frac{k}{\omega_B}\sin\phi, \\ &Y_t(t=0)\cosh[Y(t=0)]=k\cos\phi. \end{aligned} \tag{11}$$

In all cases (except when $\phi\simeq\pi/2$) the system of ODE's (8a),(8b) gives results which correspond rather well to our previous predictions concerning the critical value ε_{cr} of the force above which the breather breaks up into a kink-antikink pair.

When the initial phase ϕ is close to $\pi/2$, a new phenomenon arises. Large-amplitude inhomogeneities appear in the numerical integration of Eq. (1). In Ref. 1 we showed that ε_{cr} is extremely sensitive to any small variation of ϕ when the latter is near $\pi/2$, and we argued that the above-mentioned phenomenon might be due to this fact. Indeed, during the integration, unavoidable numerical errors contribute to a kind of "blurring" in the memory the system keeps of the initial phase. As a consequence, the system goes on hesitating between two possible futures: Either the breather disso-

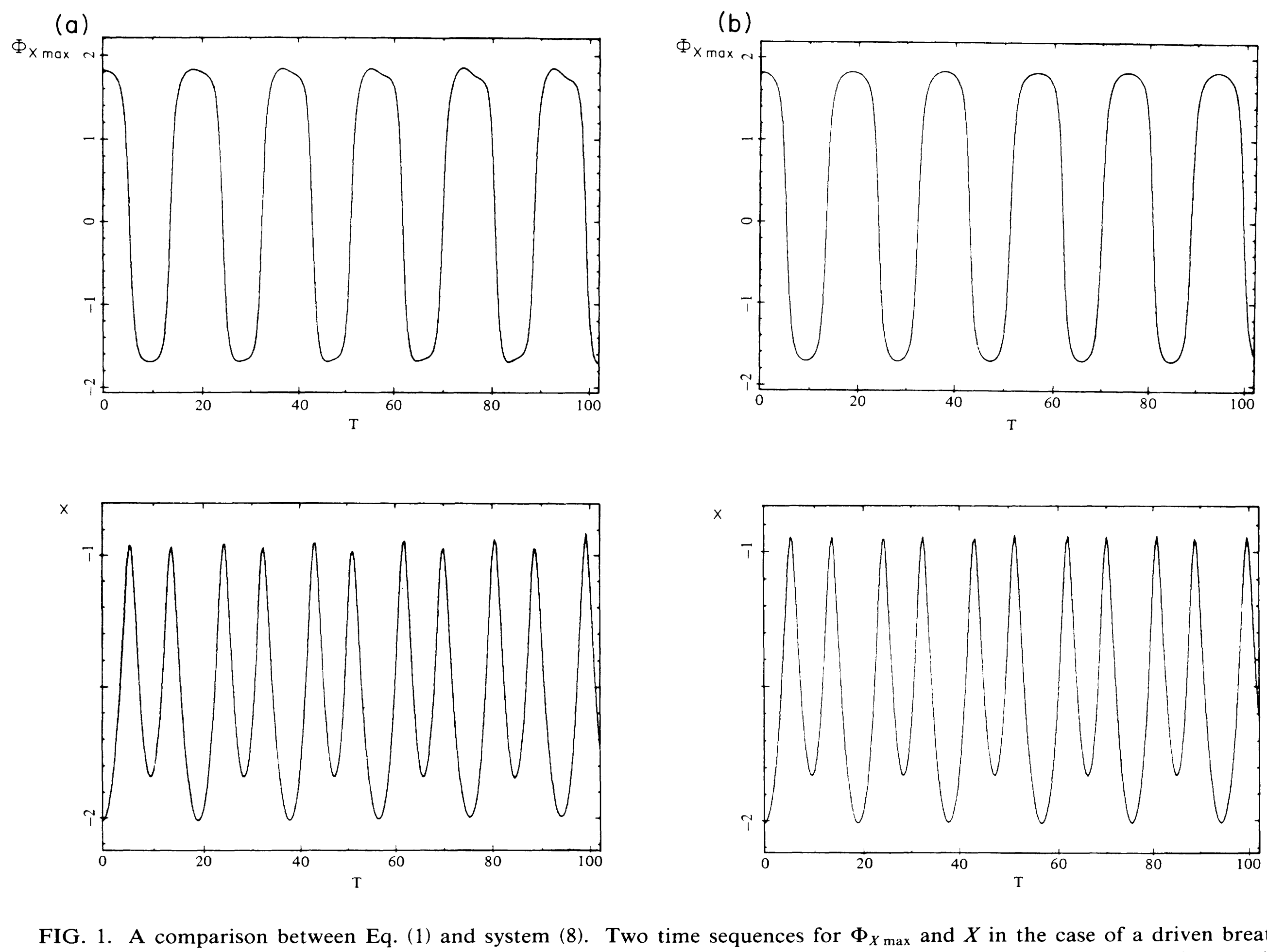

FIG. 1. A comparison between Eq. (1) and system (8). Two time sequences for $\Phi_{X\,\max}$ and X in the case of a driven breather. $\phi=\pi/2$, $\omega_B=0.3$, $\varepsilon\doteq0.01$. (a) Results from Eq. (1) obtained by only regarding the part of the solution which lies on the negative x axis. (b) The corresponding results from system (8) obtained through formulas (9a) and (9b).

