

Dynamics of spin systems with randomly asymmetric bonds: Langevin dynamics and a spherical model

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Neural networks contain, very often, asymmetric bonds. The interactions J_{ij} and J_{ji} between the i th and the j th neurons are not identical. In this paper we study the Langevin dynamics of fully connected spin systems whose interaction matrix contains a *random antisymmetric* part. The symmetric part consists of independent random bonds whose mean is either zero or ferromagnetic. We also consider a more general class of systems such as the asymmetric Hopfield model and other neural-network models. Within the framework of mean-field theory, the spin fluctuations are viewed as local, *thermally averaged, time-dependent* magnetic moments. These moments are induced by *excess* (i.e., nonthermal) internal noise which, in the presence of asymmetry, is time dependent and does not vanish even in the high-temperature phase. The mean-field equations are solved using a simplified, *spherical* model, in which the spins are linear variables except for a global constraint on the total level of their fluctuations. Random asymmetry of arbitrary strength destroys spin-glass freezing. Ferromagnetic phases, as well as “retrieval” states in neural networks, are affected only slightly by weak random asymmetry, in agreement with the conclusions of Hertz *et al.* The dynamical behavior of a system with weak asymmetry is studied in some detail. In the spin-glass case at low temperatures, when the strength of the asymmetry decreases, the internal *excess* noise does not vanish but slows down with a characteristic correlation time $\tau \propto k^{-6}$. The parameter k denotes the relative strength of the antisymmetric components of the bonds. The system behaves as a frozen symmetric spin glass on time scales $t \ll \tau$ and as a paramagnet on scales $t \gg \tau$. The *thermal* fluctuations decay with a characteristic time $\tau_T \propto k^{-4}$. The spherical model exhibits a completely frozen spin-glass state at zero temperature. As $T \rightarrow 0$, fluctuations exhibit a critical slowing down with time $\tau \propto T^{-1}$ for all values of $k > 0$. This $T=0$ spin-glass transition is probably an artifact of the spherical model and is not expected to exist in nonlinear systems. The relevance of the results to the performance of neural networks is discussed.

I. INTRODUCTION

The long-time behavior of some neural-network models have been studied by mapping them onto statistical-mechanical problems.¹⁻⁴ The mapping assumes that the synaptic connections J_{ij} between pairs of neurons i and j are symmetric, i.e., $J_{ji} = J_{ij}$. Under this assumption, the dynamics of the network can be described as a relaxation of a global energy function. The dynamic flows terminate at fixed points which are the local minima of the energy. However, the synaptic connections in biological nervous systems are usually not symmetric. Therefore it is of interest to understand the effect of that asymmetry on the long-time properties of the networks.

The asymmetry in neural networks may exhibit a well-defined structure. This is the case with layered systems⁵ where the asymmetry of the bonds determines the direction of information flow. Another example is network models of temporal pattern generation,⁶ where the asymmetry determines the direction of flows in configuration space. In this paper we focus on systems with *random* asymmetry, where it plays the role of a noise in an otherwise symmetric system.

The effect of random asymmetry has been studied in several recent works.⁷⁻¹¹ The first systematic study of

this problem is the work of Hertz *et al.*,⁷ in which the Langevin dynamics of a Hopfield model¹ with random asymmetry has been investigated. On the basis of the $n \rightarrow \infty$ limit of an n -component spin model they argue that even arbitrarily weak asymmetry destroys the “spurious” spin-glass (SG) states which exist² in the symmetric model at finite α . The parameter α denotes the ratio p/N where p and N are the number of memories and the number of neurons, respectively. On the other hand, the “retrieval” states, which are highly correlated with the memories, remain stable in the presence of weak asymmetry. The destruction of the SG states by random asymmetry has been also proposed by Parisi.¹⁰ Feigelman and Ioffe⁸ studied the Hopfield model with *strong* random asymmetry in the limit of $\alpha \rightarrow 0$. Their conclusion is that the retrieval states remain stable at sufficiently low temperature. A related work is that of Bausch *et al.*,⁹ who studied a random asymmetric model with n -component spins in the $n \rightarrow \infty$ limit. They find that thermal fluctuations destroy the SG freezing, but a fully frozen SG state exists in the absence of *thermal* noise, i.e., at $T=0$. The structure of the flows in configuration space in asymmetric Ising systems with deterministic dynamics is studied in Ref. 11.

Several suggestions have been made regarding the relevance of the random asymmetry to the *performance*

of associative-memory networks. Hertz *et al.*⁷ suggest that the absence of the spurious SG states improves the process of the retrieval of memories, i.e., the convergence to the retrieval states. Parisi¹⁰ proposed that random asymmetry is important for the learning process, in that it guarantees that only the retrieval states will be enhanced by the ‘‘Hebb’’ learning mechanism.

In this paper we study the effects of random asymmetry on the dynamics of networks. Our main aim in this work is to clarify three issues.

(1) *The limit of weak asymmetry:* What is the nature of the transition from a ‘‘symmetric’’ to a ‘‘nonsymmetric’’ behavior as the asymmetry is turned on?

(2) *The limit of zero temperature:* What is the role of thermal fluctuations in the asymmetric system? Does one expect in general an SG freezing at $T=0$?

(3) *General neural networks:* What is the source of the difference in the stability of the retrieval and SG phases in the presence of asymmetry? Can one generalize this difference to neural networks other than the Hopfield model?

A systematic analytic study of the dynamics of asymmetric networks is complicated by two factors: (1) The long-time limit has to be calculated via the full dynamic problem and cannot be evaluated by statistical-mechanical averages, and (2) Averaging over the quenched disorder of the bonds J_{ij} is difficult because in general the bonds may be correlated.

In order to circumvent these difficulties we study in detail a simplified dynamic model, a ‘‘spherical’’ asymmetric SG model. The dynamic equations are linear but a global constraint on the level of the total spin fluctuations is added. The linear nature of this model is similar to the large- n limit of Refs. 7 and 9, but it uses single-component spin variables, which is more natural in the context of neural networks. We first consider the case of Sherrington-Kirkpatrick (SK)-like¹² random asymmetric bonds with a positive mean. This enables us to study the effect of asymmetry on both spin glass and ferromagnetic (FM) ordering. The starting point is nonlinear Langevin equations for a system of ‘‘soft’’ spins with random asymmetric bonds, similar to those of Ref. 7. Using the dynamic mean-field theory of Ref. 13, these equations can be reduced to a set of local self-consistent equations. These mean-field equations are presented in the following section, Sec. II. In this section, we also introduce the notion of excess dynamic noise generated by asymmetry. The simplified, asymmetric SG spherical model is studied in Sec. III, with the main focus on the limit of weak asymmetry and the limit of zero temperature. In Sec. IV we study the spherical model with net ferromagnetic interactions. Section V extends the results regarding the SG phase to a large class of networks, including the Hopfield model and its variations. In Sec. VI we discuss the extent to which the results of the spherical model are expected to hold in nonlinear systems, and the relevance of the results to the *performance* of neural networks for associative memory.

In a second paper¹⁴ on the subject, an analytical and numerical study of the Glauber dynamics of an asymmetric Ising SG will be presented.

II. THE GENERAL FORMALISM

A. The dynamic model

We study an asymmetric spin-glass model in the mean-field limit. The model consists of N fully connected spins interacting via random asymmetric interactions. Denoting pairs of spins by (i, j) the interaction matrix J_{ij} is of the form

$$J_{ij} = J_{ij}^s + kJ_{ij}^{as}, \quad k \geq 0, \quad (2.1)$$

where J^s and J^{as} are symmetric and antisymmetric matrices, respectively,

$$J_{ji}^s = J_{ij}^s, \quad J_{ji}^{as} = -J_{ij}^{as}. \quad (2.2)$$

Each of the off-diagonal elements of J^s and J^{as} is a random Gaussian variable with zero mean and the following variance:

$$[(J_{ij}^s)^2] = [(J_{ij}^{as})^2] = \frac{J^2}{N} \frac{1}{1+k^2}. \quad (2.3)$$

Square brackets denote the ‘‘quench’’ average with respect to the distribution of J_{ij} . The diagonal elements J_{ii}^s and J_{ii}^{as} are zero. The parameter k measures the degree of asymmetry in the interactions. Equation (2.3) implies

$$N(J_{ij}^2) = J^2, \quad (2.4a)$$

$$N(J_{ij}J_{ji}) = J^2 \frac{1-k^2}{1+k^2}. \quad (2.4b)$$

Thus $k=0$ reduces to the ordinary symmetric SG model,^{12,15} whereas $k=1$ corresponds to a fully asymmetric model in which the value of the bond in the direction $i \rightarrow j$ is uncorrelated with the value of the bond in the $j \rightarrow i$ direction.

The dynamics of the model is defined by a set of Langevin equations,

$$\Gamma_0^{-1} \frac{\partial}{\partial t} \sigma_i(t) = -r_0 \sigma_i(t) - \frac{\delta V(\sigma_i)}{\delta \sigma_i(t)} + \sum_j J_{ij} \sigma_j(t) + h_i(t) + \xi_i(t). \quad (2.5)$$

Each degree of freedom is represented by a ‘‘soft’’ spin σ_i which can vary continuously from $-\infty$ to $+\infty$. The local potential $V(\sigma_i) + r_0 \sigma_i^2/2$ is an even function of σ_i which controls the fluctuations in the amplitude of σ_i . For convenience we have exhibited explicitly the term linear in σ_i so that $V'(\sigma_i)$ contains only the nonlinear parts, i.e., $V''(\sigma_i=0)=0$.

The term h_i represents a local external field, whereas ξ_i is a white noise with width

$$\langle \xi_i(t) \xi_j(t') \rangle = \frac{2T}{\Gamma_0} \delta(t-t') \delta_{ij}. \quad (2.6)$$

The parameter Γ_0^{-1} sets the scale of the microscopic

processing time, whereas T measures the level of the stochastic noise in the system. By analogy with the symmetric case we call T the “temperature” of the system even though the system does not reach thermal equilibrium.

The dynamical quantities of interest are spin correlations and response functions, in particular the average autocorrelation $C(t)$ and the average *local* response function, or susceptibility, $G(t)$,

$$C(t) = [\langle \sigma_i(t+t')\sigma_i(t') \rangle], \quad (2.7)$$

$$G(t) = \frac{\delta[\langle \sigma_i(t+t') \rangle]}{\delta h_i(t')} \Big|_{h=0}, \quad t \geq 0. \quad (2.8)$$

Angular brackets denote “thermal” average, i.e., average with respect to ξ_i , whereas square brackets denote “quench” average, i.e., average with respect to the J_{ij} 's. In the absence of asymmetry (i.e., $k=0$), the dynamical equations (2.5), obey the Callen-Welton-Kubo fluctuation-dissipation theorem (FDT),¹³

$$C(\omega) = \frac{2T}{\omega} \text{Im}G(\omega). \quad (2.9)$$

Here $C(\omega)$ and $G(\omega)$ are the Fourier transforms of Eqs. (2.7) and (2.8),

$$C(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} C(t), \quad (2.10a)$$

$$G(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} G(t). \quad (2.10b)$$

The FDT guarantees that, in the long-time limit, averaging over the ξ 's is identical to a statistical-mechanical averaging over a Boltzmann distribution $P\{\sigma\} \propto \exp[-\beta E(\sigma)]$ with an energy E ,

$$E = -\frac{1}{2} \sum_{i,j} (J_{ij} - r_0 \delta_{ij}) \sigma_i \sigma_j + \sum_i V(\sigma_i) - \sum_i h_i \sigma_i \quad (2.11)$$

and a temperature $T = \beta^{-1}$, given through Eq. (2.6). However, when $k \neq 0$ the FDT does not hold and the equilibrium limit of Eq. (2.5) does not correspond, in general, to thermal equilibrium.

