Generalized coherent states for electrons in external fields and application to potential scattering

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By continuing earlier investigations, generalized coherent states are introduced to describe the quasiclassical motion of electrons in a microwave and in a homogeneous magnetic field. The microwave is circularly polarized and propagates along the direction of the magnetic field. In view of possible applications to laboratory plasmas, the problem is treated nonrelativistically and the microwave is taken in the dipole approximation. The generalized coherent states emerge in polar coordinates (ρ,ϕ) from ordinary Landau states $|l,m>$ by an appropriate choice of unitary and displacement operations. These coherent states are shown to form a complete and orthonormal set. Consequently, these states may be conveniently used for the description of scattering processes. As an example, scattering of electrons by a screened Coulomb potential is considered in the presence of the above two external fields. For simplicity, the scattering potential is treated in the first-order Born approximation. The quasiclassical features of the corresponding cross sections for the induced nonlinear processes are discussed.

I. INTRODUCTION

The investigation of scattering processes of charged particles in strong external electromagnetic fields is of interest in fusion research and in astrophysics. There are, however, also features of these scattering phenomena which deserve attention from a fundamental point of view. In particular, potential scattering of electrons in a strong laser field has been quite intensively investigated in the past two decades.[1] If a magnetic field B is also present, additional complications arise for electron scattering.[2] This, for example, is the case in laser interactions with a magnetized plasma. The treatment of scattering in the two external fields requires the solution of the following two problems. First of all, a proper formulation of the scattering boundary conditions in a magnetic field has to be found, and secondly, the question arises as to how the limit $B \to 0$ has to be taken in order to obtain the field-free results.[3]

A few years ago, we started to investigate the possibility of describing the electron motion in the two external fields by conveniently chosen wave packets, the motion of which would follow the corresponding classical trajectories. This description would establish a connection between the quantum-mechanical boundary-value problem and the corresponding classical initial-value problem of electron motion in the above external fields. As a first step in this direction, one of the present authors (S.V.) introduced a coherent state description for electrons which only interact with a homogeneous magnetic field.[4] This investigation also showed how the limit $B \to 0$ has to be consistently formulated. At the same time, we moreover considered the classical limit of Compton scattering and electron scattering in the two external fields.[5] In these calculations we finally approached highly excited Landau states for the ingoing and outgoing electrons, respectively. Thus we were able to derive results which permitted a quasiclassical interpretation. Compton scattering, in particular, furnished valuable insight into this limit, since for this problem entirely classical cross-section calculations can be performed.

By continuing these investigations, we will show in the present paper how the quantum-mechanical motion of electrons, which are simultaneously embedded in a microwave and a constant homogeneous magnetic field, can be described by generalized coherent states. For these electrons the motion of the corresponding wave packets then follows classical trajectories. These coherent states are obtained in polar coordinates (ρ,φ) from ordinary Landau states $|l,m\rangle$ by a set of unitary and displacement operations. These states turn out to form a complete and orthonormal set. Therefore, they may be used as a basis for the description of scattering processes in the two external fields. As an example, we shall consider potential scattering of electrons. For simplicity, we choose a screened Coulomb potential, the effect of which will be treated to first-order of the Born approximation.

In Sec. II we derive the general form of electron states which are dressed by a circularly polarized electromagnetic plane wave in the dipole approximation, and by a constant homogeneous magnetic field. The magnetic field is oriented along the z axis and it is perpendicular to the directions of the polarization vector of the plane wave. From these dressed states we shall construct in Sec. III coherent states and generalized coherent states, and we shall discuss their properties in some detail. These states will then be applied to potential scattering of electrons in Sec. IV, and the quasiclassical features of the various field-induced nonlinear processes will be discussed. Potential scattering constitutes only a typical example for the application of the above generalized coherent states. We conclude our paper with a summary of the main results of our investigation in Sec. V.

The choice of the combined external fields, introduced in Sec. II, is occasionally called the Redmond configuration. Redmond[6] was the first who not only presented an elegant exact solution of the classical relativistic equation of motion for such a superposition of an electromagnetic plane wave and a magnetic field, but who also presented the corresponding exact solutions of the Klein-Gordon and Dirac equations of motion. Moreover he related these latter solutions to the classical result. In the work of Redmond,[6] however, no specific application of these solutions to a scattering problem has been considered, nor have the generalized coherent states been introduced, as we shall do in Sec. III, restricting our investigations to the nonrelativistic domain.

II. GENERAL FORM OF THE DRESSED ELECTRON STATES

We consider an electron of charge $-e$ and mass M which is embedded in an external field of the form

$$\mathbf{A}^{\rm ext}=\tfrac{1}{2}\mathbf{B}\times\mathbf{x}+\mathbf{A}_\omega(t)$$
$$=[-\tfrac{1}{2}By+(cF/\omega)\cos(\omega t),\tfrac{1}{2}Bx+(cF/\omega)\sin(\omega t),0]\ . \quad (1)$$

For convenience, we have represented the constant homogeneous magnetic field $\mathbf{B}$ in an (x,y) symmetric gauge. The circularly polarized microwave is determined by the vector potential $\mathbf{A}_\omega(t)$ and it is taken in the dipole approximation. This field has field strength F and frequency ω. The quantum-mechanical motion of an electron in these two fields will be treated nonrelativistically. Rochlin and Davidovich[7] have estimated the range of validity of the dipole approximation and of the nonrelativistic treatment of problems in magnetic fields. According to these authors, our two approximations are certainly valid for laboratory plasmas but can even be applied to problems in the astrophysical range of parameters.

For an electron in the external field (1), the Hamiltonian operator reads

$$\hat{H}=(2M)^{-1}[\hat{\mathbf{p}}+(e/c)\mathbf{A}^{\rm ext}]^2$$
$$=\hat{\mathbf{p}}^2/2M+\tfrac{1}{2}M\omega_c^2\hat{q}^2$$
$$+(eF/\omega)[(\hat{p}/M)\cos(\omega t)+\omega_c\hat{q}\sin(\omega t)]$$
$$+(2M)^{-1}(eF/\omega)^2+\hat{p}_z^2/2M\ . \quad (2)$$

In (2) we have introduced the following pair of canonical operators:

$$\hat{p}\equiv\hat{p}_x+\tfrac{1}{2}M\omega_c\hat{y},\quad M\omega_c\hat{q}\equiv\hat{p}_y+\tfrac{1}{2}M\omega_c\hat{x}\ , \quad (3)$$

which fulfill the commutation relation

$$[\hat{q},\hat{p}]=i\hbar\ . \quad (3a)$$

Moreover, we have introduced the cyclotron frequency $\omega_c=eB/Mc$. Next, it is convenient to define the following "collective" creation and annihilation operators:

$$A^\dagger\equiv(2M\hbar\omega_c)^{-1/2}(M\omega_c\hat{q}-i\hat{p})\ ,$$
$$A\equiv(2M\hbar\omega_c)^{-1/2}(M\omega_c\hat{q}+i\hat{p})\ , \quad (4a)$$

for which

$$[A,A^\dagger]=1\ . \quad (4b)$$

By means of (4a) and (4b), the Hamiltonian (2) can be brought into the form

$$\hat{H}=\hbar\omega_c(A^\dagger A+\tfrac{1}{2})-i(eF/\omega)(\hbar\omega_c/2M)^{1/2}$$
$$\times[A\exp(i\omega t)-A^\dagger\exp(-i\omega t)]$$
$$+(2M)^{-1}(eF/\omega)^2+\hat{p}_z^2/2M\ . \quad (5)$$

For convenience of our later considerations, we shall moreover introduce the creation and annihilation operators $(a^\dagger,a)$ and $(b^\dagger,b)$, corresponding, respectively, to excitations and deexcitations of the x and y components of the oscillatory motion of the electron in the magnetic field. These operators are defined by

$$a^\dagger\equiv(4M\hbar\omega_c)^{-1/2}(M\omega_c\hat{x}-2i\hat{p}_x)\ ,$$
$$a\equiv(4M\hbar\omega_c)^{-1/2}(M\omega_c\hat{x}+2i\hat{p}_x)\ , \quad (6a)$$
$$b^\dagger\equiv(4M\hbar\omega_c)^{-1/2}(M\omega_c\hat{y}-2i\hat{p}_y)\ ,$$
$$b\equiv(4M\hbar\omega_c)^{-1/2}(M\omega_c\hat{y}+2i\hat{p}_y)\ , \quad (6b)$$

with

$$[a,a^\dagger]=1,\quad [b,b^\dagger]=1\ . \quad (6c)$$

By means of (3), (4a), (6a), and (6b), the collective operators $(A^\dagger,A)$ can be expressed in terms of $(a^\dagger,b^\dagger)$ and (a,b):

$$A^\dagger=2^{-1/2}(a^\dagger+ib^\dagger),\quad A=2^{-1/2}(a-ib)\ . \quad (7)$$

Because of (7) we obtain from (5) an alternative form of the Hamiltonian

$$\hat{H}=\tfrac{1}{2}\hbar\omega_c(a^\dagger a+b^\dagger b+1)+\frac{i}{2}\hbar\omega_c(ab^\dagger-a^\dagger b)$$
$$-(eF/\omega)(\hbar\omega_c/4M)^{1/2}[(b+ia)\exp(i\omega t)$$
$$+(b^\dagger-ia^\dagger)\exp(-i\omega t)]$$
$$+(2M)^{-1}(eF/\omega)^2+\hat{p}_z^2/2M\ . \quad (8)$$

From this equation it becomes evident that the electron's transverse motion in the homogeneous magnetic field and in the microwave field can be described by two forced linear harmonic oscillators that are coupled to each other via the energy-conserving interaction term $i\hbar\omega_c(ab^\dagger-a^\dagger b)/2$. This coupling term represents the Larmor energy, which is proportional to the z component of the angular momentum of the electron. The total wave function