ciates or it enters a stationary mode (see Ref. 2). Hence, a physical uncertainty appears in the determination of ε_{cr} that produces a more or less broad region in the parameter ε where the system (8) becomes inadequate. This region lies between a value of the force slightly smaller than ε_{cr} and that of ε_{cr} itself. We qualitatively found that this region of uncertainty broadens as ω_B gets closer to unity. In fact, when $\omega_B < 1$, we are in the nonlinear Schrödinger regime in which a complicated coupling process intervenes between the extended background and the small-amplitude breather (see Ref. 5).

In Fig. 1 we compare Eq. (1) with system (8a),(8b) in the case of an initial breather with $\phi=\pi/2$, for a value of the driving term ε much smaller than $\varepsilon_{cr}(\omega_B,\pi/2)$. The agreement is undisputable. Nevertheless, the real aim of this paper is not only to corroborate the predictions we gave in Ref. 1. Indeed, we next examine the more involved case of a kink-antikink pair in the presence of a driving term plus damping.

TABLE I. A sample of ε_{th} values corresponding to four values of α lying under 0.1: (a) From the determination accounting for the u_∞ dependence of ΔH. (b) $\varepsilon_{th}=(2\alpha)^{3/2}$ from Ref. 4. (c) From the determination obtained by multiple runs of system (8), scanning in ε for each value of α.

α \ ε_{th}	(a)	(b)	(c)
0.002	3.287×10^{-4}	2.530×10^{-4}	3.15×10^{-4}
0.01	3.487×10^{-3}	2.828×10^{-3}	3.25×10^{-3}
0.03	1.744×10^{-2}	1.470×10^{-2}	1.55×10^{-2}
0.1	10.40×10^{-2}	8.944×10^{-2}	8.35×10^{-2}