B. Self-consistent dynamic equations

The analysis of Eq. (2.5) simplifies considerably in the thermodynamic limit, $N \rightarrow \infty$. In this limit the dynamics of the system can be described by a self-consistent equation involving only a single spin. This is achieved by a straightforward extension of the dynamic mean-field theory (MFT) developed by Sompolinsky and Zippelius for symmetric spin glasses.¹³ The details are delegated to Appendix A. The resultant mean-field equation of motion is

$$\begin{aligned} \Gamma_0^{-1} \frac{\partial}{\partial t} \sigma_i(t) = & -r_0 \sigma_i(t) - \frac{\delta V(\sigma_i)}{\delta \sigma_i(t)} + h_i(t) + \varphi_i(t) \\ & + J^2 \frac{1-k^2}{1+k^2} \int_{-\infty}^t dt' G(t-t') \sigma_i(t'). \end{aligned} \quad (2.12)$$

The variable $\varphi_i(t)$ is a Gaussian variable with zero mean

and variance,

$$\langle \varphi_i(t) \varphi_i(t') \rangle = \frac{2T}{\Gamma_0} \delta(t-t') + J^2 C(t-t'). \quad (2.13)$$

Here $C(t)$ and $G(t)$ are the full average autocorrelation and local response functions determined self-consistently from Eq. (2.12). The terms involving $G(t)$ and $C(t)$ represent the effect of the interactions J_{ij} on the dynamics of a single spin.

It is useful to exhibit the Fourier transform of Eqs. (2.12) and (2.13) in the following way:

$$G_0^{-1}(\omega) \sigma_i(\omega) = \varphi_i(\omega) + h_i(\omega) + \int_{-\infty}^{+\infty} dt e^{i\omega t} \frac{\delta V(\sigma_i)}{\delta \sigma_i(t)}, \quad (2.14)$$

where $G_0(\omega)$ would be the average local response function in the *absence* of the nonlinearity (i.e., $V \equiv 0$),

$$G_0^{-1}(\omega) = -i\omega \Gamma_0^{-1} + r_0 - J^2 \frac{1-k^2}{1+k^2} G(\omega), \quad (2.15)$$

and

$$\langle \varphi_i(\omega) \varphi_i(\omega') \rangle = 2\pi \delta(\omega + \omega') \left[\frac{2T}{\Gamma_0} + J^2 C(\omega) \right]. \quad (2.16)$$

C. Excess dynamic noise in asymmetric SG models

Solving Eqs. (2.14)–(2.16) is a formidable task, in particular in the $k \neq 0$ case where we do not have the powerful tool of the FDT. Nevertheless, it is useful to recall the treatment of the $k=0$ case.¹³ We write

$$C(t) = \tilde{C}(t) + q, \quad (2.17)$$

where

$$q = \lim_{t \rightarrow \infty} [\langle \sigma_i(t) \sigma_i(0) \rangle] = \lim_{t \rightarrow \infty} C(t). \quad (2.18)$$

The Edwards-Anderson¹⁶ order parameter q is nonzero only if the system undergoes a phase transition into a frozen phase. We now write the noise $\varphi_i(t)$ as a sum of two Gaussian terms,

$$\varphi_i(t) = \eta_i(t) + z_i, \quad (2.19)$$

where $\eta_i(t)$ is the dynamic component,

$$\langle \eta_i(\omega) \eta_i(\omega') \rangle = 2\pi \delta(\omega + \omega') \left[\frac{2T}{\Gamma_0} + J^2 \tilde{C}(\omega) \right] \quad (2.20)$$

and z_i the static component,

$$\langle z_i(\omega) z_i(\omega') \rangle = 2\pi \delta(\omega + \omega') \delta(\omega) J^2 q. \quad (2.21)$$

Equation (2.17) has now the following meaning: $\tilde{C}(t)$ and q are the connected and the disconnected parts of $C(t)$. As usual, the disconnected correlation is given by

$$q = \langle \langle \sigma_i(t) \rangle_\eta^2 \rangle_z. \quad (2.22)$$

Here $\langle \dots \rangle_\eta$ and $\langle \dots \rangle_z$ denote averaging with respect to the Gaussian variables $\eta_i(t)$ and z_i , respectively.

Inspecting Eqs. (2.14) and (2.15), for general k , one observes that the role of the “microscopic relaxation time” of these equations is played by $\bar{\Gamma}_0^{-1}(\omega)$, where

$$\bar{\Gamma}_0^{-1}(\omega) = -\frac{1}{\omega} \text{Im}G_0^{-1}(\omega) = \Gamma_0^{-1} + J^2 \frac{1-k^2}{1+k^2} \frac{1}{\omega} \text{Im}G(\omega). \quad (2.23)$$

It is straightforward to see that, in the $k=0$ case, $\eta_i(t)$ obeys

$$\langle \eta_i(\omega)\eta_i(\omega') \rangle = \frac{2T}{\bar{\Gamma}_0(\omega)} 2\pi\delta(\omega+\omega'), \quad (2.24)$$

which is a straightforward generalization of Eq. (2.6). It guarantees the validity of the FDT between the local response and the *connected* correlation functions,

$$\tilde{C}(\omega) = \frac{2T}{\omega} \text{Im}G(\omega). \quad (2.25a)$$

In particular, the static susceptibility $\chi \equiv G(\omega=0)$ obeys

$$\chi = \frac{1}{T} \tilde{C}(t=0) = \frac{1}{T} [C(t=0) - q]. \quad (2.25b)$$

In the *equilibrium limit*, the nontrivial ω dependence of Eq. (2.24) is irrelevant and $\eta_i(t)$ can be considered as an *ordinary thermal noise*. Thus the static limit of Eq. (2.14) (in the $k=0$ case) is equivalent to a statistical-mechanical problem of a single spin which has the following mean-field Hamiltonian:¹³

$$E = \frac{1}{2} G_0^{-1}(\omega=0) \sigma_i^2 + V(\sigma_i) - z_i \sigma_i. \quad (2.26)$$

The quantity $G_0^{-1}(\omega=0)$ equals $r_0 - J^2 G(\omega=0)$ [see Eq. (2.15)] and z_i is the *excess static noise* which has a Gaussian distribution with a width of $J^2 q$ [see Eq. (2.21)]. Averaging with respect to the dynamical thermal noise $\eta_i(t)$ is equivalent to averaging with the Gibbs distribution $P(\sigma_i) \propto \exp[-\beta E(\sigma_i)]$, whereas averaging with respect to z_i is equivalent to a quench average over the Gaussian static field z_i . This approach leads to the so-called replica symmetric SG theory. Extending it by considering time persistent parts in the local response function $G(\omega)$ yields the full static SG mean-field theory.^{13,17}

Let us now turn to the $k \neq 0$ case. Equation (2.24) does not hold any more for the dynamic noise $\eta_i(t)$ of Eq. (2.20) because of the factor $(1-k^2)/(1+k^2)$ in Eq. (2.15). Obviously the amplitude of the dynamic noise $\eta_i(t)$ is bigger than the one which is necessary to maintain thermal equilibrium at the temperature T . To express explicitly this fact let us separate the noise $\varphi_i(t)$ into three Gaussian components,

$$\varphi_i(t) = \eta_i(t) + x_i(t) + z_i. \quad (2.27)$$

The first term is a *thermal* dynamic noise which obeys the relation (2.24) [with $\bar{\Gamma}_0(\omega)$ of Eq. (2.23)]. The terms z_i and $x_i(t)$ are the *static* and *dynamic* components of the *excess noise* $\varphi_i(t) - \eta_i(t)$. The variance of $\eta_i(t)$ is, by Eqs. (2.23) and (2.24),

$$\langle \eta_i(\omega)\eta_i(\omega') \rangle = 2\pi\delta(\omega+\omega') \left[\frac{2T}{\Gamma_0} + J^2 \frac{1-k^2}{1+k^2} \hat{C}(\omega) \right]. \quad (2.28)$$

It looks similar to Eq. (2.20), but the *thermal* correlation function $\hat{C}(t)$ has a meaning different from $\tilde{C}(t)$ of Eq. (2.17). It is defined as

$$\hat{C}(t) = C(t) - \hat{q}(t), \quad (2.29)$$

where

$$\hat{q}(t) = \langle \langle \sigma_i(t') \rangle_\eta \langle \sigma_i(t+t') \rangle_\eta \rangle_{x,z}. \quad (2.30)$$

The variable $\langle \sigma_i(t) \rangle_\eta$ is the thermally averaged local magnetization induced by the excess time-dependent random field $x_i(t) + z_i$.

The variance of z_i is given by Eq. (2.21) as before, where q now is defined by

$$2\pi\delta(\omega+\omega')\delta(\omega)q = \langle \langle \sigma_i(\omega) \rangle_{\eta,x} \langle \sigma_i(\omega') \rangle_{\eta,x} \rangle_z. \quad (2.31)$$

The variance of the excess *dynamic* noise is, by Eqs. (2.16), (2.21), and (2.28),

$$\begin{aligned} \langle x_i(\omega)x_i(\omega') \rangle &= 2\pi\delta(\omega+\omega')X(\omega) \\ &= 2\pi\delta(\omega+\omega') \left[\frac{2k^2}{1+k^2} J^2 \hat{C}(\omega) \right. \\ &\quad \left. + J^2 [\hat{q}(\omega) - q\delta(\omega)] \right], \end{aligned} \quad (2.32)$$

where $\hat{C}(\omega)$ and $\hat{q}(\omega)$ are the Fourier transforms of Eqs. (2.29) and (2.30). Note that, with $\hat{C}(\omega)$ given by Eqs. (2.29) and (2.30), we have the FDT

$$\hat{C}(\omega) = \frac{2T}{\omega} \text{Im}G(\omega). \quad (2.33)$$

This justifies our identification of $\eta_i(t)$ as the dynamic thermal noise, $\hat{C}(t)$ as the thermal correlation function, and $x_i(t)$ and $\hat{q}(t)$ as the dynamic excess noise and spin correlations.¹⁸ Equation (2.34) implies for the *static susceptibility* χ a relation analogous to Eq. (2.25b),

$$\chi = \frac{1}{T} \hat{C}(t=0) = \frac{1}{T} (C - \hat{q}), \quad (2.34)$$

where

$$C \equiv C(t=0) = \langle \sigma_i^2(t') \rangle_{\eta,x,z}, \quad (2.35)$$

$$\hat{q} \equiv \hat{q}(t=0) = \langle \langle \sigma_i(t') \rangle_\eta^2 \rangle_{x,z}. \quad (2.36)$$

It is important to realize that the disconnected correlation \hat{q} is a dynamic quantity. It is nonzero even in the absence of a true static freezing (i.e., $q=0$). Therefore, unlike the $k=0$ case, $\chi < C/T$, even at high temperature. Note that since the excess noise $x_i(\omega)$ acts in general on the same time scale as $\eta_i(\omega)$, one cannot neglect its frequency dependence, even in calculating the long-time limit. Thus, one has to solve a full dynamic problem in order to derive the correct statics. Nevertheless,

the separation of the time-dependent noise into two parts $\eta_i(t)$ and $x_i(t)$ is useful, particularly for the understanding of the small- k limit. In the following sections we study a spherical model, which can be solved exactly for all k and T .