$|\Psi\rangle$, satisfying the Schrödinger equation

$$\hat{H}\,|\Psi\rangle_t = i\hbar\partial_t\,|\Psi\rangle_t\ , \tag{9}$$

can be factorized since there is no coupling between the longitudinal and the transverse components of the electronic motion,

$$|\Psi\rangle_t = |p_z\rangle_t\,|\psi\rangle_t\ , \tag{9a}$$

where $|p_z\rangle_t$ represents a momentum eigenstate,

$$\langle z\,|p_z\rangle t = (2\pi\hbar)^{-1/2}\exp\{(i/\hbar)[p_z z-(p_z^2/2M)t]\}\ . \tag{9b}$$

Equation (9b) is a free particle de Broglie wave of momentum p_z which describes the uniform longitudinal motion of the electron. On the other hand, $|\psi\rangle_t$ satisfies the Schrödinger equation of the transverse electron motion,

$$\{\hbar\omega_c(A^\dagger A+\tfrac{1}{2})-i(eF/\omega)(\hbar\omega_c/2M)^{1/2}\times[A\exp(i\omega t)-A^\dagger\exp(-i\omega t)]+(2M)^{-1}(eF/\omega)^2\}\,|\psi\rangle_t = i\hbar\partial_t\,|\psi\rangle_t\ . \tag{10}$$

In the following, we shall show that (10) can be transformed to the Schrödinger equation of an electron embedded in the magnetic field alone. As a first step towards the elimination of the interaction term in (10), we get rid of the time dependence by using the ansatz

$$|\psi\rangle_t = \exp[-i\omega t(A^\dagger A+\tfrac{1}{2})]\,|\psi_1\rangle_t\ . \tag{11}$$

If (11) is inserted into (10), we obtain for $|\psi_1\rangle_t$ the equation of motion

$$[\hbar(\omega_c-\omega)(A^\dagger A+\tfrac{1}{2})-i(eF/\omega)(\hbar\omega_c/2M)^{1/2}(A-A^\dagger)+(2M)^{-1}(eF/\omega)^2]\,|\psi_1\rangle_t = i\hbar\partial_t\,|\psi_1\rangle_t\ . \tag{11a}$$

It is interesting to note that in the case of exact resonance ($\omega=\omega_c$), the first term in (11a) will vanish. In classical terms this corresponds to an aperiodic unbounded transverse electron motion, as discussed by Redmond[6] and Varró *et al.*[8] In the present paper we shall always assume $\omega\neq\omega_c$. Now we introduce the displacement operator

$$D(\sigma)=\exp(\sigma A^\dagger-\sigma^* A)\ , \tag{12}$$

and we define the transformed wave function $|\psi_2\rangle_t$ by means of the relation

$$|\psi_1\rangle_t = D(\sigma)\,|\psi_2\rangle_t\ . \tag{13}$$

Introducing (13) into (11a), we can easily show that the interaction term is eliminated if we choose the parameter σ as

$$\sigma = i(eF/\omega)(\hbar\omega_c/2M)^{1/2}[\hbar(\omega-\omega_c)]^{-1}\ . \tag{13a}$$

This yields the transformed equation for $|\psi_2\rangle_t$:

$$[\hbar(\omega-\omega_c)(A^\dagger A+\tfrac{1}{2}-|\sigma|^2)+(2M)^{-1}(eF/\omega)^2]\,|\psi_2\rangle_t = i\hbar\partial_t\,|\psi_2\rangle_t\ . \tag{13b}$$

Finally, we make the ansatz

$$|\psi_2\rangle_t = \exp[i\omega t(A^\dagger A+\tfrac{1}{2})]\,|\Phi\rangle_t\ , \tag{14}$$

to obtain from (13b) the following equation for $|\Phi\rangle_t$:

$$\hbar\omega_c(A^\dagger A+\tfrac{1}{2})\,|\Phi\rangle_t = i\hbar\partial_t\,|\Phi\rangle_t\ . \tag{14a}$$

According to the definitions of $(A^\dagger, A)$ in (4a), we see, however, that (14a) is the Schrödinger equation for an electron in the presence of the constant homogeneous magnetic field alone. Consequently, taking into account (14), (13), (11), (9a), and (13a), the total wave function $|\Psi\rangle_t$ of the electron motion in both external fields can be cast into the form

$$|\Psi\rangle_t = |p_z\rangle_t D(\sigma)\,|\Phi\rangle_t \exp(-i\Delta E t/\hbar)\ , \tag{15}$$

where

$$D(\sigma)\equiv\exp\{(i/\hbar)\lambda_0(M\hbar\omega_c/2)^{1/2}\times[A\exp(i\omega t)+A^\dagger\exp(-i\omega t)]\}\ , \tag{15a}$$

with

$$\lambda_0\equiv(eF/Mc\omega)(\omega/\Delta\omega)\lambda\!\!\!^{-},\quad \lambda\!\!\!^{-}=c/\omega,\quad \Delta\omega=\omega-\omega_c\ , \tag{15b}$$

and

$$\Delta E\equiv(2M)^{-1}(eF/\omega)^2+\hbar\Delta\omega\,|\sigma|^2\ . \tag{15c}$$

As we shall see later, λ_0 represents the amplitude of that part of the classical oscillatory motion of the electron which oscillates with the microwave frequency ω. In (15c), the first part of the energy shift ΔE is the usual intensity-dependent ac Stark shift due to the presence of the microwave. This shift can be expressed in the form $Mc^2\mu^2/2$, where

$$\mu\equiv eF/Mc\omega\ . \tag{16}$$

This is the dimensionless intensity parameter which measures the electric field strength of the microwave. As we can see from (15b), the effective intensity parameter, $\mu(\omega/\Delta\omega)$, can be much larger than the intensity parameter μ if the detuning $\Delta\omega$ is much smaller than the microwave frequency ω. Moreover, we have introduced in (15b) the reduced wavelength $\lambda\!\!\!^{-}$ of the microwave. Therefore, the amplitude λ_0 may be expressed in the form $\lambda_0=\mu(\omega/\Delta\omega)\lambda\!\!\!^{-}$.

For convenience in our later considerations, we shall present an alternative expression for the displacement operator $D(\sigma)$, as defined in (15a). By means of (7), $D(\sigma)$ can be expressed as a product of two displacement operators acting on Hilbert subspaces which belong to the x and y components of the transverse electron motion. This representation reads

$$D(\sigma)=D_a(\alpha)D_b(\beta)\ , \tag{17}$$

where

$$D_a(\alpha)=\exp(\alpha a^\dagger-\alpha^* a),\quad D_b(\beta)=\exp(\beta b^\dagger-\beta^* b)\ , \tag{18a}$$

and

$$\alpha=i\lambda_0(M\omega_c/2\hbar)^{1/2}\exp(-i\omega t)\ ,$$
$$\beta=-\lambda_0(M\omega_c/2\hbar)^{1/2}\exp(-i\omega t)\ . \qquad (18b)$$

The operators (18a) have the displacement properties

$$D_a^{-1}(\alpha)aD_a(\alpha)=a+\alpha,\quad D_b^{-1}(\beta)bD_b(\beta)=b+\beta\ . \qquad (18c)$$

With reference to (14a), the state vector $|\Phi\rangle_t$ satisfies the Schrödinger equation in the presence of the homogeneous magnetic field alone. The stationary solutions of this equation are the well-known Landau states $|\Phi\rangle_t = |l,m\rangle_t = |l,m\rangle\exp(-i\omega_{lm}t)$, where l and m are the radial and azimuthal quantum numbers, respectively. By introducing the polar coordinates (ρ,φ) through the definitions $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, these Landau states have the following form in coordinate representation (Landau and Lifshitz[9]):

$$\langle \mathbf{x}|\Phi\rangle_t=\langle x,y\,|l,m\rangle_t=\Phi_{l,m}(\rho,\varphi)\exp(-i\omega_{l,m}t)\ , \qquad (19a)$$

where

$$\Phi_{l,m}(\rho,\varphi)=(\gamma/\pi)^{1/2}\left[\frac{l!}{(l+|m|)!}\right]^{1/2}\exp(im\varphi)$$
$$\times\exp(-\xi/2)\xi^{|m|/2}L_l^{|m|}(\xi)\ . \qquad (19b)$$

In (19b) the $L_l^{|m|}$ are associated Laguerre polynomials (Gradshteyn and Ryzhik[10]) and we have introduced the abbreviations

$$\xi=\gamma\rho^2,\quad \gamma=eB/2\hbar c=M\omega_c/2\hbar\ . \qquad (19c)$$

Correspondingly, in (19a) the discrete energy levels $E_{l,m}$ of the transverse electron motion (Landau levels) are known to be

$$E_{l,m}=\hbar\omega_{l,m},\quad \omega_{l,m}=\omega_c[l+\tfrac{1}{2}(m+|m|+1)]\ ,$$
$$l=0,1,2,\ldots,\quad m=0,\pm1,\pm2,\ldots\ . \qquad (19d)$$

III. COHERENT AND GENERALIZED COHERENT STATES OF ELECTRONS IN EXTERNAL FIELDS

Coherent states of charged particles in a homogeneous magnetic field have been introduced by Malkin and Man'ko.[11] These states have been thoroughly studied by Varró,[4] in particular, from the point of view of the limit of zero magnetic field. Moreover, in a paper by Varró *et al.*,[8] a special class of coherent states of electrons simultaneously interacting with a homogeneous magnetic field and a microwave field have been presented. These states were applied to a semiclassical treatment of magneto-Raman scattering in a microwave field.

In this section we shall construct generalized coherent states of an electron which is simultaneously embedded in a constant homogeneous magnetic field and in a circularly polarized electromagnetic plane wave. These states will form a complete set of solutions of the Schrödinger equation (9), where the corresponding Hamiltonian is given by (8). These states, dressed by the external fields, will be used as a basis for calculations in perturbation theory, to be presented in Sec. IV.