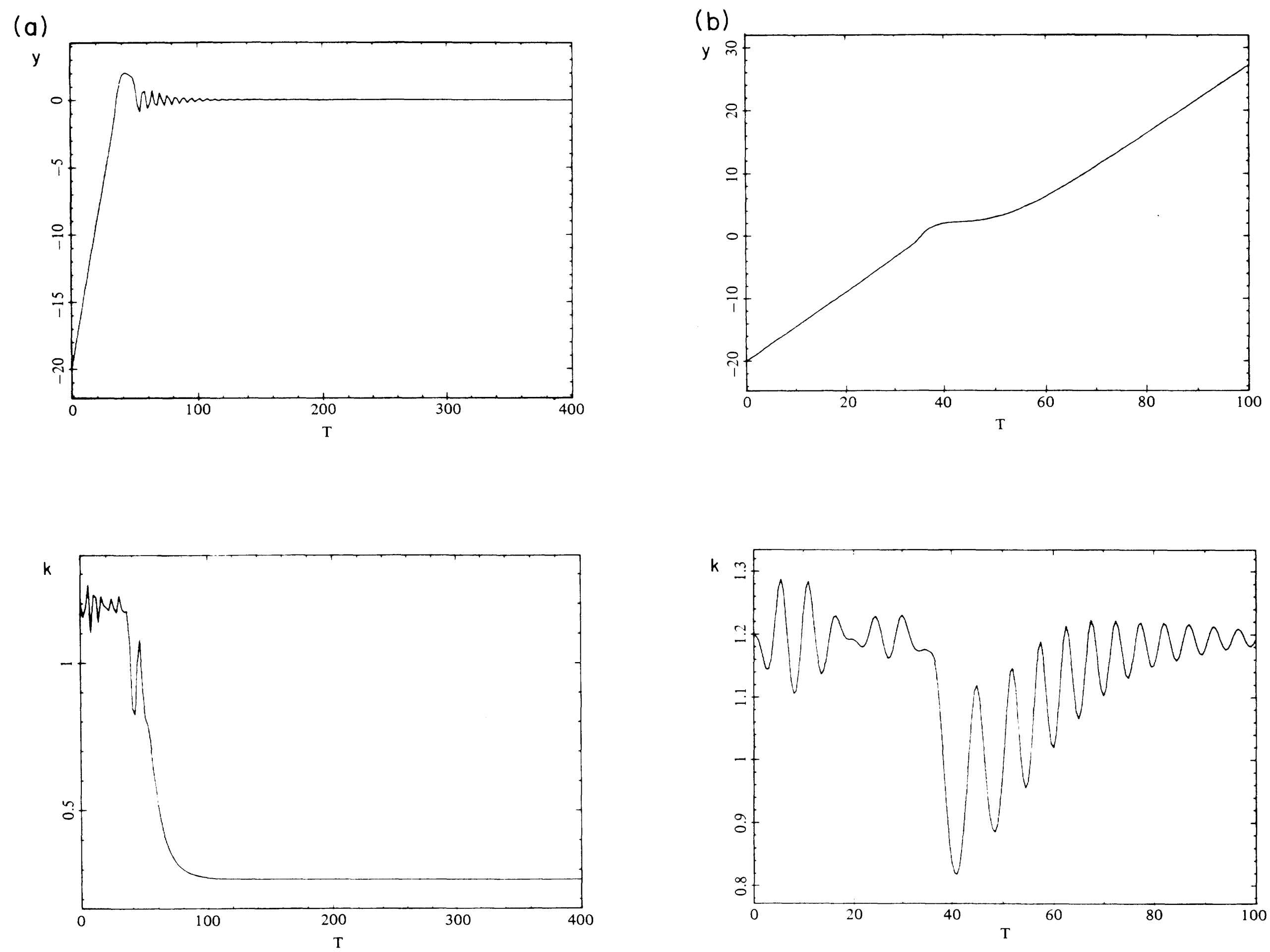

FIG. 2. Typical time sequences from system (8) intended to the determination of ε_{th}. Here we have $\alpha=0.1$. y and k are plotted vs time. (a) $\varepsilon=0.083<\varepsilon_{th}$. y decreases down to zero as k goes under unity: annihilation. (b) $\varepsilon=0.084>\varepsilon_{th}$. The kink-antikink is restored after collision. The shift in y due to the collision is obvious. k eventually returns to its original value corresponding to the Lorentz factor $\gamma=(1-u_\infty)^{-1/2}$.

IV. KINK-ANTIKINK ANNIHILATION

In Ref. 4, a perturbational method is used to determine the threshold driving term below which a kink and an antikink annihilate each other on the infinite line. Before going further into our investigations, let us briefly summarize this method.

For the unperturbed case, the kink-antikink solution of the SG equation is given by

$$\Phi_{K\bar{K}}=4\tan^{-1}\left[\frac{1}{u}\frac{\sinh[\gamma(u)ut]}{\cosh[\gamma(u)x]}\right], \tag{12a}$$

where $\gamma(u)$ is the Lorentz factor

$$\gamma(u)=(1-u^2)^{-1/2} . \tag{12b}$$

Including the loss and driving terms, the time rate of change of the energy H is evaluated as

$$\frac{dH}{dt}=\int_{-\infty}^{+\infty}(\varepsilon\Phi_t-\alpha\Phi_t^2)dx . \tag{13}$$

Next we assume that Φ is the kink-antikink solution given in formula (12). We note that far from each other (i.e., in the case of vanishing mutual interaction), the kink and the antikink reach power-balance velocities which are opposite in sign and the absolute value of which is determined by

$$u_\infty=\left[1+\left[\frac{4\alpha}{\pi\varepsilon}\right]^2\right]^{-1/2} . \tag{14}$$