III. THE ASYMMETRIC SG SPHERICAL MODEL

A. The model

We define a spherical model of an asymmetric SG by the following Langevin equations:

$$\Gamma_0^{-1} \frac{\partial}{\partial t} \sigma_i(t) = -r \sigma_i(t) + \sum_j J_{ij} \sigma_j(t) + h_i(t) + \xi_i(t), \quad (3.1)$$

where J_{ij} is defined by Eqs. (2.1)–(2.4) and ξ_i obeys Eq. (2.6). The source h_i is an external field, as before. Equation (3.1) is just a linear version of the model (2.5). However, unlike r_0 of Eq. (2.5), the parameter r is *not* a free parameter but is determined by the global condition¹⁹

$$\frac{1}{N} \sum_{i=1}^N \sigma_i^2(t) = 1. \quad (3.2)$$

In the thermodynamic limit this condition can be written as

$$\langle \sigma_i^2 \rangle = C = 1, \quad (3.3)$$

where, as before, $C \equiv C(t=0)$.

In the absence of explicit nonlinear terms, the mean-field equations (2.14) read simply

$$\sigma_i(\omega) = G(\omega) [\varphi_i(\omega) + h_i(\omega)], \quad (3.4)$$

where $G(\omega)$ equals $G_0(\omega)$ of Eq. (2.15) and $\varphi_i(\omega)$ is the self-consistent noise, given by Eq. (2.16). Solving Eq. (2.15) with $G(\omega) \equiv G_0(\omega)$ yields

$$G^{-1}(\omega) = \frac{1}{2} \{ r - i\omega \Gamma_0^{-1} + [(r - i\omega \Gamma_0^{-1})^2 - 4(1 - k^2)/(1 + k^2)]^{1/2} \}. \quad (3.5)$$

Here and in the following we substitute for simplicity $J = 1$. One also finds the useful relations

$$\text{Re}G(\omega) = \frac{r}{|G(\omega)|^{-2} + (1 - k^2)/(1 + k^2)}, \quad (3.6)$$

$$\text{Im}G(\omega) = \frac{\omega \Gamma_0^{-1}}{|G(\omega)|^{-2} - (1 - k^2)/(1 + k^2)}. \quad (3.7)$$

The parameter r is determined via the constraint on C , Eq. (3.3). Using Eqs. (3.4) and (2.16) one obtains the following equation:

$$(|G(\omega)|^{-2} - 1)C(\omega) = 2T\Gamma_0^{-1}, \quad (3.8)$$

where

$$\langle \sigma_i(\omega) \sigma_i(\omega') \rangle = 2\pi \delta(\omega + \omega') C(\omega). \quad (3.9)$$

Substituting

$$C(\omega) = \tilde{C}(\omega) + 2\pi q \delta(\omega) \quad (3.10)$$

yields

$$\tilde{C}(\omega) = \frac{2T\Gamma_0^{-1}}{|G(\omega)|^{-2} - 1} \quad (3.11)$$

for the finite- ω correlations, and

$$q = \chi^2 q \quad (3.12)$$

for the static part. Recall that $\chi = G(\omega=0)$. Defining T_g as the first temperature for which q is nonzero, Eq. (3.12) implies that

$$\chi = 1, \quad T \leq T_g. \quad (3.13)$$

This equation determines the value of r below T_g [see Eq. (3.6)]

$$r = 1 + (1 - k^2)/(1 + k^2), \quad T \leq T_g. \quad (3.14)$$

The value of q is determined by the constraint (3.3), i.e.,

$$q = 1 - \tilde{C}(t=0). \quad (3.15)$$

Before discussing the asymmetric case let us recall briefly the behavior in the $k=0$ case. The FDT [Eq. (2.25b)] implies that

$$\chi = \frac{1}{T}, \quad T > T_g, \quad (3.16)$$

yielding an SG transition at $T_g = 1$ [see Eq. (3.13)]. Below T_g , $r = 2$ and $q = 1 - T$. As for the dynamic properties, both $\tilde{C}(\omega)$ and $G(\omega)$ have a low-frequency singularity at $T \leq T_g$. In fact, Eq. (3.5) (with $k=0$ and $r=2$) yields

$$\frac{\text{Im}G(\omega)}{\omega} \sim \tilde{C}(\omega) \sim \omega^{-1/2}, \quad \omega \rightarrow 0, \quad T \leq T_g. \quad (3.17)$$

B. Absence of an SG phase at finite T

In the $k \neq 0$ case, Eq. (3.13) cannot be satisfied at finite T . The reason is that $G(\omega)$ has a low-frequency singularity only when $\chi^2 = (1 - k^2)/(1 + k^2)$ [see Eq. (3.7)], whereas $C(\omega)$ has a singularity when $\chi = 1$ [see Eq. (3.8)]. Thus, if $\chi = 1$, $|G(\omega)|^2$ is still analytic at $\omega = 0$ and is of the form $|G(\omega)|^2 \sim 1 - O(\omega^2)$. This, in turn, would mean, as noted in Refs. 7 and 9, that when $\chi = 1$

$$\tilde{C}(\omega) \sim \omega^{-2}, \quad \omega \rightarrow 0, \quad (3.18)$$

violating the constraint that

$$\tilde{C}(t=0) = \int \frac{d\omega}{2\pi} \tilde{C}(\omega) \quad (3.19)$$

be finite. Note that in the $k=0$ case the singularity of $\tilde{C}(\omega)$ is much weaker and is integrable, see Eq. (3.17). Consequently, we conclude that, in the asymmetric spherical model

$$q = 0, \quad T > 0, \quad (3.20)$$

and both $C(\omega)$ and $G(\omega)$ are analytic at $\omega=0$ at all finite T .

C. A zero-temperature SG phase

Let us now consider the zero-temperature case. Substituting $T=0$ in Eqs. (3.8) and (3.11) yields

$$\tilde{C}(\omega)=0, \quad T=0, \quad (3.21)$$

whereas the constraint (3.15) implies that

$$q=1, \quad \chi=1, \quad T=0, \quad (3.22)$$

for all k . Thus at $T=0$ the system is completely frozen in an SG state despite the asymmetry. To understand the nature of the transition from a paramagnetic state ($q=0$) at $T>0$ to an ordered phase ($q=1$) at $T=0$, we study in more detail the limit of $T \rightarrow 0$. Expanding Eq. (3.5) in powers of ω and T , with the constraint $C=1$, we obtain

$$\text{Re}G(\omega) \cong 1 - \frac{\Gamma^{-2}}{2k^2} \omega^2 - \frac{2k^6}{1+k^2} T^2 + \dots, \quad (3.23)$$

$$\text{Im}G(\omega) \cong \Gamma^{-1} \omega + \dots, \quad (3.24)$$

$$r \cong \frac{2}{1+k^2} + \frac{4k^8}{(1+k^2)^4} T^2 + \dots, \quad (3.25)$$

where

$$\Gamma^{-1} = \frac{1+k^2}{2k^2} \Gamma_0^{-1}. \quad (3.26)$$

Substituting these results in Eq. (3.8) yields a correlation function with Lorentzian shape,

$$C(\omega) \cong \frac{2\tau}{1+(\omega\tau)^2}, \quad T, \omega \rightarrow 0, \quad (3.27)$$

where

$$\tau = \left[\frac{1+k^2}{2k^3} \right]^2 \frac{\Gamma_0^{-1}}{T}. \quad (3.28)$$

The zero- T SG phase is recovered by taking the limit $T \rightarrow 0$ keeping $\omega \neq 0$, which results in

$$\lim_{\tau \rightarrow \infty} \left[\frac{2\tau}{1+(\omega\tau)^2} \right] = 2\pi\delta(\omega), \quad (3.29)$$

i.e., $q=1$. Note that the limits of $T \rightarrow 0$ and $\omega \rightarrow 0$ are noncommuting. The result (3.29) represents the behavior in the regime $\omega/T \gg 1$, whereas in the opposite limit $\omega/T \ll 1$, the system behaves paramagnetically with $C(\omega) \sim 2\tau \sim T^{-1}$. This is a typical case of a critical slowing down with $\tau \sim 1/T$. In fact, the discontinuity of the order parameter (in our case q), together with a divergence of the correlation time (and length), is characteristic of ordinary zero-temperature critical points. Note that the response function $G(\omega)$ remains analytic at $\omega=0$ even at $T=0$.

We conclude this paragraph by considering the particularly simple case of a fully asymmetric system. Substituting $k=1$ in Eq. (3.5) one observes that in this case the form of the response function is unaffected by the in-

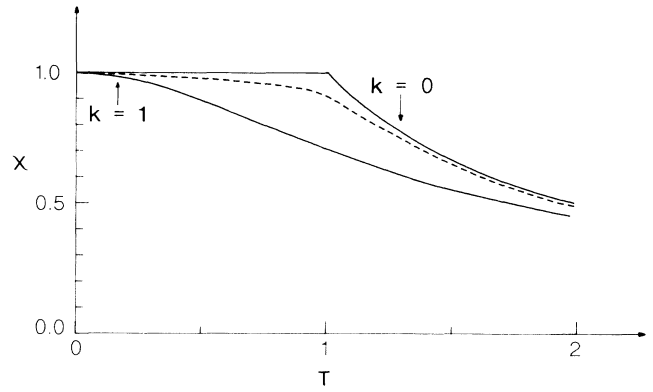


FIG. 1. Static susceptibility χ of the SG spherical model vs T for different values of k . The dashed line shows the qualitative shape of χ for a fixed small value of k .

teraction J_{ij} , i.e.,

$$G^{-1}(\omega) = r - i\omega\Gamma_0^{-1}, \quad k=1. \quad (3.30)$$

The correlation function $C(\omega)$ [see Eq. (3.8)] has a Lorentzian shape for all T and ω ,

$$C(\omega) = \frac{2T}{(r^2-1)^{1/2}} \frac{\tau}{1+(\omega\tau)^2}, \quad k=1 \quad (3.31)$$

$$\tau = \frac{\Gamma_0^{-1}}{(r^2-1)^{1/2}}, \quad k=1. \quad (3.32)$$

The constraint $C=1$ yields

$$\chi^{-1} = r = (1+T^2)^{1/2}, \quad k=1. \quad (3.33)$$

The low- T limit of these equations is, of course, consistent with Eqs. (3.23)–(3.26). The result for $\chi(T)$ is plotted in Fig. 1, together with the value of $\chi(T)$ at $k=0$. It is seen that the sharp cusp of the symmetric $\chi(T)$ at T_g is replaced by a gradual saturation of $\chi(T)$ as $T \rightarrow 0$. We also present, in Fig. 1, the qualitative shape of $\chi(T)$ for a fixed small value of k . This case will be discussed in some detail in the following paragraph.

D. The limit of weak asymmetry

The conclusion that the system remains ergodic at all $T>0$, even for arbitrarily weak asymmetry, raises the following question: Is there a mechanism for a smooth recovery of the SG phase (below $T=1$) in the $k \rightarrow 0$ limit, despite the fact that $q=0$ for all $k \neq 0$? To answer this question one needs to understand better the effect of the asymmetry on the SG phase. To achieve this we use the concept of *excess dynamic noise* introduced in Sec. II, and study its behavior in the limit of $k \rightarrow 0$ (and $T>0$).

Since $q=0$, the static part of the noise, $z_i(\omega)$, Eq. (2.21), vanishes. The variance of the excess dynamic noise is now given by [see Eq. (2.33)]

$$X(\omega) = \frac{2k^2}{1+k^2} \hat{C}(\omega) + \hat{q}(\omega) \quad (3.34)$$

(where as before we have put $J = 1$).