We begin our investigation with a short review of the ordinary coherent states for electrons in a homogeneous magnetic field. Perhaps the simplest method for constructing such coherent states is based on the one-dimensional harmonic oscillator algebra.[4] Instead of starting from (14a), it is more convenient to consider the equivalent Schrödinger equation, which is obtained by means of (7), introducing the operators $(a^\dagger,a)$ and $(b^\dagger,b)$ of particle oscillations in the x and y directions, respectively. This equation reads

$$\left[\tfrac{1}{2}\omega_c(a^\dagger a+b^\dagger b+1)+\frac{i}{2}\omega_c(ab^\dagger-a^\dagger b)\right]|\Phi\rangle_t$$
$$=i\partial_t|\Phi\rangle_t\ . \qquad (20)$$

In (20) we eliminate the coupling between the two oscillatory motions by means of the ansatz

$$|\Phi\rangle_t=R(\tfrac{1}{2}\omega_c t)|\Phi_1\rangle_t\ , \qquad (21)$$

where the operator $R(\lambda)$ is defined by

$$R(\lambda)\equiv\exp[\lambda(ab^\dagger-a^\dagger b)]\ . \qquad (21a)$$

This operator has the interesting property of rotating the linear combinations of a and b in the (a,b) plane,

$$R^{-1}(\lambda)aR(\lambda)=a\cos\lambda-b\sin\lambda\ ,$$
$$R^{-1}(\lambda)bR(\lambda)=b\cos\lambda+a\sin\lambda\ . \qquad (21b)$$

After having eliminated the coupling term by means of (21), we are left with the equation

$$\tfrac{1}{2}\omega_c(a^\dagger a+b^\dagger b+1)|\Phi_1\rangle_t=i\partial_t|\Phi_1\rangle_t\ . \qquad (21c)$$

This is simply the Schrödinger equation of two independent linear harmonic oscillators of the same frequency $\omega_c/2$. The coherent states of this system can be easily generated from the ground state by means of the displacement operations

$$|\Phi_1\rangle_t=|\alpha_1,\beta_1\rangle_t=|\alpha_1\exp(-i\omega_c t/2)\rangle_a\,|\beta_1\exp(-i\omega_c t/2)\rangle_b\exp(-i\omega_c t/2)$$
$$=D_a[\alpha_1\exp(-i\omega_c t/2)]D_b[\beta_1\exp(-i\omega_c t/2)]|0\rangle_a\,|0\rangle_b\exp(-i\omega_c t/2)\ , \qquad (22)$$

where D_a and D_b are the displacement operators defined in (18), and α_1 and β_1 are arbitrary complex parameters. By using (21), the coherent states of the original system (20) can be immediately obtained. After some algebra, we thus get

$$|\Phi\rangle_t = |\alpha,\beta\rangle_t = D_a[\alpha \exp(-i\omega_c t)+\beta] \times D_b[i(\alpha \exp(-i\omega_c t)-\beta)]|0,0\rangle \times \exp(-i\omega_c t/2) , \quad (23)$$

where

$$\alpha \equiv \tfrac{1}{2}(\alpha_1 - i\beta_1), \quad \beta \equiv \tfrac{1}{2}(\alpha_1 + i\beta_1) , \quad (23a)$$

and $|0,0\rangle = |0\rangle_a |0\rangle_b$ is the ground state of the coupled harmonic oscillators corresponding to a Landau state with radial and azimuthal quantum numbers $l = m = 0$.

If an electron is in one of its coherent states $|\alpha,\beta\rangle_t$, then the probability distribution for the position of the electron in a plane perpendicular to the magnetic induction lines is a Gaussian wave packet of constant width. The center of this packet gyrates along one of the possible classical trajectories in the magnetic field. The characteristic parameters of these trajectories, namely, initial position and initial velocity of the electron, are implicitly contained in the complex numbers α and β.[4] It will be useful to present an alternative form of the coherent states $|\alpha,\beta\rangle_t$ in terms of the time-independent one-dimensional oscillator coherent states $|\alpha_1\rangle$ and $|\beta_1\rangle$:

$$|\alpha,\beta\rangle_t = \exp(-iH_B t/\hbar) D_a(\alpha_1) D_b(\beta_1)|0,0\rangle = \exp(-iH_B t/\hbar)|\alpha_1\rangle_a |\beta_1\rangle_b . \quad (24)$$

In this equation, H_B is the Hamiltonian of an electron in a homogeneous magnetic field alone. This Hamiltonian can be inferred from (20) to be

$$H_B \equiv \tfrac{1}{2}\hbar\omega_c(a^\dagger a + b^\dagger b + 1) + \frac{i}{2}\hbar\omega_c(ab^\dagger - a^\dagger b) . \quad (24a)$$

From (24) it is clear that the states $|\alpha,\beta\rangle_t$ form an overcomplete set, as the usual coherent states of a linear harmonic oscillator do. For these states the completeness relation reads

$$\pi^{-2}\int d^2\alpha_1 \int d^2\beta_1 |\alpha,\beta\rangle_t\,{}_t\langle\alpha,\beta| = 1 , \quad (25)$$

where

$$d^2\alpha_1 \equiv d(\mathrm{Re}\alpha_1)d(\mathrm{Im}\alpha_1), \quad d^2\beta_1 \equiv d(\mathrm{Re}\beta_1)d(\mathrm{Im}\beta_1) , \quad (25a)$$

and in (25) the integration is extended over the whole complex α_1 and β_1 plane. In (25a) Re and Im denote the real and imaginary parts, respectively.

Since the coherent states $|\alpha,\beta\rangle_t$ form a complete set on the Hilbert space belonging to the transverse part of the electronic motion in the magnetic field, they could in principle, be used as a basis set in perturbation theory, in order to describe scattering processes in the presence of a homogeneous magnetic field. This would allow us to interpret the quantum-mechanical results in terms of classical particle trajectories. Since, however, the coherent states are not orthogonal to one another, even in Born approximation we encounter a set of integro-differential equations for the scattering amplitudes. As we shall show in Sec. IV, we can get rid of this difficulty by using a set of generalized coherent states, which we shall now introduce.

To this end, we recall the expression (24) for the coherent states $|\alpha,\beta\rangle_t$, and we define the generalized coherent states as a natural generalization of (24) in the following manner

$$|\alpha,\beta;l,m\rangle_t \equiv \exp(-iH_B t/\hbar) D_a(\alpha_1) D_b(\beta_1)|l,m\rangle , \quad (26)$$

where $|l,m\rangle$ are ordinary Landau states of the form 19(a)–19(d). By analogy with (23), we can obtain from (26), after some algebra, an alternative form of the generalized coherent states

$$|\alpha,\beta;l,m\rangle = D_a[\alpha \exp(-i\omega_c t)+\beta] \times D_b[i(\alpha \exp(-i\omega_c t)-\beta)] \times |l,m\rangle \exp(-i\omega_{l,m} t) . \quad (26a)$$

We now consider an electron, which is simultaneously embedded in a constant homogeneous magnetic field and in a microwave field, and we follow the general outline of solution of the Schrödinger equation which was presented in Sec. II. Accordingly, if we want to take into account the effect of the radiation field on the electron, we have to determine the total wave function $|\Psi\rangle_t$, which is given by (15). In the present case, however, the solution $|\Phi\rangle_t$ is not an ordinary Landau state (19a) and (19b), but instead has to be considered as a generalized coherent state $|\alpha,\beta;l,m\rangle_t$. Hence, treating only the transverse part of the electron motion, we obtain from (26a) by applying the displacement operator $D(\sigma)$ given by (15a), (17), (18a), and (18b):

$$D|\Phi\rangle_t = D|\alpha,\beta;l,m\rangle_t = D_a[ig\exp(-i\omega t)+\alpha\exp(-i\omega_c t)+\beta] D_b\{i[ig\exp(-i\omega t)+\alpha\exp(-i\omega_c t)-\beta]\}|l,m\rangle \times \exp(-i\omega_{l,m}t)\exp(2ig\{\alpha\exp[i(\omega-\omega_c)t]+\alpha^*\exp[-i(\omega-\omega_c)t]\}) , \quad (27)$$

where

$$g=\lambda_0(\gamma/2)^{1/2} \tag{27a}$$

with λ_0 and γ defined by (15b) and (19c), respectively. If we introduce the two-dimensional vector $\mathbf{u}_\perp(t)\equiv(u_x(t),u_y(t),0)$ with components

$$u_x(t)\equiv(2/\gamma)^{1/2}\mathrm{Re}[ig\exp(-i\omega t)+\alpha\exp(-i\omega_c t)+\beta] , \tag{28a}$$

$$u_y(t)\equiv(2/\gamma)^{1/2}\mathrm{Im}[ig\exp(-i\omega t)+\alpha\exp(-i\omega_c t)-\beta] , \tag{28b}$$

then we can show that the probability distributions for the transverse electron position $\mathbf{x}_\perp\equiv(x,y)$ read

$$|\langle\mathbf{x}_\perp|D|\alpha,\beta;l,m\rangle_t|^2=|\Phi_{l,m}(\mathbf{x}_\perp-\mathbf{u}_\perp(t))|^2 . \tag{29}$$

Here the electrons move in two external fields and they are described by generalized coherent states (27), while the $\Phi_{l,m}(\mathbf{x}_\perp)$ are ordinary Landau states (19b) in coordinate representation. As we shall discuss below, the transverse parts of the classical electron trajectories in our external field configuration have the same structure as $\mathbf{u}_\perp(t)=(u_x(t),u_y(t))$ of (28a) and (28b). Consequently, if we properly adjust the complex parameters α and β, we can always construct generalized coherent states, the probability distributions of which exactly follow the possible classical trajectories. The parameters α and β can be expressed as certain combinations of the initial position and the initial velocity of a classical electron. This means that, up to an irrelevant phase factor in the wave function, there is a one-to-one correspondence between the classical trajectories and the generalized coherent states, if a fixed pair of quantum numbers (l,m) is considered.