Inserting u_∞ into the total amount of energy ΔH dissipated during the collision, yields

$$\Delta H=\int_{-\infty}^{+\infty}\frac{dH}{dt}dt=16\alpha\int_{-\infty}^{+\infty}\left[u_\infty-\frac{(1+u_\infty^2\sinh^2Y)^{1/2}}{\cosh Y}\left[1+\frac{2Y}{\sinh 2Y}\right]\right]dY , \tag{15}$$

which can be analytically evaluated in the limiting case $u_\infty=1$:

$$\Delta H=-4\pi^2\alpha . \tag{16}$$

Otherwise, for $u_\infty<1$, the latter integral is easily evaluated by numerical means. Then, the threshold driving term ε_{th} corresponding to annihilation is determined by equating the initial energy of the pair $E_i=16(1-u_\infty^2)^{-1/2}$ to the energy loss plus twice the energy of a kink at rest, i.e.,

$$16(1-u_\infty^2)^{-1/2}=\Delta H+16 . \tag{17}$$

We stress that the authors of Ref. 4 had *a priori* no right to discard the u_∞ dependence of ΔH in the evaluation of ε_{th}. Their approximate result (which they assert to be valid for small velocities) reads

$$\varepsilon_{\text{th}}=(2\alpha)^{3/2} . \tag{18}$$

By numerically evaluating the integral given in (15), we solve Eq. (17) and find a second determination of ε_{th} which in fact agrees with Eq. (18) only for intermediate velocities. Then, through a series of numerical integrations of system (8a),(8b) for different values of ε and α, we obtain a third evaluation of the threshold force. In Fig. 2 we give results obtained through two of these integrations for ε values enclosing ε_{th} as precisely as possible.

Table I shows a comparison of the three above determination for values of α ranging from 0.002 to 0.1. When α is smaller than 0.1, the results obtained through scanning with system (8) fit those issued from the exact solution of Eq. (17) better than those given by formula (18). Naturally, before using system (8) to evaluate ε_{th},

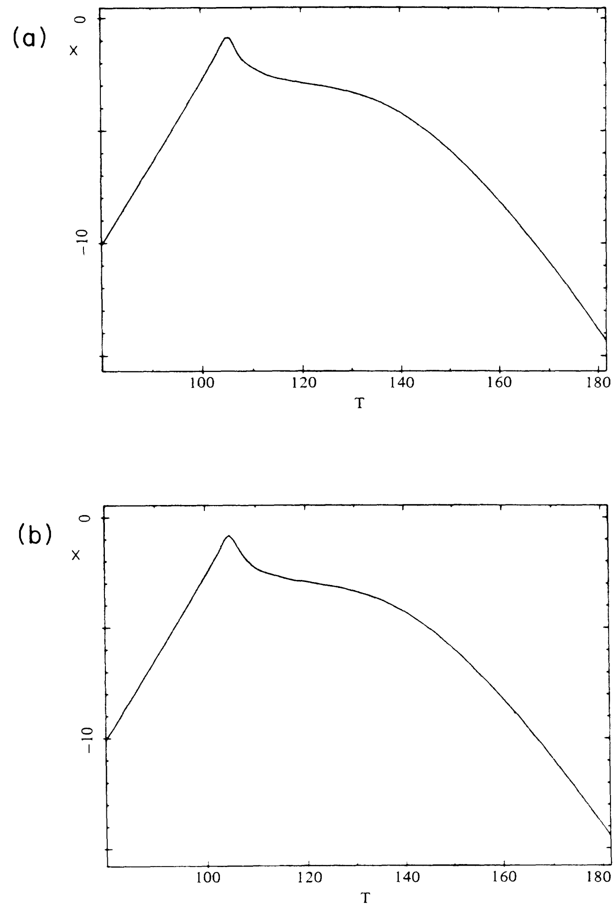

FIG. 3. Same as in Fig. 1 but for a kink-antikink collision in the presence of a force and dissipation. $\varepsilon=0.016>\varepsilon_{\text{th}}$, $\alpha=0.03$.

we checked that it correctly describes the real system for the values of ε and α in question. Figure 3 exhibits a typical case of an extremely good agreement between the reduced model and the complete description given by Eq. (1), in the case of an ε value just above the threshold.

The reason for such a good agreement essentially rests with the important role played by the loss term in smoothing away any inhomogeneity that would otherwise grow unimpeded. We noticed that these inhomogeneities become all the more important as the corresponding coordinate y gets larger, especially in the absence of damping. Indeed, if so, the interaction between the kink and the antikink is negligible in comparison to the effects of the driving and dissipation terms. The problem then becomes almost equivalent to that of an isolated kink (or antikink) in the presence of the hitherto considered perturbative terms (see for instance Ref. 6). Thanks to the loss term, the reduced description proceeding from system (8a),(8b) still remains valid for long enough times (see Fig. 3).

In a word, we have here elaborated a useful collective-coordinate transformation in view of a detailed study of the internal dynamical features displayed by a perturbed bion. The reliability of the ensued model proves to be very satisfactory in the case of close or intermediate interaction, thus providing a suitable representation of bion systems in terms of everlasting kink-antikink profiles.

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