In the limit $k \rightarrow 0$, the first contribution to $X(\omega)$ disappears. Therefore, one would expect naively that in this limit $X(\omega)$ will vanish, which will self-consistently make $\hat{q}(\omega)$ go to zero. This is indeed the case for $T > 1$. In this case,

$$X(\omega) \rightarrow 0, \quad \hat{q}(\omega) \rightarrow 0 \quad \text{as } k \rightarrow 0, \quad T > 1 \quad (3.35)$$

and

$$\chi = \frac{1}{T} \hat{C}(t=0) \rightarrow \frac{1}{T} \quad \text{as } k \rightarrow 0, \quad T > 1. \quad (3.36)$$

However, since χ necessarily remains smaller than 1 at all T , $\hat{q}(\omega)$ cannot be too small when $T < 1$. Thus, below $T = 1$ both $X(\omega)$ and $\hat{q}(\omega)$ stay finite even as $k \rightarrow 0$. This almost "spontaneous" appearance of $\hat{q}(\omega)$ occurs at a frequency range which becomes more and more concentrated around $\omega = 0$ as k decreases to zero, so that

$$\hat{q}(\omega) \rightarrow \hat{q} \delta(\omega) \quad \text{as } k \rightarrow 0, \quad T < 1. \quad (3.37)$$

In fact, writing

$$X(\omega) \rightarrow \hat{q}(\omega), \quad k \rightarrow 0, \quad (3.38)$$

it is straightforward to see that the equation for \hat{q} be-

$$\hat{C}(\omega) = \frac{2T\Gamma_0^{-1}}{|G(\omega)|^2 - (1-k^2)/(1+k^2)} = \frac{2T}{\omega} \text{Im}G(\omega), \quad (3.41)$$

$$\hat{q}(\omega) = \frac{2k^2}{1+k^2} \frac{2T\Gamma_0^{-1}}{(|G(\omega)|^{-2} - 1)[|G(\omega)|^{-2} - (1-k^2)/(1+k^2)]}, \quad (3.42)$$

where $G(\omega)$ is given by Eq. (3.5).

For $T > 1$, $\hat{q}(\omega)$ vanishes as $k \rightarrow 0$, while the denominators of Eqs. (3.41) and (3.42) do not develop singularity since the $k = 0$ value of χ is less than unity. Thus one recovers in the $k \rightarrow 0$ limit the symmetric paramagnetic phase with $\chi = 1/T$. On the other hand, for $T < 1$, care must be taken in handling the singularity of the denominators at $\omega = 0$ as $k \rightarrow 0$.

Constructing a self-consistent solution of Eqs. (3.5), (3.41), and (3.42) in the $k \rightarrow 0$ limit and $T < 1$ we find two relevant scales of characteristic frequencies. For $G(\omega)$ and $\hat{C}(\omega)$, the characteristic frequency scale is

$$\omega_T \sim k^4 \quad \text{as } k \rightarrow 0, \quad T < 1. \quad (3.43)$$

For $\omega \gg \omega_T$ the system behaves as a symmetric system, i.e.,

$$1 - G(\omega) \sim \omega^{1/2}, \quad \hat{C}(\omega) \sim \omega^{-1/2}, \quad \omega \gg k^4. \quad (3.44)$$

On the other hand, when $\omega \ll \omega_T$ the effect of asymmetry is strong. In this case we find

$$G^{-1}(\omega) \cong 1 + (\lambda - 1)k^6 + \frac{(\omega\Gamma_0^{-1})^2}{8k^6} - \frac{i\omega\Gamma_0^{-1}}{2k^2}, \quad \omega \ll k^4. \quad (3.45)$$

comes, in the $k \rightarrow 0$ limit, identical to the equation for q in the $k = 0$ case, with $x_i(\omega)$ playing the role of the excess static field z_i , see Eq. (2.21). From this we conclude that, below $T = 1$, the excess dynamic noise does not vanish as k decreases, but slows down giving rise to a gradual freezing of the system. In the limit of $k = 0$ the excess noise becomes time persistent and the SG phase of the symmetric case is recovered.

The above qualitative analysis is quite general and holds also in the nonlinear case, Eq. (2.5). We will use the spherical model to explicitly calculate the limit $k \rightarrow 0$. In this linear model the local (unaveraged) magnetization is $\sigma_i(\omega) = G(\omega)[\eta_i(\omega) + x_i(\omega)]$. Therefore, the thermal and excess correlations, Eqs. (2.29) and (2.30), are, respectively,

$$\begin{aligned} 2\pi\hat{C}(\omega) &= |G(\omega)|^2 \langle \eta_i(\omega)\eta_i(-\omega) \rangle \\ &= 2\pi |G(\omega)|^2 \left[\frac{2T}{\Gamma_0} + \frac{1-k^2}{1+k^2} \hat{C}(\omega) \right], \end{aligned} \quad (3.39)$$

and

$$\hat{q}(\omega) = |G(\omega)|^2 X(\omega), \quad (3.40)$$

where $X(\omega)$ is given by Eq. (2.33) and use has been made of Eq. (2.28). Solving Eqs. (3.39) and (3.40) one obtains

This corresponds to a value of r which is

$$\frac{1}{2}r \cong 1 - k^2 + k^4 - k^6 + \lambda k^8, \quad \omega \ll k^4, \quad (3.46)$$

and the value of $\lambda, \lambda > 1$, will be determined self-consistently below.

The result (3.45) implies that the thermal correlation function behaves in this regime as a Lorentzian,

$$\hat{C}(\omega) \cong \frac{T\Gamma_0^{-1}k^{-2}}{1 + \left[\frac{\omega\Gamma_0^{-1}}{\sqrt{8}k^4} \right]^2}, \quad \omega \ll k^4. \quad (3.47)$$

Note, however, that the integral of Eq. (3.47) up to $\omega \sim k^4$ has a negligible weight $\sim k^2$ which means that as far as the constraint (3.3) is concerned the dominant contribution of $\hat{C}(\omega)$ comes from the high-frequency regime, Eq. (3.44). This contribution is the same as in the symmetric case, i.e.,

$$\hat{C}(t=0) \cong T + O(k^2), \quad k \rightarrow 0. \quad (3.48)$$

A second, smaller, characteristic frequency scale, ω_0 , appears in the dynamic behavior of the excess spin correlations $\hat{q}(\omega)$. Substituting Eq. (3.45) in Eq. (3.42), one obtains

$$\hat{q}(\omega) \cong \frac{2T\Gamma_0^{-1}k^{-6}}{\left[1 + \left[\frac{\omega\Gamma_0^{-1}}{\sqrt{8}k^4}\right]^2\right] \left[(\lambda-1) + \left[\frac{\omega\Gamma_0^{-1}}{2k^6}\right]^2\right]}, \quad \omega \ll k^4 \quad (3.49)$$

from which we conclude that the smallest relaxation frequency of $\hat{q}(\omega)$ is

$$\omega_0 \sim k^6 \text{ as } k \rightarrow 0, \quad T < 1. \quad (3.50)$$

Equation (3.49) implies that $\hat{q}(\omega)$ gives a *finite* contribution to $\hat{q} = \hat{q}(t=0)$ from the regime $\omega \sim k^6$. In fact, $\hat{q}(\omega)$ can be written *in this regime* as

$$\hat{q}(\omega) \cong \frac{T}{\sqrt{\lambda-1}} \frac{2\tau}{1+(\omega\tau)^2}, \quad (3.51)$$

$$\tau = \frac{\Gamma_0^{-1}}{2\sqrt{\lambda-1}} \frac{1}{k^6}, \quad (3.52)$$

which, upon integration, yields

$$\hat{q} = 2T(\lambda-1)^{-1/2}. \quad (3.53)$$

Using the constraint $\hat{q} = 1 - T$ [see Eq. (3.48)] fixes λ ,

$$\lambda = 1 + \left[\frac{2T}{1-T}\right]^2, \quad (3.54)$$

from which one finds for τ of Eq. (3.52) the result

$$\tau = \frac{(1-T)\Gamma_0^{-1}}{4T} \frac{1}{k^6} \text{ as } k \rightarrow 0, \quad T < 1. \quad (3.55)$$

Note that the limit $T \rightarrow 0$ of Eq. (3.55) agrees with the $k \rightarrow 0$ limit of the result (3.28) for the relaxation time of $C(\omega)$ in the zero-temperature limit. Thus, the limits $k \rightarrow 0$ and $T \rightarrow 0$ commute. As for the static susceptibility, Eqs. (3.45) and (3.54) yield

$$\chi \cong 1 - \left[\frac{2T}{1-T}\right]^2 k^6 \text{ as } k \rightarrow 0, \quad T < 1. \quad (3.56)$$

The results (3.55) and (3.56) are valid only outside the neighborhood of $T=1$. In fact, at $T=1$ the results (3.55) and (3.56) are replaced by

$$\tau \sim k^{-5}, \quad 1 - \chi \sim k^4 \text{ as } k \rightarrow 0, \quad T = 1. \quad (3.57)$$

The crossover from the behavior (3.55) and (3.56) to the critical behavior (3.57) occurs at temperatures such that

$$1 - T \sim k. \quad (3.58)$$

IV. THE ASYMMETRIC SG WITH FERROMAGNETIC INTERACTION

A. The general model

In this section we investigate the effect of asymmetry on the long-range ferromagnetic (FM) order. We introduce infinite-range ferromagnetic interactions simply by adding a constant positive term J_0/N to the off-diagonal elements of the interaction matrix J_{ij} . The Langevin dy-

namic equations are now

$$\Gamma_0^{-1} \frac{\partial}{\partial t} \sigma_i(t) = -r_0 \sigma_i(t) - \frac{\delta V(\sigma_i)}{\delta \sigma_i(t)} + \sum_j J_{ij} \sigma_j(t) + J_0 m(t) + h_i(t) + \xi_i(t), \quad (4.1)$$

where the J_{ij} 's are given, as before, by Eqs. (2.1)–(2.4), and $J_0 > 0$. The magnetization $m(t)$ is

$$m(t) = \frac{1}{N} \sum_{j(\neq i)} \sigma_j(t) = [\langle \sigma_j(t) \rangle], \quad (4.2)$$

where the last equality holds in the thermodynamic limit.

The quantities of interest, besides the local $G(t)$ and $C(t)$, are the *uniform* response and correlation functions defined by

$$G_{\text{FM}}(t) = \frac{1}{N} \sum_{i,j} \left. \frac{\delta[\langle \sigma_i(t+t') \rangle]}{\delta h_j(t')} \right|_{h=0} = \left. \frac{\delta m(t+t')}{\delta h(t')} \right|_{h=0}, \quad t \geq 0 \quad (4.3)$$

and

$$C_{\text{FM}}(t) = \frac{1}{N} \sum_{i,j} [\langle \sigma_i(t+t') \sigma_j(t') \rangle - \langle \sigma_i(t+t') \rangle \langle \sigma_j(t') \rangle], \quad (4.4)$$

where $h(t)$ is a *uniform* external field.

The dynamic mean-field equations are now

$$G_0^{-1}(\omega) \sigma_i(\omega) = \varphi_i(\omega) + J_0 m(\omega) + h_i(\omega) + \int_{-\infty}^{+\infty} dt e^{i\omega t} \frac{\delta V(\sigma_i)}{\delta \sigma_i(t)}, \quad (4.5)$$

where $G_0(\omega)$ and $\varphi_i(\omega)$ are given, as before, by Eqs. (2.15) and (2.16). The self-consistent equation for $m(t)$ is just

$$\langle \sigma_i(t) \rangle_{\varphi} = m(t). \quad (4.6)$$

Note that in the case of a static uniform external field $h_i(\omega) = 2\pi h \delta(\omega)$ Eq. (4.6) has the form

$$\langle \sigma_i(\omega) \rangle_{\varphi} = 2\pi m \delta(\omega), \quad (4.7)$$

i.e., the magnetization is a static quantity. In Appendix B we show that $G_{\text{FM}}(\omega)$ and $C_{\text{FM}}(\omega)$ are related to $G(\omega)$ and $C(\omega)$ by

$$G_{\text{FM}}(\omega) = \frac{G(\omega)}{1 - J_0 G(\omega)}, \quad (4.8)$$

$$C_{\text{FM}}(\omega) = \frac{C(\omega)}{|1 - J_0 G(\omega)|^2}. \quad (4.9)$$

In the $k=0$ case the FDT, Eq. (2.25a), implies via Eqs. (4.8) and (4.9) a similar theorem relating $G_{\text{FM}}(\omega)$ and $C_{\text{FM}}(\omega)$. In the $k \neq 0$ case, the relations (4.8) and (4.9) still hold, but the FDT does not exist. From Eqs.