In order to derive the relations between the parameters α and β and the initial values of the corresponding classical trajectory, it is convenient to introduce the complex position $u(t)$ and the corresponding complex velocity $v(t)$ through the definitions

$$u(t)\equiv x_c(t)-iy_c(t) , \tag{30a}$$

$$v(t)\equiv\dot{x}_c(t)-i\dot{y}_c(t) , \tag{30b}$$

where $x_c(t)$ and $y_c(t)$ are, respectively, the x and y components of the classical transverse electron trajectory, and the dot denotes as usual derivation with respect to time. The nonrelativistic classical Lorentz equation of motion for an electron in the two external fields reads

$$M\dot{\mathbf{v}}=-e\mathbf{E}_\omega(t)-(e/c)\mathbf{v}\times\mathbf{B} , \tag{31a}$$

where

$$\mathbf{E}_\omega(t)=F(\sin(\omega t),-\cos(\omega t),0),\quad \mathbf{B}=(0,0,B) , \tag{31b}$$

which are the electric field of the microwave and the magnetic induction field, respectively. They follow from the vector potential (1). The transverse components of the classical trajectory can be easily determined by employing the complex combinations $u(t)$ and $v(t)$ defined in (30a) and (30b). Thus we obtain for $u(t)$:

$$u(t)=x_c(t)-iy_c(t)=i\lambda_0\exp(-i\omega t)+i(v_0/\omega_c)\exp(-i\omega_c t)+u_0 , \tag{32}$$

where λ_0 has been defined in (15b). As we can see, it is in fact the amplitude of that part of the oscillatory motion of the electron, which oscillates with the microwave frequency. Moreover, the parameters v_0 and u_0 in (32) are determined by the initial velocity $v(t_0)$ and the initial position $u(t_0)$, where t_0 is some arbitrary initial time. We get

$$v_0\equiv v(t_0)\exp(i\omega_c t_0)-\omega\lambda_0\exp(-i\Delta\omega t_0) , \tag{32a}$$

$$u_0\equiv u(t_0)-i\lambda_0\exp(-i\omega t_0)-i(v_0/\omega_c)\exp(-i\omega_c t_0) , \tag{32b}$$

where $\Delta\omega=\omega-\omega_c$. From (32), (32a), (32b), (28a), and (28b) it is obvious that $u(t)$ and $u_x(t)-iu_y(t)$ have the same structure. Therefore, if we choose

$$\alpha=i(v_0/\omega_c)(\gamma/2)^{1/2},\quad \beta=u_0^*(\gamma/2)^{1/2} , \tag{33}$$

then $u_x(t)-iu_y(t)$ coincides with the classical complex trajectory $u(t)=x_c(t)-iy_c(t)$. Consequently, the vector $\mathbf{u}_\perp(t)=(u_x(t),u_y(t))$ coincides with the classical transverse position vector $\mathbf{x}_{c\perp}\equiv(x_c(t),y_c(t),0)$ of the electron

$$\mathbf{u}(t)=\mathbf{x}_{c\perp}(t) . \tag{33a}$$

Henceforth we shall denote the generalized coherent state $D|\alpha,\beta;l,m\rangle_t$, defined in (27), and corresponding to a classical complex electron trajectory $u(t)$, by

$$|u(t);l,m\rangle\equiv D|\alpha,\beta;l,m\rangle_t . \tag{34}$$

Here $u(t)$ is given by (32), (32a), and (32b). The relations between the classical parameters (u_0,v_0) and the original parameters (α,β) are shown in (33). With the compact notation introduced by (34), we can rewrite (29) in the form

$$|\langle\mathbf{x}_\perp|u(t);l,m\rangle|^2=|\Phi_{l,m}(\mathbf{x}_\perp-\mathbf{x}_{c\perp}(t))|^2 . \tag{35}$$

The ordinary coherent states for electrons in our external field configuration are special cases of the generalized coherent states $|u(t);l,m\rangle$ with $l=m=0$. Hence they are shifted ground states of the Landau type. We shall denote them by

$$|u(t)\rangle\equiv|u(t);0,0\rangle . \tag{36}$$

As has been mentioned earlier, these coherent states form an overcomplete set with

$$\int d^4u\,|u(t)\rangle\langle u(t)|=1 , \tag{37a}$$

$$|\langle u'(t)|u(t)\rangle|^2=\exp[-\gamma|u_0'-u_0|^2-(\gamma/\omega_c^2)|v_0'-v_0|^2] , \tag{37b}$$

where

$$d^4u\equiv\pi^{-2}d(\mathrm{Re}\alpha_1)d(\mathrm{Im}\alpha_1)d(\mathrm{Re}\beta_1)d(\mathrm{Im}\beta_1) . \tag{37c}$$

According to the definition of the parameter γ in (19c),

$\gamma^{-1/2}$ is a characteristic length in our magnetic field problems.

The complex parameters α_1 and β_1 are related to the initial values of the classical trajectories via (33), (32a), (32b) and (23a). Therefore, (37a) says that the integral of the dyad $|u(t)\rangle\langle u(t)|$ taken over all classical trajectories (more precisely, taken over all possible initial values of the classical motion) yields the unit operator of the Hilbert space belonging to the two-dimensional quantum motion of an electron in the two external fields. Furthermore, (37b) shows that the overlap of two coherent states is small, if the initial data (u_0,v_0) and (u_0',v_0') of the two corresponding classical trajectories are far from each other in the classical parameter space. This overlap also decreases with increasing value of the magnetic parameter γ.

In the present paper we shall not enter into a detailed discussion of the formal properties of the generalized coherent states. We shall merely present the completeness and orthogonality properties, which we shall need in Sec. IV. These two properties of the generalized coherent states originate in the corresponding properties of the ordinary Landau states $|l,m\rangle$ of (19a) and (19b). From (34), (25), (19a), and (19b) immediately follows that for any classical trajectory $u(t)$ the corresponding coherent states $|u(t);l,m\rangle$ satisfy the following completeness and orthogonality relations:

$$\sum_{l,m} |u(t);l,m\rangle\langle u(t);l,m| = 1 , \qquad (38a)$$

$$\langle u(t);l,m|u(t);l',m'\rangle = \delta_{l,l'}\delta_{m,m'} . \qquad (38b)$$

Finally, if we choose in (15) for $D|\Phi\rangle_t$ a generalized coherent state (34), then the total wave function of (15) has the form

$$|\Psi\rangle_t = |p_z\rangle_t\, |u(t);l,m\rangle \exp(-i\Delta E t/\hbar) , \qquad (39)$$

where ΔE is defined by (15a). In Sec. IV we shall apply the complete orthonormal set of this type of wave functions to perturbation calculations.

IV. APPLICATION OF GENERALIZED COHERENT STATES TO POTENTIAL SCATTERING

In Sec. II we have derived the exact solution (15) and (15a)–(15c) of the Schrödinger equation for an electron embedded in a magnetic field and in a microwave. We shall consider in this section scattering of electrons by a screened Coulomb potential $V(r)=(A/r)\exp(-k_D r)$, where k_D is the Debye wave number. In order to describe electron scattering in our external field configuration, we represent the initial and final states by dressed states of the form (15). In (15) we may choose for $D|\Phi\rangle_t$ either coherent states $|u(t)\rangle$, or generalized coherent state, $|u(t);l,m\rangle$ as introduced in Sec. III, and we shall discuss both possibilities. For simplicity, we shall treat the scattering potential to lowest order of the Born approximation. Consequently, the matrix element corresponding to the diagram of Fig. 1 has to be evaluated. In this figure the double line indicates dressed electron states.

We employ the time-dependent perturbation theory of Dirac to evaluate the scattering matrix element. The total wave function $|\chi\rangle_t$ satisfies the Schrödinger equation

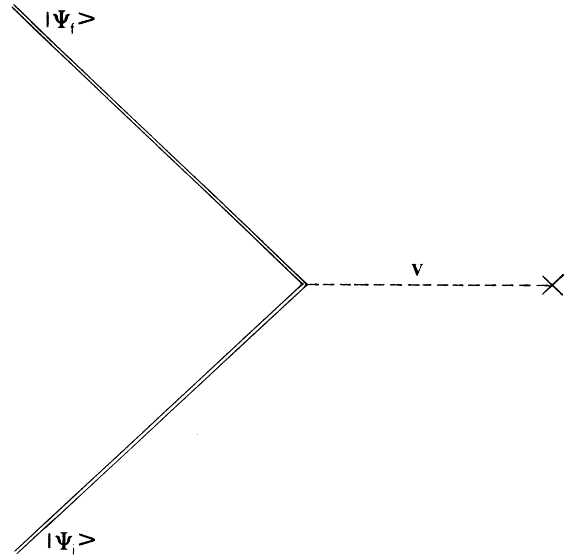

FIG. 1. Feynman diagram of electron scattering in a screened Coulomb potential V in the first-order Born approximation. The double lines represent the ingoing and outgoing electrons, which are described by the generalized coherent states $|\Psi_i\rangle \equiv |u(t);l,m\rangle|p_z\rangle_t$ and $|\Psi_f\rangle \equiv |u'(t);l',m'\rangle|p_z'\rangle_t$, respectively. This diagram corresponds to the transition matrix element (46).

$$(H+V)|\chi\rangle_t = i\hbar\partial_t|\chi\rangle_t , \qquad (40)$$

where H is given by (2). We split $|\chi\rangle_t$ into a sum of the initial state vector $|\Psi_i\rangle_t$ and a correction term $|\Psi_K\rangle_t$:

$$|\chi\rangle_t = |\Psi_i\rangle_t + |\Psi_K\rangle_t . \qquad (41)$$

For $|\Psi_i\rangle_t$ we take one of the solutions of the unperturbed Schrödinger equation (9). The correction vector $|\Psi_K\rangle_t$ can be expressed as a superposition of unperturbed states satisfying (9). In what follows, we have two possibilities.