(4.8) and (4.9) one concludes that the temperature T_c below which spontaneous magnetization appears is determined by the condition

$$1 = J_0 G_0(\omega=0) = J_0 \chi, \quad T \leq T_c. \quad (4.10)$$

B. The spherical model with FM interactions

In the spherical model, introduced in Sec. III A, the mean-field equations (4.5) reduce to

$$\sigma_i(\omega) = G(\omega) [\varphi_i(\omega) + 2\pi\delta(\omega)(h + J_0 m)], \quad (4.11)$$

where $G(\omega)$ is given by Eq. (3.5). The parameter r has to be determined via the constraint (3.3). Although explicit solution of this self-consistent linear problem is possible for all values of k , the case $k = 1$ offers a particularly simple example. Using Eqs. (3.3) and (3.5), the equation for T_c , Eq. (4.10), yields

$$T_c = (J_0^2 - 1)^{1/2}, \quad J_0 \geq 1, \quad k = 1. \quad (4.12)$$

For general k , the equation for the magnetization below T_c yields, $m = J_0 m \chi$, i.e.,

$$\chi = J_0^{-1}, \quad T \leq T_c. \quad (4.13)$$

Below T_c , both m and q are nonzero. Separating the self-consistent noise $\varphi_i(\omega)$, as in Eq. (2.27), one obtains

$$q = \langle \langle \sigma_i(t) \rangle_{\eta,x}^2 \rangle_z = \chi^2 (q + J_0^2 m^2), \quad (4.14)$$

or,

$$q = \frac{m^2}{1 - J_0^{-2}}, \quad T \leq T_c. \quad (4.15)$$

Using the constraint $1 = \tilde{C}(t=0) + q$, together with the result (3.11), which implies that $\tilde{C}(t=0) = T/T_c$, one has

$$q = 1 - T/T_c, \quad (4.16)$$

$$m^2 = (1 - T/T_c)(1 - J_0^{-2}), \quad (4.17)$$

valid for $T < T_c$ and $J_0 > 1$. Note that in the $T \rightarrow 0$ limit, $q \rightarrow 1$, meaning that the system is completely frozen at zero temperature, as in the SG case.

Equations (4.16) and (4.17) are independent of k , and the only effect of k is to reduce the value of T_c . Thus a weak asymmetry generates only a slight perturbation of the FM phase. In the language of Sec. III, the excess noise $x_i(\omega)$ and hence the excess spin correlations $\hat{q}(\omega)$ vanish in the $k \rightarrow 0$ limit for all ω , and $T < T_c$. The complete phase diagram of the asymmetric spherical model with nonzero J_0 is shown in Fig. 2(a). For comparison, we present in Fig. 2(b) the corresponding phase diagram in the $k = 0$ case.

V. ABSENCE OF AN SG PHASE IN GENERAL ASYMMETRIC NETWORKS

In this section we study the effects of random Gaussian antisymmetry on the SG freezing in systems whose interaction matrices are not necessarily Gaussian variables. We consider an interaction matrix J_{ij} of the following form:

$$J_{ij} = J_{ij}^s + k J_{ij}^{as}, \quad (5.1)$$

where J_{ij}^{as} is an antisymmetric matrix with independent Gaussian elements, i.e.,

$$P(J_{ij}^{as}) = \frac{1}{\sqrt{2\pi/N}} \exp \left[-\frac{(J_{ij}^{as})^2}{2/N} \right], \quad i < j \quad (5.2)$$

and $J_{ji}^{as} = -J_{ij}^{as}$. The symmetric part J_{ij}^s is a random matrix which exhibits an SG phase in the absence of the asymmetry (i.e., in the case $k = 0$). The matrix elements of J_{ij}^s may have correlations so that the SG phase is not necessarily identical to the SK model.¹² Specifically, we assume that the spectrum of the eigenvalues J_λ of J_{ij}^s forms a continuous band with a sharp edge, at least in the upper part of the spectrum. The average density of eigenvalues is assumed to vanish at this edge. As usual the matrix J_{ij}^s is normalized so that $J_{ij}^s = O(N^{-1/2})$. This

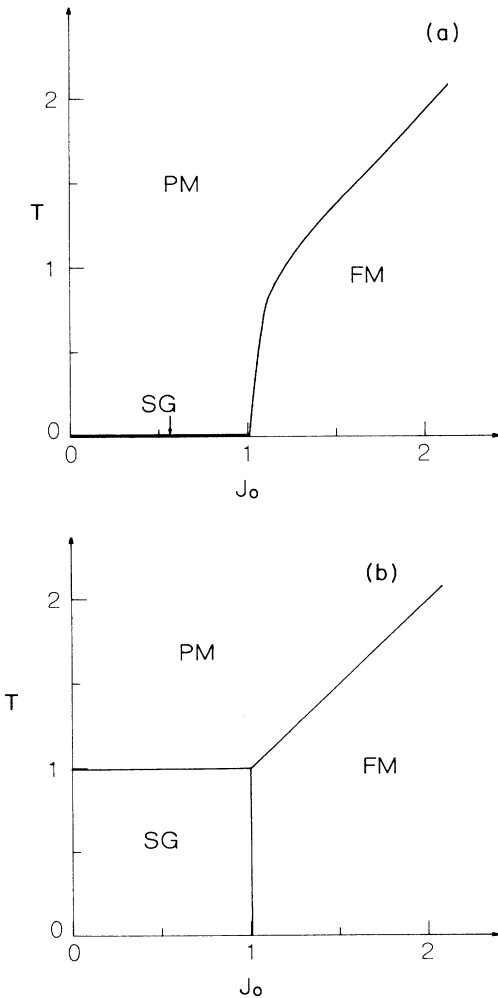


FIG. 2. Phase diagram of the SG spherical model: (a) $k \neq 0$. The line separating the FM and PM phases is qualitative. The case of $k = 1$ is given in Eq. (4.12); (b) $k = 0$.

ensures that the eigenvalues are of order 1.

An example is the Hopfield model of associative memory¹

$$J_{ij}^s = \frac{1}{N} \sum_{\mu=1}^p S_i^\mu S_j^\mu, \quad (5.3)$$

where $\{S_i^\mu\}_{\mu=1,2,\dots,p}$ are Np independent random (± 1) variables and $\alpha = p/N$ is finite. If $k=0$, the system undergoes a second-order SG transition² at finite T . The spectrum of J_{ij}^s , Eq. (5.3), is given by the following average density of eigenvalues:²⁰

$$\rho(J_\lambda) = (1-\alpha)\delta(J_\lambda + \alpha) + \rho_0(J_\lambda), \quad \alpha < 1, \quad (5.4a)$$

$$\rho(J_\lambda) = \rho_0(J_\lambda), \quad \alpha > 1, \quad (5.4b)$$

$$\rho_0(J_\lambda) = \frac{[4\alpha - (J_\lambda - 1)^2]^{1/2}}{2\pi(J_\lambda + \alpha)}, \quad (5.4c)$$

$$1 - 2\sqrt{\alpha} \leq J_\lambda \leq 1 + 2\sqrt{\alpha}.$$

Hence, for any finite α , the upper part of the spectrum of J_{ij}^s is continuous with an edge at

$$J_\lambda^{\max} = 1 + 2\sqrt{\alpha}. \quad (5.5)$$

In order to investigate the SG freezing in such systems we use again the dynamic spherical model of Sec. III with the general J_{ij} matrix of Eq. (5.1). The full mean-field equations depend on the form of J_{ij}^s which we have not fully specified. Instead, we use the mean-field procedure to write down the dynamic equations which result from averaging only over the Gaussian antisymmetric part. These equations are

$$\Gamma_0^{-1} \frac{\partial}{\partial t} \sigma_i(t) = -r \sigma_i(t) + \sum_j' J_{ij}^s \sigma_j(t) - k^2 \int dt' G(t-t') \sigma_i(t') + \phi_i(t), \quad (5.6)$$

where the prime, as usual, means that the sum is over all $j \neq i$. The stochastic variable $\phi_i(t)$ has a Gaussian distribution with zero mean and variance

$$\langle \phi_i(t) \phi_j(t') \rangle = \left[\frac{2T}{\Gamma_0} \delta(t-t') + k^2 C(t-t') \right] \delta_{ij}. \quad (5.7)$$

$C(t)$ and $G(t)$ are the average local correlation and response functions given by Eqs. (2.7) and (2.8), which have to be calculated self-consistently through (5.6). As before, r is determined by the constraint (3.3).

The matrix J_{ij}^s is symmetric, so that it has real eigenvalues J_λ and its eigenvectors ψ_λ^i (which are real) form an orthogonal base. Thus Eq. (5.6) can be diagonalized in terms of

$$\sigma_\lambda(t) = \frac{1}{N} \sum_{i=1}^N \sigma_i(t) \psi_\lambda^i. \quad (5.8)$$

Denoting

$$\sigma_\lambda(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} \sigma_\lambda(t), \quad (5.9)$$

then Eq. (5.6) reads

$$\sigma_\lambda(\omega) = G_\lambda(\omega) \phi_\lambda(\omega), \quad (5.10)$$

$$G_\lambda^{-1}(\omega) = r - J_\lambda - i\omega \Gamma_0^{-1} + k^2 G(\omega), \quad (5.11)$$

$$\langle \phi_\lambda(\omega) \phi_\lambda(\omega') \rangle = \left[\frac{2T}{\Gamma_0} + k^2 C(\omega) \right] 2\pi \delta(\omega + \omega'), \quad (5.12)$$

where $C(\omega)$ and $G(\omega)$ are the Fourier components of the average local correlation and response functions. By definition one has

$$\langle \sigma_\lambda(\omega) \sigma_\lambda(\omega') \rangle = 2\pi \delta(\omega + \omega') C_\lambda(\omega), \quad (5.13)$$

so that from Eqs. (5.10) and (5.12) it follows that

$$C_\lambda(\omega) = |G_\lambda(\omega)|^2 \left[\frac{2T}{\Gamma_0} + k^2 C(\omega) \right]. \quad (5.14)$$

The functions $C(\omega)$ and $G(\omega)$ are related to $C_\lambda(\omega)$ and $G_\lambda(\omega)$ through

$$C(\omega) = \int dJ_\lambda \rho(J_\lambda) C_\lambda(\omega), \quad (5.15)$$

$$G(\omega) = \int dJ_\lambda \rho(J_\lambda) G_\lambda(\omega), \quad (5.16)$$

where, as before, $\rho(J_\lambda)$ is the (average) density of eigenvalues of J_{ij}^s . The integrals are over the whole range of eigenvalues.