According to (39), (36), and (37a), the coherent states $|u(t)\rangle$ form a complete set. Therefore, we can express $|\Psi_K\rangle_t$ as an *integral* over the states $|u(t)\rangle|p_z\rangle_t$. The term containing the energy shift ΔE, given by (15c), does not need to be written down explicitly, since it drops out in the evaluation of transition probabilities. Hence we write

$$|\Psi_K\rangle_t = \int dp_z \int d^4u\, C_{p_z,u}(t)\, |u(t)\rangle\, |p_z\rangle_t , \qquad (42)$$

where $C_{p_z,u}(t)$ are the unknown transition amplitudes which have to be determined. On the other hand, if we use the relation (38a) for the completeness of the generalized coherent states, then we may express $|\Psi_K\rangle_t$ as a *sum* over generalized coherent states belonging to some classical trajectory $u'(t)$. Thus we may write

$$|\Psi_K\rangle_t = \int dp_z \sum_{l,m} C_{p_z,l,m}(t;u')\, |p_z\rangle\, |u'(t);l,m\rangle . \qquad (43)$$

In the first case, where $|\Psi_K\rangle_t$ is given by (42), we obtain in the first-order Born approximation for the unknown amplitudes $C_{p_z,u}(t)$ the following integro-differential equation:

$$i\hbar \int dp_z \int d^4u\, \dot{C}_{p_z,u}(t)\,|u(t)\rangle\,|p_z\rangle_t = V\,|\Psi_i\rangle_t \ . \tag{44}$$

Apparently, we cannot transform (44) into an ordinary differential equation for the amplitudes $C_{p_z,u}(t)$ by projecting each side of (44) on $\langle u'(t)|\,{}_t\langle p_z'|$, since according to (37b) the set of states $|u(t)\rangle$ is not orthogonal. We can overcome this difficulty, however, if we use the representation (43) of $|\Psi_K\rangle_t$ as a sum over generalized coherent states $|u'(t);l,m\rangle$. In fact, in the first-order Born approximation the amplitudes $C_{p_z,l,m}(t;u')$ satisfy the equation

$$i\hbar \int dp_z \sum_{l,m} \dot{C}_{p_z,l,m}(t;u')\,|p_z\rangle_t\,|u'(t);l,m\rangle = V\,|\Psi_i\rangle_t \tag{45}$$

and this equation can be simplified by means of the orthonormality relation (38b) of the generalized coherent states. Thus we obtain for the $C_{p_z,l,m}(t;u')$ the following ordinary differential equation

$$i\hbar\dot{C}_{p_z',l',m'}(t,u') = {}_t\langle p_z'|\,\langle u'(t);l',m'|\,V\,|\Psi_i\rangle_t \ , \tag{45a}$$

which can be immediately solved.

Consequently, if we choose for the transverse part of the initial state $|\Psi_i\rangle_t$ a generalized coherent state $|u(t);l,m\rangle$, then we can evaluate from (45a) the transition matrix elements T_{fi}, which are equivalent to the amplitudes $C_{p_z',l',m'}(t=\infty,u')$. Hence,

$$T_{fi}(p_z',u',l',m' \leftarrow p_z,u,l,m) = T_{fi} = (i\hbar)^{-1}\int_{-\infty}^{+\infty} dt\ {}_t\langle p_z'|\,\langle u'(t);l',m'|\,V\,|u(t);l,m\rangle\,|p_z\rangle_t \ . \tag{46}$$

In evaluating T_{fi} we should pay attention to the fact that we have expressed in (43) the correction state $|\Psi_K\rangle_t$ in terms of generalized coherent states $|u'(t);l',m'\rangle$ corresponding to some classical trajectory $u'(t)$. The trick of our procedure is that in $|\Psi_K\rangle_t$ we use the family of "out states" $|u'(t);l',m'\rangle$, whereas the initial states $|u(t);l,m\rangle$ belong to the family of "in states" of the generalized coherent states.

The transition matrix elements (46) are the probability amplitudes for transitions during which the classical trajectories change from $u(t)$ to $u'(t)$, and, at the same time, the generalized coherent wave packets change their shape, which is accounted for by the change in the quantum numbers l and m and can be recognized in (35). In the present paper, we shall not consider the change of the shape of the wave packets. Moreover, for clarity we shall study transitions between the ordinary coherent states $|u(t)\rangle = |u(t);l=0,m=0\rangle$ of (36). Accordingly, the transition matrix elements, which we shall analyze below, are the following special cases of (46):

$$T_{fi} = (i\hbar)^{-1}\int_{-\infty}^{+\infty} dt\ {}_t\langle p_z'|\,\langle u'(t)|\,V\,|u(t)\rangle\,|p_z\rangle_t \ . \tag{47}$$

Now we shall choose in (47) for V a screened Coulomb potential, which we will represent by the Fourier integral

$$V(r) = (A/r)\exp(-k_D r) = [4\pi A/(2\pi)^3]\int d^2k_\perp dk_z \frac{\exp[i(\mathbf{k}_\perp\cdot\mathbf{x}_\perp + k_z z)]}{k_D^2 + k_\perp^2 + k_z^2} \ . \tag{48}$$

In (48) we have introduced $\mathbf{k}_\perp \equiv (k_x,k_y)$ and $\mathbf{x}_\perp \equiv (x,y)$ as two dimensional vectors in a plane perpendicular to the direction of the magnetic field $\mathbf{B}=(0,0,B)$. For clarity, we summarize from (39), (36), (34), (27), (27a), (25), (24), (24a), (23a), (18a), (18b), (17), (9b), (32), and (33) the meaning of those quantities which appear in the matrix elements (47):

$$\begin{aligned}
&|u(t)\rangle = D_a[ig\exp(-i\omega t)]D_b[-g\exp(-i\omega t)]\exp(-iH_B t/\hbar)\,|\alpha_1\rangle_a\,|\beta_1\rangle_b \ ,\\
&H_B = \tfrac{1}{2}\hbar\omega_c(a^\dagger a + b^\dagger b + 1) + \frac{i}{2}\hbar\omega_c(ab^\dagger - a^\dagger b) \ ,\\
&\langle z\,|p_z\rangle_t = (2\pi\hbar)^{-1/2}\exp\{-(i\hbar)^{-1}[zp_z - (p_z^2/2M)t]\} \ ,\\
&g = \lambda_0(\gamma/2)^{1/2},\ \ \lambda_0 = (eF/Mc\omega)(\omega/\Delta\omega)\lambda\!\!\!^{-},\ \ \gamma = M\omega_c/2\hbar,\ \ \Delta\omega = \omega - \omega_c,\ \ \lambda\!\!\!^{-} = c/\omega,\ \ \omega_c = eB/Mc \ ,\\
&\alpha_1 = \alpha + \beta,\ \ \beta_1 = i(\alpha - \beta),\ \ \alpha = i(v_0/\omega_c)(\gamma/2)^{1/2},\ \ \beta = u_0^*(\gamma/2)^{1/2} \ ,\\
&u(t) = x_c(t) - iy_c(t) = i\lambda_0\exp(-i\omega t) + i(v_0/\omega_c)\exp(-i\omega_c t) + u_0 \ .
\end{aligned} \tag{49}$$

After having carried out the k_z integration, we obtain from (47) and (48)

$$T_{fi} = (i\hbar)^{-1}[4\pi A/(2\pi)^3]\int_{-\infty}^{+\infty} dt\,\exp[(-i\hbar 2M)^{-1}(p_z'^2 - p_z^2)t]\int d^2k_\perp \frac{\langle u'(t)\,|\exp(i\mathbf{k}_\perp\cdot\hat{\mathbf{x}}_\perp)\,|u(t)\rangle}{k_D^2 + q^2 + k_\perp^2} \ , \tag{50}$$

where

$$q=\hbar^{-1}(p_z'-p_z)\ , \tag{50a}$$

corresponding to the longitudinal momentum transfer. Next we apply the displacement properties (18c) of D_a and D_b and take into account the relations (6a)–(6c). Then the matrix elements $\langle u'(t)\,|\exp(i\mathbf{k}_\perp\cdot\hat{\mathbf{x}}_\perp)|\,u(t)\rangle$ in (50) can be brought to the form

$$\langle u'(t)\,|\exp(i\mathbf{k}_\perp\cdot\hat{\mathbf{x}}_\perp)|\,u(t)\rangle={}_b\langle\beta_1'|\,{}_a\langle\alpha_1'|\exp(iH_Bt/\hbar)\exp\{i(2\gamma)^{-1/2}[k_x(a+a^\dagger)+k_y(b+b^\dagger)]\}$$
$$\times\exp(-iH_Bt/\hbar)\,|\alpha_1\rangle_a\,|\beta_1\rangle_b\exp[ik_\perp\lambda_0\sin(\omega t-\chi)]\ , \tag{51}$$

where we have introduced the polar angle of the vector $\mathbf{k}_\perp$ through the definition

$$\mathbf{k}_\perp=(k_x,k_y)=k_\perp(\cos\chi,\sin\chi)\ . \tag{51a}$$

The last factor in (51) represents the generating function of the ordinary Bessel functions J_n and can be expanded into a Fourier series using the Jacobi-Anger formula:[10]

$$\exp[ik_\perp\lambda_0\sin(\omega t-\chi)]=\sum_n J_n(k_\perp\lambda_0)\exp[in(\omega t-\chi)]\ . \tag{52}$$

From (52), (51), and (50) we thus obtain

$$T_{fi}=(i\hbar)^{-1}[4\pi A/(2\pi)^3]\sum_n\int_{-\infty}^{+\infty}dt\,\exp\left[(-i\hbar)^{-1}\left[\frac{p_z'^2-p_z^2}{2M}+n\hbar\omega\right]t\right]$$
$$\times\int_0^\infty k_\perp dk_\perp\int_0^{2\pi}d\chi\frac{J_n(k_\perp\lambda_0)\exp(-in\chi)}{k_D^2+q^2+k_\perp^2}M_{\alpha_1'\beta_1';\alpha_1\beta_1}\ , \tag{53}$$

where we have introduced an abbreviation for the following matrix element:

$$M_{\alpha_1',\beta_1';\alpha_1,\beta_1}={}_b\langle\beta_1'|\,{}_a\langle\alpha_1'|\exp(iH_Bt/\hbar)$$
$$\times\exp\{i(2\gamma)^{-1/2}[k_x(a+a^\dagger)+k_y(b+b^\dagger)]\}\exp(-iH_Bt/\hbar)\,|\alpha_1\rangle_a\,|\beta_1\rangle_b\ . \tag{53a}$$

Using some of the formulas collected in (49), we obtain, after a lengthy but straightforward calculation, for the matrix element (53a)

$$M_{\alpha_1';\beta_1',\alpha_1,\beta_1}=\exp\{\tfrac{1}{2}k_\perp[(v_0'^*/\omega_c)e^{i(\omega_ct-\chi)}-(v_0/\omega_c)e^{-i(\omega_ct-\chi)}]\}$$
$$\times\exp\left[\frac{i}{2}k_\perp(u_0'e^{i\chi}+u_0^*e^{-i\chi})\right]\exp(-k_\perp^2/4\gamma)\,{}_a\langle\alpha_1'|\alpha_1\rangle_a\,{}_b\langle\beta_1'|\beta_1\rangle_b\ . \tag{53b}$$

In (53b) we have taken into account the relations between $(\alpha_1,\beta_1)(\alpha,\beta)$, and (u_0v_0) presented in (49). The last two factors on the right-hand side of (53b) represent the overlap of the initial and final coherent states, $|\,u(t)\rangle$ and $|\,u'(t)\rangle$, respectively. The square of the modulus of the product of these states has been presented in (37b).

In order to carry out in (53) the time integration and the integration over χ, we expand the first two factors on the right-hand side of (53b) into Laurent series by applying the generating formula of Bessel functions:[10]

$$\exp\{\tfrac{1}{2}k_\perp[(v_0'^*/\omega_c)e^{i(\omega_ct-\chi)}-(v_0/\omega_c)e^{-i(\omega_ct-\chi)}]\}=\sum_s J_s[k_\perp(v_0'^*v_0)^{1/2}/\omega_c](v_0'^*/v_0)^{s/2}\exp[is(\omega_ct-\chi)]\ , \tag{53c}$$

$$\exp\left[\frac{i}{2}k_\perp(u_0'e^{i\chi}+u_0^*e^{-i\chi})\right]=\sum_l J_l[k_\perp(u_0'u_0^*)^{1/2}](-u_0'/u_0^*)^{l/2}\exp(il\chi)\ . \tag{53d}$$

After we have inserted (53b)–(53d) into (53), we can easily perform the integrations over time and χ and we thus obtain for T_{fi} an infinite double sum over matrix elements for various incoherent nonlinear scattering processes,

$$T_{fi}=-2\pi i\sum_{n,s=-\infty}^{+\infty}\delta[(2M)^{-1}(p_z'^2-p_z^2)+n\hbar\omega+s\hbar\omega_c]M_{fi}^{(n,s)}\ , \tag{54}$$

where

$$M_{fi}^{(n,s)}\equiv(A/\pi\hbar)(v_0'^*/v_0)^{s/2}(-u_0'/u_0^*)^{(n+s)/2}\,{}_a\langle\alpha_1'|\alpha_1\rangle_a\,{}_b\langle\beta_1'|\beta_1\rangle_b$$
$$\times\int_0^\infty k_\perp dk_\perp\frac{\exp(-k_\perp^2/4\gamma)}{k_D^2+q^2+k_\perp^2}J_n(k_\perp\lambda_0)J_{n+s}[k_\perp(u_0'u_0^*)^{1/2}]J_s[k_\perp(v_0'^*v_0)^{1/2}/\omega_c]\ . \tag{54a}$$

For the evaluation of the scattering cross sections we need the squares of the moduli of the matrix elements (54a). These quantities can be written in the form

$$|M_{fi}^{(n,s)}|^2=(A/\pi\hbar)^2\,|u_0'/u_0|^{n+s}\,|v_0'/v_0|^5\exp[-\gamma\,|u_0'-u_0|^2-(\gamma/\omega_c^2)\,|v_0'-v_0|^2]\,|I_{ns}|^2\ , \tag{54b}$$

where

$$I_{ns}=\int_0^\infty k_\perp dk_\perp \frac{\exp(-k_\perp^2/4\gamma)}{k_D^2+q^2+k_\perp^2} J_n(k_\perp\lambda_0)J_{n+s}[k_\perp(u_0'u_0^*)^{1/2}]J_s[k_\perp(v_0'^*v_0)^{1/2}/\omega_c]\ . \tag{54c}$$

In the case of head-on collisions, for which $u_0=0$, $v_0=0$, we obtain for the transition matrix element a simpler formula

$$T_{fi}(\mathrm{HO})=-2\pi i\sum_{n,s=-\infty}^{+\infty}\delta[(2M)^{-1}(p_z'^2-p_z^2)+n\hbar\omega+s\hbar\omega_c]M_{fi}^{(n,s)}(\mathrm{HO})\ , \tag{55}$$

where

$$M_{fi}^{(n,s)}(\mathrm{HO})\equiv(A/\pi\hbar)\,{}_a\langle\alpha_1'|\alpha_1\rangle_a\,{}_b\langle\beta_1'|\beta_1\rangle_b[s!(n+s)!]^{-1}$$
$$\times(v_0'^*/2\omega_c)^s(iu_0'/2)^{n+s}\int_0^\infty k_\perp dk_\perp\frac{\exp(-k_\perp^2/4\gamma)}{k_D^2+q^2+k_\perp^2}J_n(k_\perp\lambda_0)k_\perp^{2s+n}\ . \tag{55a}$$

For our further discussion it is convenient to visualize the essential parameters of our scattering process. Since the electrons move freely in the z direction before and after the scattering by the potential V, this part of the problem causes little difficulties. We shall therefore consider, in particular, the transverse part of the quasiclassical electron motion in the (x,y) plane. Here we have chosen to describe the electrons before and after the scattering by coherent states $|u(t)\rangle$ and $|u'(t)\rangle$, respectively, given by (36). To these states correspond Gaussian wave packets, which according to (35) follow classical trajectories, namely, $|\langle \mathbf{x}_\perp|u(t)\rangle|^2=|\Phi_{00}(\mathbf{x}_\perp-\mathbf{x}_{c\perp}(t))|^2$. We have indicated this behavior in Fig. 2. In this figure the range of interaction of the scattering potential is determined according to (48) by the screening length k_D^{-1}. The distance of the center of gyration of the electron from the scattering center is given by the parameter $|u_0|$. With reference to (49), the initial value u_0 has moreover a particular phase in the (x,y) plane. Furthermore, the motion of the electron wave packet in the magnetic field alone is characterized by the cyclotron velocity $|v_0|$ and by the cyclotron radius of gyration $|v_0|/\omega_c$, where v_0 has also some initial phase. To this motion are superimposed the oscillations with amplitude λ_0 of the wave packet in the microwave field. As is apparent from (35), while an electron is moving in the two fields, its wave packet does not change its shape. The width of the electron wave packet is determined by the magnetic length $\gamma^{-1/2}$, which was introduced in (19c). During a collision process, an electron makes a transition from the initial coherent state $|u(t)\rangle$ to the final state $|u'(t)\rangle$. In this transition, the initial values of the classical trajectory (u_0,v_0) change in amplitude and phase to take on the final values (u_0',v_0').

For the evaluation of the scattering cross sections of the various field-induced nonlinear processes, we first introduce a finite normalization length L in the z direction. Then we can calculate from (54) the transition probabilities per unit time to be

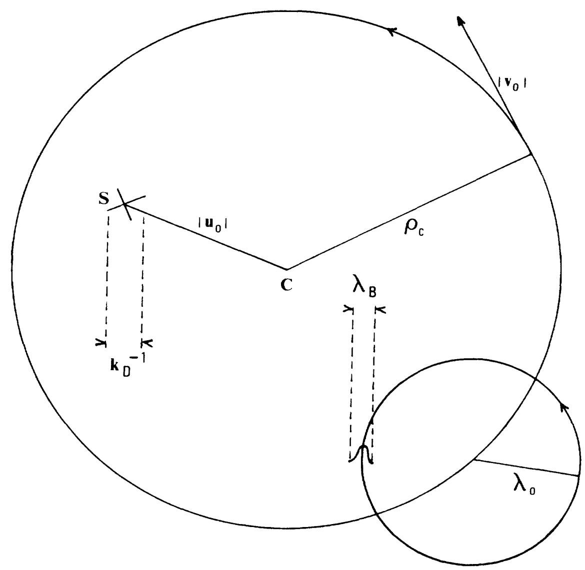

FIG. 2. Illustration of the essential parameters that characterize our quasiclassical scattering process in the transverse direction. The electron is described by a wave packet, the probability distribution of which has the width $\lambda_B=\gamma^{-1/2}$, which is our unit of magnetic length. This packet oscillates in the microwave field with the amplitude λ_0 and the frequency ω. At the same time, the packet gyrates in the magnetic field with the cyclotron velocity $|v_0|$ and the frequency ω_c on a circle of radius $\rho_c=|v_0|/\omega_c$, which has its center of gyration at C. The center of gyration is located at a distance $\rho_0=|u_0|$ from the scattering center at S, and the range of interaction of the scattering potential V is determined by the screening length k_D^{-1}. The phases of the initial values u_0 and v_0 are not indicated. In general, the complex initial values u_0 and v_0 change in amplitude and phase during the scattering process, thus changing in particular the position of the center of gyration C and the radius of gyration ρ_c.

$$T^{-1}W_{fi}^{(n,s)}=(2\pi/\hbar)(M/p_z')\,|M_{fi}^{(n,s)}|^2(2\pi\hbar/L)\ ,$$

from which we obtain the cross sections, if we divide by the current density of incoming particles, $j_{\rm inc}=(p_z/ML)(\gamma/\pi)$, where π/γ is the area of the transverse cross section of the Gaussian wave packets of the ingoing electrons. This finally yields

$$\sigma^{(n,s)}=(\pi/\gamma)[(2\pi)^2/v_z'v_z]\,|M_{fi}^{(n,s)}|^2\ , \tag{56}$$

where the matrix elements $M_{fi}^{(n,s)}$ are given by (54a). Moreover, we have introduced $v_z=p_z/M$ and $v_z'=p_z'/M$, which are the z components of the initial and final electron velocities.