Let us define

$$g(\omega) = \int dJ_\lambda \rho(J_\lambda) |G_\lambda(\omega)|^2. \quad (5.17)$$

Then from Eqs. (5.14) and (5.15) we find

$$C(\omega) = \frac{2T\Gamma_0^{-1}}{[g(\omega)]^{-1} - k^2}. \quad (5.18)$$

On the other side from Eq. (5.11) it follows that

$$- |G_\lambda(\omega)|^{-2} \text{Im} G_\lambda(\omega) = -\omega + k^2 \text{Im} G(\omega), \quad (5.19)$$

so that

$$g(\omega) = \frac{1}{\omega - k^2 \text{Im} G(\omega)} \text{Im} \int dJ_\lambda \rho(J_\lambda) G_\lambda(\omega) = \frac{\text{Im} G(\omega)}{\omega - k^2 \text{Im} G(\omega)} = \frac{1}{\frac{\omega}{\text{Im} G(\omega)} - k^2}. \quad (5.20)$$

Inserting Eq. (5.20) into Eq. (5.18) we find

$$C(\omega) = \frac{2T\Gamma_0^{-1}}{\frac{\omega}{\text{Im} G(\omega)} - 2k^2}. \quad (5.21)$$

From Eqs. (5.11), (5.16), and (5.21) one can calculate $G(\omega)$ and $C(\omega)$ if the average density of eigenvalues of the symmetric matrix $\rho(J_\lambda)$ is known. The parameter r is determined by the constraint (3.3). In the $k=0$ case, as T decreases, r decreases until it reaches the value $r = J_\lambda^{\max}$, signaling an SG transition. Because of the form of the constraint, the transition temperature will in fact be identical to its value in the corresponding Ising case. For example, solving the spherical version of the symmetric Hopfield model, one finds²⁰ an SG phase for all $\alpha > 0$, below $T_g = 1 + \sqrt{\alpha}$, as in the Ising case.²

However, if $k \neq 0$, Eq. (5.21) shows that a divergence of $\text{Im}G(\omega)/\omega$ would lead to instability of $C(\omega)$. On the other hand a divergence of $C(\omega=0)$ implies

$$\left. \frac{\text{Im}G(\omega)}{\omega} \right|_{\omega=0} = \frac{1}{2k^2}, \quad (5.22)$$

but in this case $G(\omega)$ is still analytic in ω , so that for small ω we find the behavior $C(\omega) \sim \omega^{-2}$ which means that $C(t=0)$ will diverge and the constraint (3.3) cannot be satisfied.

Our conclusion, therefore, is that the random Gaussian asymmetry, even if it is arbitrarily weak, destroys the SG phase in general fully connected networks. It should be emphasized that we have used the spherical model *only* to study the possibility of *SG freezing*. In general, the networks may possess other types of ordered states. Unlike the SG or FM states discussed above, some of these ordered states may be reached only via a *first-order* transition and cannot be investigated within the spherical model. For instance, in an Ising system with the Hopfield interactions, Eq. (5.3), the retrieval states (states with a large overlap with the patterns $\{S_i^\mu\}$) exist at low T for $\alpha < \alpha_c \sim 0.14$.² They appear (upon varying α or T) in a discontinuous manner,² indicating that their existence depends on the strong nonlinearity of the system. Indeed, it is found²⁰ that retrieval states do not exist in the “spherical version” of the symmetric Hopfield model for any finite α . Therefore, the spherical model is useless in studying the effect of asymmetry on the retrieval states. Further discussion of the retrieval states is presented in Sec. VI.

VI. SUMMARY AND DISCUSSION

We have studied the effect of randomly asymmetric bonds on fully connected spin systems governed by Langevin nonlinear dynamics. Applying mean-field theory, the dynamics of the system has been described by single-spin self-consistent equations. In this paper the mean-field equations have been solved within the *spherical model*. In this model the starting equations are *linear* Langevin equations in which the linear coefficient is determined by the global constraint (3.2). In Sec. VIA we summarize the results of the spherical model. In Sec. VIB we discuss the general, nonlinear case.

A. Summary of results: The spherical model

1. Absence of an SG phase at all $T > 0$

In the presence of the asymmetry, the fluctuation dissipation theorem is violated and the system does not relax to *thermal equilibrium*. In mean-field theory this effect can be described by an excess local noise which is generated by the random asymmetry. The static local susceptibility χ is then

$$\chi = \beta(1 - \hat{q}), \quad \beta = 1/T, \quad (6.1)$$

where \hat{q} is the amplitude of the local spin correlations induced by the excess noise. It is given by

$$\hat{q} = \langle \langle \sigma_i(t) \rangle_\eta^2 \rangle_{x,z}, \quad (6.2)$$

where $\langle \sigma_i(t) \rangle_\eta$ is the local spin average over the (renormalized) *thermal* noise $\eta(t)$ [see Eq. (2.28)] but not over the static and dynamic components [denoted, respectively, by z and x (t)] of the *excess* noise.

The condition for the appearance of an SG phase below $T = T_g$ is

$$\chi = J^{-1}, \quad T \leq T_g, \quad (6.3)$$

where J (assumed equal to 1 in the preceding sections) is defined by Eq. (2.4a). But the presence of the dynamic component of the excess noise suppresses χ so that for any strength of asymmetry, $\chi < J^{-1}$ for all $T > 0$. Therefore, the Edwards-Anderson (EA) order parameter

$$q = \langle \langle \sigma_i(t) \rangle_{\eta,x}^2 \rangle_z, \quad (6.4)$$

is zero for all $T > 0$. The SG interactions induce local dynamic fields instead of the static ones which appear in symmetric SG's below T_g .

2. Kinetic freezing at small k

At high T (i.e., $T > J$), when the value of k decreases so does \hat{q} . On the other hand, at low T ($T < J$) \hat{q} remains finite as k decreases. This is because the amplitude of the dynamic component of the excess noise does *not* vanish as $k \rightarrow 0$. Instead, it becomes increasingly slower at small k . The characteristic relaxation time for the decay of the dynamic correlations in the excess noise [and hence the decay of $\hat{q}(t)$] diverges as

$$\tau \propto k^{-6}, \quad k \rightarrow 0, \quad T < J. \quad (6.5a)$$

Thus, at $T < J$, the system behaves as a frozen symmetric SG in the “*high*”-frequency range $\omega \gg k^6$. In this regime the excess spin correlations $\hat{q}(\omega)$ play the role of the EA order parameter. As ω becomes smaller than k^6 these correlations decay and the system behaves “paramagnetically.” The crossover from a frozen SG state to a paramagnetic behavior is depicted schematically in Fig. 3. At the critical temperature regime of the symmetric system, i.e., $T - J \sim k$, the relaxation time τ diverges as

$$\tau \propto k^{-5}, \quad k \rightarrow 0, \quad T \sim J. \quad (6.5b)$$

The “thermal” noise [and hence also $G(\omega)$ and the thermal spin correlations $\hat{C}(\omega)$] decays much faster than the excess noise. Its characteristic frequency is

$$\omega_T \propto k^4, \quad k \rightarrow 0, \quad T < J, \quad (6.6)$$

see Eqs. (3.43)–(3.46).

Note that purely static quantities have a smooth limit as $k \rightarrow 0$ (e.g., $\chi \rightarrow 1$ for $T \leq T_g$).

3. The limit of zero temperature

In the spherical model, the system has a zero-temperature SG transition for all values of k , signaled by

$$\chi(T=0) = J^{-1}, \quad (6.7)$$

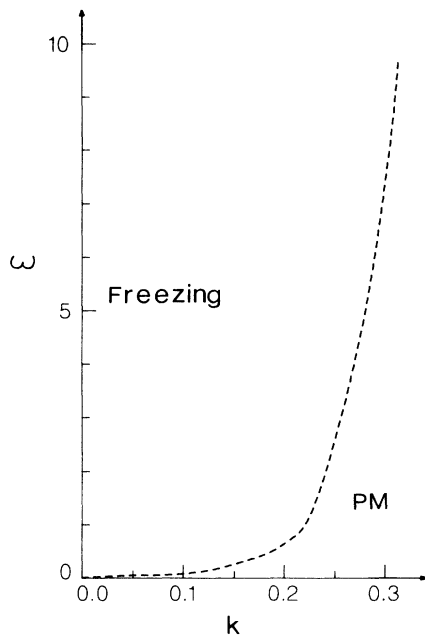


FIG. 3. Crossover from a frozen SG to a PM state at $T < J$. For frequencies *above* the dashed line the system behaves as a frozen symmetric SG. On the other hand, for frequencies *below* the dashed line the correlations decay and the system behaves paramagnetically. ω is in arbitrary units.

see Fig. 1. In terms of the EA order parameter q the transition is discontinuous: As $T \rightarrow 0$, q jumps from zero to 1, implying the complete freezing of the system at $T=0$. This is consistent with the large- n result of Bausch *et al.*⁹ As $T \rightarrow 0$ the system exhibits a critical slowing down with a characteristic relaxation time which diverges as

$$\tau \propto 1/T, \quad T \rightarrow 0 \quad (6.8)$$

for all $k \neq 0$.

4. Long-range ferromagnetic order

The condition for an onset of long-range FM order at $T = T_c$ is, according to the mean-field theory,

$$\chi = J_0^{-1}, \quad T = T_c, \quad (6.9)$$

where $J_0 > 0$ is (N times) the mean FM interaction. Since in the paramagnetic phase, χ is *independent* of J_0 , it is clear that for all k an FM transition is induced at sufficiently large J_0 . A weak asymmetry has, thus, a negligible effect on the FM state. The main effect of the asymmetry is to reduce somewhat the value of T_c , see Figs. 2. Once again, in the spherical model the system freezes completely at $T=0$ with a value of magnetization m which is less than unity but $q = 1$.

B. General nonlinear systems

We now discuss the properties of nonlinear systems governed by the Langevin equations (2.5).

1. Absence of an SG phase at $T > 0$

The mechanism which leads to the absence of SG freezing, for all $k \neq 0$, in the spherical model is expected to apply also in the general nonlinear case, as discussed in Ref. 7. From the structure of the general mean-field equations (2.12)–(2.16) it follows that the spin autocorrelation function $C(\omega)$ is of the form

$$C(\omega) = \frac{\Lambda(\omega)}{J^{-2} |G(\omega)|^{-2} - 1}, \quad (6.10)$$

where $G(\omega)$ is the dynamic susceptibility and $\Lambda(\omega)$ is the renormalized noise vertex equal to $2T\Gamma_0^{-1}$ plus contributions from the nonlinear terms [compare with Eq. (3.11)]. Examining these contributions by perturbation theory indicates that the dominant singularity (at $\omega=0$) in $C(\omega)$ comes only from the vanishing of the denominator. This implies that the condition (6.3) for the onset of long-time autocorrelation holds generally.

Likewise,

$$\text{Im}G(\omega) = \frac{\omega[\Gamma_0^{-1} + \Sigma(\omega)]}{J^{-2} |G(\omega)|^{-2} - (1-k^2)/(1+k^2)}, \quad (6.11)$$

where again $\Sigma(\omega)$ is the contribution of the nonlinear terms [compare with Eq. (3.7)]. Again, one expects that $\Sigma(\omega)$ will not diverge at $\omega=0$. Thus the low-frequency singularity in $G(\omega)$ occurs when

$$J\chi = [(1+k^2)/(1-k^2)]^{1/2} > 1. \quad (6.12)$$

Therefore, $J\chi \rightarrow 1$ would imply a singularity in $C(\omega)$ of the form $C(\omega) \sim \omega^{-2}$ which is inconsistent since *local* spin fluctuations [$\langle \sigma_i^2(t) \rangle$] should remain finite.

From this argument one concludes that SG transition is suppressed and $q = 0$ as soon as asymmetry is turned on. The qualitative description of the dynamic behavior of the system in terms of an excess dynamic noise and excess spin correlations applies in the general case as well, and so does Eq. (6.1). Similarly, the behavior of the system with small asymmetry is similar to the one described above. At $k=0$ the EA order parameter jumps discontinuously from zero to its value in the symmetric SG. This transition is generated by a critical slowing down of the excess noise and the spin correlations as $k \rightarrow 0$. Most probably, the divergence of the characteristic relaxation times remains as in Eqs. (6.5) and (6.6) even in the presence of nonlinearities. These conclusions are supported by numerical simulations of an asymmetric Ising SG.¹⁴

2. Absence of an SG phase at zero temperature

The most important difference between the spherical model and the behavior of nonlinear systems concerns with the $T \rightarrow 0$ limit. We believe that, *the $T=0$ SG freezing predicted by the spherical model is an artifact of the linearity of the system.* In explicitly nonlinear systems, the PM phase (i.e., the $q=0$ state) will remain stable even at $T=0$.

In order to understand the origin of the $T=0$ freezing in the spherical model let us consider again the dynamic equations (3.1). Let us denote by $\{\psi_\lambda^i\}$ the λ th *right*

eigenvector of the *asymmetric* matrix J_{ij} and by J_λ the corresponding eigenvalue. In the absence of thermal noise (i.e., $\xi=0$) the general solution of Eqs. (3.1) is

$$\sigma_i^{(t)} = \sum_{\lambda=1}^N a_\lambda \psi_\lambda^i e^{-t\omega_\lambda}, \quad (6.13)$$

where

$$\omega_\lambda = \Gamma_0(r - J_\lambda), \quad (6.14)$$

and a_λ are arbitrary.

The *long-time* behavior is then

$$\sigma_i^{(t)} \sim a_{\lambda_0} \psi_{\lambda_0}^i e^{-t\omega_{\lambda_0}}, \quad (6.15)$$

where J_{λ_0} is the eigenvalue with the largest real part. Taking into account the constraint (3.3) which, in $T=0$, reads

$$\lim_{t \rightarrow \infty} [\sigma_i^2(t)] = 1, \quad (6.16)$$

leads to $\omega_{\lambda_0} = 0$, i.e.,

$$r = J_{\lambda_0}. \quad (6.17)$$

Comparison with Eq. (3.25) implies that J_{λ_0} is in fact real and has the value

$$J_{\lambda_0} = \frac{2}{1+k^2}. \quad (6.18)$$

(These predictions are in agreement with a recent study of the spectrum of real random asymmetric matrices.²¹)

The above exercise clearly demonstrates that in the absence of external sources a (stable) linear system must relax to a static limit. On the other hand, the nonlinear equations (2.5) can have solutions with nontrivial long-time behavior, even in the absence of external sources. These solutions are “chaotic” in the sense that $\lim_{t \rightarrow \infty} [\sigma_i^2(t)]$ is finite, whereas the average correlations between $\sigma_i(t)$ and $\sigma_i(t')$ decay to zero as $|t-t'| \rightarrow \infty$. This will stabilize the PM phase even at $T=0$.

It is important to note that the spherical model predicts that the PM phase is unstable at $T=0$, not only for small k but for all values of k . On the other hand, studying the dynamics of Ising systems with asymmetric bonds, at $T=0$, one finds¹⁴ that for $k > 1$ stable states do not exist at all. This already suggests that the instability of the PM phase at $T=0$ is an artifact of the spherical model. This conclusion is also supported by numerical simulations of an asymmetric Ising SG with $k < 1$, as will be discussed in detail in Ref. 14.

3. The stability of the ferromagnetic order

The relations (4.8) and (4.9) between the local response and correlation functions and the uniform ones hold generally for all systems in the mean-field limit. Hence the condition for the onset for an FM order is given by Eq. (6.9) in the nonlinear case as well. Similarly, in all mean-field systems χ is independent of J_0 in the paramagnetic phase. Therefore, one concludes that ran-

dom asymmetry does not destroy completely the onset of FM order at finite temperatures for sufficiently large values of J_0 . As $k \rightarrow 0$ the value of $T_c(J_0)$ approaches smoothly its value in the symmetric case.

4. Replica symmetry breaking

A central feature of the SG phase in symmetric nonlinear systems is the well-known replica symmetry breaking (RSB).^{15,22} This feature does not exist in the symmetric spherical model. RSB is associated with the structure of the $\omega=0$ singularities in the $C(\omega)$ and $G(\omega)$.^{13,17} In the asymmetric case, both $C(\omega)$ and $G(\omega)$ are not singular at $\omega=0$ and therefore do not exhibit the effects of RSB. This is most probably true also for the FM phase, although in the symmetric case, replica symmetry is broken in the FM phase at sufficiently low T . Nevertheless, the precursors of RSB will probably appear as k becomes sufficiently small. In particular, as $k \rightarrow 0$, the slow relaxation of the excess noise may not be characterized by one or two relaxation times, but instead by a hierarchical distribution of large relaxation times. Likewise, anomalously slow components will appear as $k \rightarrow 0$, not only in $C(\omega)$, but in $G(\omega)$ as well.^{13,17} We do not know yet how small k must be (as function of the size N of the system) for these phenomena to be observed. Studying the appearance of RSB in the $k \rightarrow 0$ limit might shed new insight on the physical meaning of this phenomenon.

C. Order in asymmetric neural networks

In Sec. V we have shown, using the spherical model, that the SG phase is destroyed by weak asymmetry not only in SK-type systems but also in neural networks with correlated bonds. In particular, the SG phase which exists² in the symmetric Hopfield model (5.3) (for all values of $\alpha > 0$) disappears when random asymmetry (with arbitrary strength) is added. These results are expected to hold not only in the spherical model but in nonlinear models as well.

An important question is whether the retrieval phases, i.e., the phases which are highly correlated with the “memories” of the network, are also destroyed by weak asymmetry. Explicit calculations of these states are difficult, in particular, since they may appear via first-order transitions. It is therefore useful to try to generalize the difference between the effect of weak asymmetry on the SG and FM phases. Within the local mean-field theory the essential difference between the two cases is the following. In the symmetric SG phase the local fields are static random fields whose correlations are determined self-consistently. When asymmetry is turned on, random dynamic fields replace the static ones thus, stabilizing the PM phase at low T . These dynamic fields are extremely slow when the asymmetry is weak, so that at a given finite period of time, this PM state behaves in a manner similar to that of a frozen SG. On the other hand, the local fields in the symmetric FM phase have a uniform component. This component cannot be replaced by dynamic fields since the random asymmetry

generates only random fields. This explains why (at least in mean-field theory) the PM phase remains stable at low T in the asymmetric SG case and not in the FM case.

This distinction can be easily generalized to other types of ordered phases. Those phases that are described by global order parameters cannot be destroyed by weak asymmetry. In particular, the “retrieval states” of the symmetric Hopfield model are characterized by the global overlap

$$m = \frac{1}{N} \sum_i S_i^0 S_i^\mu > 0, \quad (6.19)$$

where $\{S_i^0\}$ is the retrieval state and $\{S_i^\mu\}$ is the nearest memory. Similarly, the static local fields in this phase have a component which is proportional to $\xi_i^\mu m$. This order cannot be replaced by the Gaussian random fields generated by the random asymmetry and therefore m will remain positive if k is sufficiently small. The same applies to the ordering of other neural network models (see, e.g., Ref. 4).

The above discussion supports the conclusion of Hertz *et al.*⁷ and others^{9,10} that the retrieval phases but not the SG phase remain stable in the presence of weak asymmetry. Note that in the limit of $\alpha \rightarrow 0$, studied by Feigelman and Ioffe,⁸ the Hopfield model is a special case of the asymmetric SG with FM interactions, since the correlations among the bonds vanish in this limit.

Note, however, that the retrieval phases, in the asymmetric networks, do *not* represent fixed (“persistent”) configurations of spins. According to the above results (regarding the FM phase), spins will continue to fluctuate even at $T=0$. Nevertheless, the average overlap m will have a finite long-time limit.

Naively one would think that the disappearance of the spurious SG states enlarges the basins of attraction of the “memories” or speeds up the retrieval process. This, however, may not be the case, since the PM phase may attract as much volume of configuration space as the original SG one. The destruction of the SG phases may be more relevant to the “Hebb” learning process as suggested by Parisi.¹⁰ If the synaptic changes ΔJ_{ij} are proportional to the product of the time averages of $S_i(t)$ and $S_j(t)$ then the “spurious” PM phase will not contribute to ΔJ_{ij} because of the fluctuations of $S_i(t)$. However, our result, Eq. (6.5), indicates that the time correlations in the PM phase decay extremely slowly if the asymmetry is weak. Therefore, in order that this learning mechanism will work in practice, *strong* asymmetry

needs to be present. On the other hand, strong asymmetry increases the level of noise in the retrieval process (e.g., the capacity α_c will be substantially reduced while the level of errors will increase significantly).

We now comment on a few related issues. First, we have discussed so far only *Gaussian* asymmetry. Our results apply, however, to other distributions of asymmetry as well as long as the antisymmetric components are independent random variables. Since in the thermodynamic limit, only the first two moments of J_{ij} are relevant, the parameter k can still be defined in the general case via Eqs. (2.4). An example of a network whose asymmetry is due to an asymmetric dilution of bonds is presented in Appendix C.

It should be emphasized that the paper discusses only fully connected systems, i.e., systems where the average number of neighbors per spin is of $O(N)$. An interesting question is whether the instability of the SG phase in the presence of random asymmetry holds also in *short-range* systems. The behavior of short-range asymmetric systems or infinite-range systems with finite, coordination number is beyond the scope of the present paper. We point out, however, that a recent work by Derrida²³ found a dynamic phase transition at finite temperature in fully asymmetric SG. The system studied is a highly diluted infinite-range SG. This phase transition is apparently not associated with the onset of freezing but rather with the sensitivity of the dynamics flows to small changes in the initial conditions. This behavior, which is not reflected in a singularity of $G(\omega)$ and $C(\omega)$, cannot be investigated by the methods of the present paper.

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APPENDIX A: THE DYNAMIC MEAN-FIELD EQUATIONS OF MOTION

To treat the stochastic equation of motion (2.5) we use a functional integral formalism¹³ which is very convenient for the discussion of quenched-random systems.²⁴ We define a stochastic generating functional for the correlation and response functions

$$Z_J[\phi, \hat{\phi}] = \int D\sigma \int D\hat{\sigma} \exp \left[\int dt \sum_i [\phi_i(t)\sigma_i(t) + i\hat{\phi}_i(t)i\hat{\sigma}_i(t)] + L_J[\hat{\sigma}, \sigma] \right], \quad (A1)$$

$$L_J[\hat{\sigma}, \sigma] = \int dt \sum_i \left[i\hat{\sigma}_i(t) \left[-\Gamma_0^{-1} \partial_t \sigma_i(t) - r_0 \sigma_i(t) - \frac{\delta V(\sigma_i)}{\delta \sigma_i(t)} + \sum_j J_{ij} \sigma_j(t) + h_i(t) + T \Gamma_0^{-1} i\hat{\sigma}_i(t) \right] + K(\sigma_i) \right]. \quad (A2)$$

The term $K(\sigma_i)$, which arises from the functional Jacobian, is given by^{25,26}

$$K(\sigma_i) = -\frac{1}{2} \int dt \frac{\delta^2 V(\sigma_i)}{\delta \sigma_i^2(t)}, \quad (\text{A3})$$

and ensures the proper normalization of Z_J ,

$$Z_J[\phi = \hat{\phi} = 0] = 1. \quad (\text{A4})$$

Taking the average over J_{ij} yields the functional which

generates the average correlation and response functions

$$Z[\phi, \hat{\phi}] = \int d[J] P[J] \frac{Z_J[\phi, \hat{\phi}]}{Z_J[\phi = \hat{\phi} = 0]}. \quad (\text{A5})$$

Since the Z_J is normalized [Eq. (A4)] the quenched average is done directly²⁴ on Z_J . This is particularly easy due to the fact that Eq. (2.5) is linear in J_{ij} . A straightforward integration then yields

$$Z[\phi, \hat{\phi}] = \int D\sigma \int D\hat{\sigma} \exp \left[\sum_i L_0(\hat{\sigma}_i, \sigma_i) + \frac{J^2}{2N} \sum_{j=1}^N \sum_{\substack{i=1 \\ (i \neq j)}}^N \int dt dt' [i\hat{\sigma}_i(t)\sigma_j(t)i\hat{\sigma}_i(t')\sigma_j(t') + \frac{(1-k^2)}{(1+k^2)} i\hat{\sigma}_i(t)\sigma_j(t)i\hat{\sigma}_j(t')\sigma_i(t')] \right], \quad (\text{A6})$$

$$L_0(\hat{\sigma}_i, \sigma_i) = \int dt \left[i\hat{\sigma}_i(t) \left(-\Gamma_0^{-1} \partial_t \sigma_i(t) - r_0 \sigma_i(t) - \frac{\delta V(\sigma_i)}{\delta \sigma_i(t)} + h_i(t) + T\Gamma_0^{-1} i\hat{\sigma}_i(t) \right) + \phi_i(t)\sigma_i(t) + i\hat{\phi}_i(t)i\hat{\sigma}_i(t) + K(\sigma_i) \right]. \quad (\text{A7})$$

Note that in deriving Eq. (A6) we have used the symmetry of J_{ij}^s and the antisymmetry of J_{ij}^{as} .