According to (56), the cross sections $\sigma^{(n,s)}$ are proportional to $|M_{fi}^{(n,s)}|^2$ which are given by (54b) and (54c), and which contain the integrals I_{ns} as the only nontrivial factors. In the following we shall concentrate on the evaluation of the integrals I_{ns} for particular cases.

Since the scattering potential is spherically symmetric, it is intuitively clear that during the scattering process there will be only a comparatively small, if any, change in phase of the parameter u_0 describing the position of the center of gyration. The scatterer essentially attracts or repulses the particle in the radial direction, so it very likely only modifies the distance $|u_0|$ between the center of gyration and the origin. Therefore, we may use in (54c) the approximation $u_0'u_0^*=|u_0'u_0|$. This relation is certainly valid, if the electron does not get too close to the scattering center. Moreover, the change of the phase of the velocity v_0 can also be neglected. In order to see this, we shall define the phase difference ϵ by the relation $v_0'^*v_0=|v_0'v_0|\exp(i\epsilon)$. By taking into account (32), we can estimate ϵ to be roughly equal to $\omega_c\tau$, where τ is the collision time. It is natural to approximate τ by the ratio between the screening length k_D^{-1} and the average velocity $\bar{v}_z=(v_z'+v_z)/2$, hence $\tau=(\bar{v}_zk_D)^{-1}$. In order to estimate τ we take as an example $\bar{v}_z\simeq10^9$ cm/sec and $k_D^{-1}\simeq100$ Å. In this case we get $\tau\simeq10^{-15}$ sec, which is of the order of magnitude of ω^{-1} of the optical frequencies. For magnetic fields available nowadays in laboratories, ω_c is usually much smaller than the optical frequencies, so that $\epsilon\simeq\omega_c\tau\ll1$. Therefore, in the case under discussion, $v_0'^*v_0\simeq|v_0'v_0|$ seems to be a very good approximation. This conclusion is supported by the following consideration. In (54b) appears the factor $\exp[-\gamma|v_0'-v_0|^2\omega_c^{-2}]$, the argument of which may be rewritten in the form $\gamma|v_0'-v_0|^2\omega_c^{-2}=(B_c/B)|v_0'-v_0|^2c^{-2}$, where $B_c\equiv M^2c^3/e\hbar$ ($\simeq10^{13}$ G) is the critical field strength. Even though $|v_0'-v_0|^2c^{-2}$ may be very small, this cannot compensate the huge factor B_c/B and therefore we obtain

$$\exp(-\gamma|v_0'-v_0|^2\omega_c^{-2})\cong\begin{cases}0 & \text{for } v_0'\neq v_0\\ 1 & \text{for } v_0'=v\ .\end{cases}$$

With these simplifications in (54c), the integrals I_{ns} take a relatively simpler form:

$$I_{ns}=\int_0^\infty x\,dx\frac{\exp(-x^2)}{\alpha^2+x^2}J_n(ax)J_{n+s}(bx)J_s(cx)\ , \tag{57}$$

where

$$\alpha^2\equiv(4\gamma)^{-1}(k_D^2+q^2),\quad a\equiv2\lambda_0\gamma^{1/2}\ ,$$
$$b\equiv2\,|u_0'u_0|^{1/2}\gamma^{1/2},\quad c\equiv2\,|v_0'v_0|^{1/2}\gamma^{1/2}/\omega_c\ . \tag{57a}$$

To discuss the physical meaning of the parameters (57a), we remember from (19c) that $\gamma^{-1}=2\hbar/M\omega_c=\lambda_B^2$, where λ_B is the characteristic length of the probability distributions belonging to the coherent states. Therefore, if we denote the geometric mean value of the initial and final radial position of the center of gyration by $\rho_0^{\rm tr}\equiv|u_0'u_0|^{1/2}$, and, similarly, call the "transition cyclotron radius" $\rho_c^{\rm tr}\equiv|v_0'v_0|^{1/2}/\omega_c$, then a, b, and c in (57a) measure $2\lambda_0$, $\rho_0^{\rm tr}$, and $\rho_c^{\rm tr}$, respectively, in units of λ_B, where λ_0 is the amplitude of oscillations in the microwave. Finally, for $q=0$, α^{-1} is a measure of the Debye length k_D^{-1} of the potential V in terms of λ_B. On the basis of these parameters, we may now consider the following two limiting cases.

(i) If the magnetic field is strong and/or the screening of the potential is weak, then in (57a) we shall have $\alpha^2\ll1$, and, consequently, in (57) the Lorentzian profile $(\alpha^2+x^2)^{-1}$ will be much sharper than the Gaussian profile $\exp(-x^2)$. Therefore, in this case I_{ns} of (57) can be well approximated by

$$I_{ns}=\int_0^\infty\frac{x\,dx}{\alpha^2+x^2}J_n(ax)J_{n+s}(bx)J_s(cx),\quad\alpha^2\ll1\ . \tag{58}$$

By means of formula 6.5411 of Gradshteyn and Ryzhik,[10] the integral (58) can be evaluated to yield

$$I_{ns}=\begin{cases}K_n(\alpha a)I_{n+s}(\alpha b)I_s(\alpha c)(-1)^s, & a>|b+c|\\ I_n(\alpha a)I_{n+s}(\alpha b)K_s(\alpha c), & c>b,\quad a<|c-b|\\ I_n(\alpha a)K_{n+s}(\alpha b)I_s(\alpha c), & c<b,\quad a<|c-b|\ ,\end{cases} \tag{59}$$

where $I_n(z)$ and $K_n(z)$ are modified Bessel and Hankel functions, respectively. The cross sections obtained from (56), (54b), and (59) represent natural generalizations of the results of our earlier investigations, in which the classical limit of Compton scattering and electron scattering in external fields has been considered.[5] In that paper we used ordinary Landau states (19a) and (19b) for the states $|\Phi\rangle_t$ of (15), as a basis of perturbation theory in (46), instead of the generalized coherent states $|u(t);l,m\rangle$ introduced in the present investigation. Finally, in the aforementioned paper, the classical limit of highly excited Landau states with ($l\to\infty$, $m\to\infty$) was discussed. For a detailed analysis of this limit and its consequences for the reaction rates, the interested reader is referred to Sec. 4.2 of that investigation.[5]

If electron scattering takes place in the presence of a magnetic field alone, in which case in (15b) the field amplitude F and therefore λ_0 are zero, we obtain from (57a) and (59)

$$I_{ns}=\delta_{n,0}\begin{cases}I_s(\alpha b)K_s(\alpha c), & c>b\\ I_s(\alpha c)K_s(\alpha b), & b>c\ ,\end{cases} \tag{60}$$

which is the generalized form of a result derived by Ventura.[12] The difference between our result and that of Ventura comes from the fact that in our calculations we have taken into account the changes of the particle trajectories during the scattering process from $u(t)$ to $u'(t)$, as is also apparent from (54b). Hence, if we completely neglect the changes in the particle trajectories (corresponding to a neglect of transverse recoil effects), our results coincide with those of Ventura. In Ventura's paper highly excited Landau states have also been considered and therefore this investigation is close in spirit to our earlier paper.[5]

It is interesting to note that, for large arguments (αb) and (αc) in (59) or (60), we may derive from the asymptotic expressions for the functions I_n and K_n a factor of the form $\exp[-(k_D^2+q^2)^{1/2}|\rho_c-\rho_0|]$. The maximum of this factor at $\rho_c=\rho_0$ can be easily understood by inspecting Fig. 3. In this situation the particle orbit may hit the scattering center during the collision process.

(ii) Next, we consider the other extreme, in which the magnetic field is very weak and/or the screening is very strong. In (57a) we then shall have $\alpha^2 \gg 1$, and therefore in (57) the Gaussian profile $\exp(-x^2)$ will be much sharper than the Lorentzian $(\alpha^2+x^2)^{-1}$. Hence, in this case the integral in (57) can be approximated by

$$I_{ns}=\alpha^{-2}\int_0^\infty x\,dx\,\exp(-x^2)J_n(ax)J_{n+s}(bx)J_s(cx)\,, \quad \alpha^2\gg 1\,. \tag{61}$$

Now we use Graf's theorem for the generating function of the integrals I_{ns}, and employ the formula 6.633.2 of Gradshteyn and Ryzhik[10] to obtain from (61):

$$I_{ns}=\tfrac{1}{2}\alpha^{-2}\exp[-(a^2+b^2+c^2)/4] \times \sum_k I_{n+s+k}(\tfrac{1}{2}ab)I_{s+k}(\tfrac{1}{2}ac)I_k(\tfrac{1}{2}bc)(-1)^{s+k}\,, \tag{62}$$

and if there is no microwave present $(a=0)$, we get

$$I_{ns}=\tfrac{1}{2}\alpha^{-2}\exp[-(b^2+c^2)/4]I_s(\tfrac{1}{2}bc)\delta_{n,0}\,. \tag{62a}$$

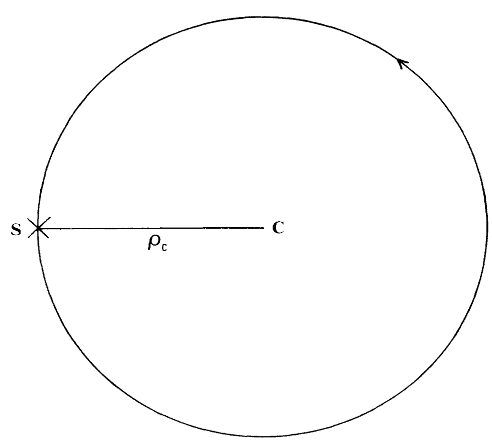

FIG. 3. During scattering in a magnetic field alone, the interaction with the scattering potential is particularly effective, if the orbit of gyration of the particle in the magnetic field hits the scattering center at S, in which case $|u_0|=\rho_c=|v_0|/\omega_c$.

The corresponding cross section formulas are again obtained from (56) and (54b), inserting for I_{ns}, (62) or (62a), respectively.

Finally, we investigate the case of head-on collisions. To this end we return to (55a). Taking into account the definitions (57a), we get

$$|M_{fi}^{(n,s)}(\mathrm{HO})|^2=(A/\pi\hbar)^2[(n+s)!s!]^{-2} \times(\gamma|u_0'|^2)^{n+s}(\gamma|v_0'|^2\omega_c^{-2})^s \times\exp[-\gamma(|u_0'|^2+|v_0'|^2\omega_c^{-2})] \times I_{ns}^2(\mathrm{HO})\,, \tag{63}$$

where

$$I_{ns}(\mathrm{HO})\equiv\int_0^\infty dx\frac{\exp(-x^2)}{\alpha^2+x^2}x^{n+2s+1}J_n(ax)\,, \tag{63a}$$

with $n,s\geq 0$.

For weak magnetic fields and/or strong screening $(\alpha^2\gg 1)$, (63a) can be brought to a closed analytic form. Using formula 6.631.10 of Gradshteyn and Ryzhik[10] we get from (63a)

$$I_{ns}(\mathrm{HO})\simeq\alpha^{-2}\int_0^\infty dx\,e^{-x^2}x^{n+2s+1}J_n(ax) =(s!/2\alpha^2)e^{-a^2/2}(a^2/2)^{n/2}L_s^n(a^2/2),\quad \alpha^2\gg 1\,, \tag{64a}$$

where L_s^n are associated Laguerre polynomials. Using in (62) the power-series expansions of the modified Bessel functions I_n, we can show that the general expression for $|M_{fi}^{(n,s)}|^2$ of (54b) and (62) exactly reduces to $|M_{fi}^{(n,s)}(\mathrm{HO})|^2$, given by (63) and (64a), if we take the limits $b,c\to 0$ (corresponding to $|u_0|,|v_0|\to 0$). Consequently, although (62) and (64a) apparently look very different, (62) in fact contains (64a) as a special case.

In the other extreme of strong magnetic fields and/or weak screening $(\alpha^2\ll 1)$, we obtain from (54b) and (59)

$$I_{ns}(\mathrm{HO})\simeq\alpha^{2(n+2s)}K_n(\alpha a),\quad \alpha^2\ll 1\,. \tag{64b}$$

If we consider the limit of small microwave intensities $(a\ll 1)$, then $J_n(ax)$ in (63a) can be approximated by $(ax)^n/n!$, and we can obtain $I_{ns}(\mathrm{HO})$ for any value of α, using formula 3.383.10 of Gradshteyn and Ryzhik:[10]

$$I_{ns}(\mathrm{HO})\simeq\frac{(n+s)!}{n!}a^n\alpha^{2(n+s)}e^{\alpha^2}\Gamma(-n-s,\alpha^2)\,, \quad a\ll 1\,, \tag{65}$$

where $\Gamma(-n,x)$ is the incomplete gamma function. In the absence of the microwave during scattering $(a=0)$ we get from (65):

$$I_{ns}(\mathrm{HO})=\delta_{n,0}s!\alpha^{2s}e^{\alpha^2}\Gamma(-s,\alpha^2),\quad a=0\,. \tag{65a}$$

If we assume that the transverse part of the electron trajectory does not change during the scattering, the matrix elements of the head-on collision become particularly simple. From (53) and (53a) we recognize that this corresponds to elastic transmission $(p_z'=p_z)$ and reflection $(p_z'=-p_z)$. From (63), (64a), and (64b) we obtain:

$$|M^{\rm el}(\mathrm{HO})|^2 = \left[\frac{A}{\pi\hbar}\right]^2 \begin{cases} (2\alpha^2)^{-2}e^{-\alpha^2}, & \alpha^2 \gg 1 \\ K_0^2(\alpha a), & \alpha^2 \ll 1 \; . \end{cases} \tag{66}$$

V. SUMMARY AND CONCLUDING REMARKS

In the foregoing sections we have introduced generalized coherent states to describe the motion of charged particles in the simultaneous presence of a constant homogeneous magnetic field and in a microwave field in the dipole approximation. These states form a complete and orthonormal set and they are thus convenient as a basis for perturbation calculations. The probability distributions of these states follow classical trajectories and, thus, allow the investigation of the quasiclassical features of a quantum-mechanical process. As an example, we have applied these states to the treatment of potential scattering of electrons, embedded in a homogeneous magnetic field and in a microwave.

Section II was devoted to an explicit solution of the Schrödinger equation for a charged particle in the two external fields. It was shown that this problem can be reduced to the solution of the Schrödinger equation in a magnetic field alone, yielding the well-known Landau states (19a)–(19d). The coupling of the particle to the microwave was then accomplished by an appropriate shift operation given by (15a) and (15b), by means of which we arrived at the desired solution (15), (15c). Next we constructed in Sec. III in a first step coherent and generalized coherent states for charged particles in a magnetic field alone. This was achieved by defining appropriate displacement operators (18a) which have to be applied to the Landau number states (19a)–(19d) to yield the generalized coherent states (26a) for particles in a magnetic field. The additional coupling of a particle to the microwave field was then accomplished by the displacement operator (15a) and (15b). This led to the generalized coherent states (27) and (27a), which describe the desired particle motions in both external fields, and which have the quasiclassical property (29). To relate the quantum-mechanical particle motion in a generalized coherent state, characterized by the quantum numbers l,m of a particular Landau state, and by the arbitrary complex parameters (α,β), to the classical particle motion, we introduced the classical trajectories (32) and we derived the one-to-one correspondence (33) between the initial values (v_0,u_0) of the classical motion and the complex parameters (α,β) of the generalized coherent states. This correspondence is summarized by the definition (34). The orthogonality and completeness of the generalized coherent states were shown to be given by (38a) and (38b).

In Sec. IV the complete quantum states (39) were applied to potential scattering of electrons by a screened Coulomb potential. The potential was treated in the first-order Born approximation, while the states (39) were used to define a set of "in" and "out" states for the perturbation treatment of the scattering problem. The corresponding transition matrix element (46) was considered, in particular, for the Landau ground state $l=m=0$, whereas changes of the classical particle orbit (32) were permitted to take place during the scattering event. The evaluation of the scattering matrix elements (54a)–(54c) and cross sections (56) was carried through in detail for limiting cases of the magnetic field strength and screening of the scattering potential. If the magnetic field is strong and/or the screening of the potential weak, the integral (54c) reduced to the closed analytic expressions (59), which took the simpler form (60), if no microwave was present. For weak magnetic fields and/or strong screening, on the other hand, (54c) reduced to (62), and, in the absence of a microwave, to (62a). Finally, head-on collisions were considered, which yield the formulas (63), (64a), and (64b).

In conclusion, we should again point out that the treatment of scattering problems of charged particles in the two external fields by means of generalized coherent states stresses the quasiclassical features of these processes and elucidates the close interrelation between the quantum-mechanical boundary-value problem and the classical initial-value problem. The corresponding results are of particular interest for applications in plasma physics. While the present problem of electron scattering by a screened ion potential in the simultaneous presence of a radiation field and a constant magnetic field is of interest for the investigation of the heating of a magnetized plasma by the absorption of radiation, the process of magneto-Raman scattering has applications in plasma diagnostics. In the latter problem, high-frequency radiation is scattered by free electrons, which are embedded in a microwave and in a magnetic field.[5,8] This problem can be treated with the same methods described here.

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[1]M. Gavrila, *Atomic Physics 9,* edited by N. Fortsan (Plenum, New York, 1985); G. Ferrante, *Physics of Ionized Gases,* edited by M. M. Popovic (Springer, Berlin, 1985); L. Rosenberg, Adv. At. Mol. Phys. **18**, 1 (1982).

[2]J. Bergou, F. Ehlotzky, and S. Varró, Phys. Rev. A **26**, 470 (1982); G. Ferrante, S. Nuzzo, and M. Zarcone, J. Phys. B **14**, 1565 (1981); J. F. Seely, Phys. Rev. A **10**, 1963 (1974).

[3]S. Bivona and G. Ferrante, J. Phys. B **17**, L411 (1984); M. Zarcone, M. R. C. McDowell, and F. H. M. Faisal, *ibid.* **16**, 4005 (1983); F. H. M. Faisal, *ibid.* **15**, L739 (1982).

[4]S. Varró, J. Phys. A **17**, 1631 (1984).

[5]S. Varró and F. Ehlotzky, J. Phys. B **18**, 3395 (1985).

[6]P. J. Redmond, J. Math. Phys. **6**, 1163 (1965).

[7]H. Rochlin and L. Davidovich, Phys. Rev. A **28**, 2329 (1983).

[8]S. Varró, F. Ehlotzky, and J. Bergou, J. Phys. B **17**, 483 (1984).
[9]L. D. Landau and L. M. Lifshitz, *Quantum Mechanics* (Pergamon, London, 1958).
[10]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980).
[11]I. A. Malkin and V. I. Man'ko, Zh. Eksp. Teor. Fiz. **55**, 1014 (1968) [Sov. Phys.—JETP **28**, 527 (1969)].
[12]J. Ventura, Phys. Rev. A **8**, 3021 (1973).