The four-spin interactions can be decoupled by using Gaussian transformations and introducing four auxiliary fields $Q_\alpha(t, t')$ ($\alpha = 1, 2, 3, 4$). In the mean-field limit ($N \rightarrow \infty$) the integral over the fields Q_α can be done by the steepest-descent method.¹³ One then obtains the following effective local generating functional:

$$Z[\phi, \hat{\phi}] = \int D\sigma \int D\hat{\sigma} \exp \left[L_0(\hat{\sigma}_i, \sigma_i) + \frac{J^2}{2} \int dt \int dt' \left[C(t-t') i\hat{\sigma}_i(t) i\hat{\sigma}_i(t') + 2 \frac{1-k^2}{1+k^2} G(t-t') i\hat{\sigma}_i(t) \sigma_i(t') \right] \right]. \quad (\text{A8})$$

The average local correlation and response functions $C(t)$ and $G(t)$ are given by $C(t) = [\langle \sigma_i(t+t') \sigma_i(t') \rangle]$ and $G(t) = [\langle i\hat{\sigma}_i(t') \sigma_i(t+t') \rangle]$, which have to be calculated self-consistently through Eq. (A8). Carrying out the integration over $\hat{\sigma}_i$ in Eq. (A8), the generating functional reduces to the mean-field equations, Eqs. (2.12) and (2.13).

APPENDIX B: THE AVERAGE FM CORRELATION AND RESPONSE FUNCTIONS IN THE MEAN-FIELD LIMIT

In this appendix we show how the FM correlation and response functions, defined by Eqs. (4.3) and (4.4), can be calculated in the mean-field theory. We use the functional method described in the Appendix A. Let us consider the general stochastic equations of motion (4.1). This can be written as

$$\Gamma_0^{-1} \frac{\partial}{\partial t} \sigma_i(t) = f_i(\sigma) + \frac{J_0}{N} \sum_{i=1}^N \sigma_j(t) + h_i(t) + \xi_i(t), \quad (\text{B1})$$

where

$$f_i(\sigma) = -r_0 \sigma_i(t) - \frac{\delta V(\sigma_i)}{\delta \sigma_i(t)} + \sum_j J_{ij} \sigma_j(t), \quad (\text{B2})$$

and the prime means that the sum is over all $j \neq i$.

We shall derive the mean-field equations for the FM correlation and response functions above the ferromagnetic transition, i.e., $m = \langle \sigma \rangle = 0$. In the FM phase the derivation follows the same line with few minor changes.

In terms of the fields σ and $\hat{\sigma}$ the correlation and the response functions are given, respectively, by the correlations $\langle \sigma \sigma \rangle$ and $\langle i\hat{\sigma} \sigma \rangle$. Thus we define the generating functional²⁵

$$Z_\xi[\phi, \hat{\phi}] = \int D\sigma \int D\hat{\sigma} \exp \left[L(\sigma, \hat{\sigma}, \xi) + \int dt \left[i\hat{\phi}(t) \sum_i i\hat{\sigma}_i(t) + \phi(t) \sum_i \sigma_i(t) \right] \right], \tag{B3}$$

$$L(\sigma, \hat{\sigma}, \xi) = \int dt \sum_i \left[i\hat{\sigma}(t) \left[-\Gamma_0^{-1} \partial_i \sigma_i(t) + f_i(\sigma) + \frac{J_0}{N} \sum_j' \sigma_j(t) + h_i(t) + \xi_i(t) \right] + V(\sigma_i(t)) \right]. \tag{B4}$$

Since $Z_\xi[\phi = \phi = 0] = 1$, one has [see Eqs. (4.3) and (4.4)]

$$C_{\text{FM}}(t - t') = \frac{1}{N} \frac{\delta^2 Z[\phi, \hat{\phi}]}{\delta \phi(t) \delta \phi(t')} \Big|_{\phi = \hat{\phi} = h = 0}, \tag{B5}$$

$$G_{\text{FM}}(t - t') = \frac{1}{N} \frac{\delta^2 Z[\phi, \hat{\phi}]}{\delta \phi(t) \delta i\hat{\phi}(t')} \Big|_{\phi = \hat{\phi} = h = 0}, \tag{B6}$$

where

$$Z[\phi, \hat{\phi}] = [\langle Z_\xi[\phi, \hat{\phi}] \rangle]. \tag{B7}$$

The two averages can be easily carried out leading to

$$Z[\phi, \hat{\phi}] = \int D\sigma \int D\hat{\sigma} \exp \left[\int dt \left[\frac{J_0}{N} \sum_{j=1}^N \sum_{\substack{i=1 \\ (i \neq j)}}^N i\hat{\sigma}_i(t) \sigma_j(t) + \phi(t) \sum_i \sigma_i(t) + i\hat{\phi}(t) \sum_i i\hat{\sigma}_i(t) \right] + L_e(\sigma, \hat{\sigma}) \right], \tag{B8}$$

$$L_e(\sigma, \hat{\sigma}) = \int dt \sum_i \{ i\hat{\sigma}_i(t) [-\Gamma_0^{-1} \partial_i \sigma_i(t) + f_i^e(\sigma) + h_i(t) + \Gamma_0^{-1} T i\hat{\sigma}_i(t)] + K(\sigma_i(t)) \}, \tag{B9}$$

where f_i^e consists of the first two terms in Eq. (B2) and the terms generated by the averaging over J_{ij} .

We are interested in the mean-field limit ($N \rightarrow \infty$); thus, we can replace in Eq. (B8) \sum' with \sum , the difference being of $O(1/N)$. Then, with the help of a Gaussian transformation, the generating functional takes the form

$$Z[\phi, \hat{\phi}] = \frac{N}{J_0} \int Dm \int D\hat{m} \exp \left\{ \int dt \left[-\frac{N}{J_0} i\hat{m}(t)m(t) + \frac{N}{J_0} \phi(t)m(t) + \frac{N}{J_0} i\hat{\phi}(t)i\hat{m}(t) - \frac{N}{J_0} i\hat{\phi}(t)\phi(t) \right] \right. \\ \left. \times \int D\sigma \int D\hat{\sigma} \exp \left[\int dt \left[m(t) \sum_i i\hat{\sigma}_i(t) + i\hat{m}(t) \sum_i \sigma_i(t) \right] + L_e(\sigma, \hat{\sigma}) \right] \right\}. \tag{B10}$$

Inserting Eq. (B10) into Eqs. (B5) and (B6) we find

$$C_{\text{FM}}(t) = \frac{N}{J_0^2} \langle \langle m(t+t')m(t') \rangle \rangle, \tag{B11}$$

$$G_{\text{FM}}(t) = \frac{N}{J_0^2} \langle \langle i\hat{m}(t')m(t+t') \rangle \rangle - \frac{1}{J_0} \delta(t), \quad t \geq 0 \tag{B12}$$

where

$$\langle \langle (\dots) \rangle \rangle = \frac{N}{J_0} \int Dm \int D\hat{m} (\dots) e^{N\Omega(m, \hat{m})}, \tag{B13}$$

$$\Omega(m, \hat{m}) = -\frac{1}{J_0} \int dt i\hat{m}(t)m(t) + \frac{1}{N} \ln \int D\sigma \int D\hat{\sigma} \exp \left[\int dt \left[m(t) \sum_i i\hat{\sigma}_i(t) + i\hat{m}(t) \sum_i \sigma_i(t) \right] + L_e^0(\sigma, \hat{\sigma}) \right], \tag{B14}$$

$$L_e^0(\sigma, \hat{\sigma}) = L_e(\sigma, \hat{\sigma}) \Big|_{h=0}. \tag{B15}$$

In the limit $N \rightarrow \infty$, Ω can be replaced by

$$\Omega(m, \hat{m}) \cong \Omega(\underline{x}_0) + \frac{1}{2} \int dt \int dt' [\underline{x}(t') - \underline{x}_0(t')]^T A(t, t') [\underline{x}(t) - \underline{x}_0(t)]. \tag{B16}$$

We have defined

$$\underline{x}^T(t) = (m(t), i\hat{m}(t)), \tag{B17}$$

$$A_{\alpha\beta}(t, t') = \frac{\delta^2 \Omega(\underline{x})}{\delta x_\alpha(t) \delta x_\beta(t')} \Big|_{\underline{x}=\underline{x}_0}, \quad \alpha, \beta = 1, 2 \quad (\text{B18})$$

where \underline{x}_0 is the value of \underline{x} at the saddle point

$$\frac{\delta \Omega(\underline{x})}{\delta \underline{x}} \Big|_{\underline{x}=\underline{x}_0} = \underline{0}. \quad (\text{B19})$$

A straightforward calculation yields

$$\underline{x}_0 = \underline{0}, \quad (\text{B20})$$

above the FM phase, and

$$\underline{A}(\omega) = \begin{pmatrix} 0 & \frac{1}{J_0} - G(-\omega) \\ \frac{1}{J_0} - G(\omega) & -C(\omega) \end{pmatrix}, \quad (\text{B21})$$

for the Fourier components of A . $C(\omega)$ and $G(\omega)$ are the Fourier components of the average *local* correlation and response functions. The correlations $\langle mm \rangle$ and $\langle i\hat{m}m \rangle$ are then obtained by means of the generating functional

$$\begin{aligned} Z[\underline{\psi}] &= \frac{N}{J_0} e^{N\Omega(\underline{0})} \int D\underline{x} \exp \left[-\frac{N}{2} \int \frac{d\omega}{2\pi} [\underline{x}^T(-\omega) \underline{A}(\omega) \underline{x}(\omega) + 2\underline{\psi}^T(-\omega) \underline{x}(\omega)] \right] \\ &= \exp \left[\frac{N}{2} \int \frac{d\omega}{2\pi} \underline{\psi}^T(-\omega) \underline{A}^{-1}(\omega) \underline{\psi}(\omega) \right], \end{aligned} \quad (\text{B22})$$

since $Z[0]=1$. By definition we have

$$\langle\langle x_\alpha(\omega) x_\beta(\omega') \rangle\rangle = \left[\frac{2\pi}{N} \right]^2 \frac{\delta^2 Z[\underline{\psi}]}{\delta \psi_\alpha(-\omega) \delta \psi_\beta(-\omega')} \Big|_{\underline{\psi}=\underline{0}}; \quad (\text{B23})$$

therefore, from Eqs. (B11), (B12), and (B23) we have

$$C_{\text{FM}}(\omega) = \frac{C(\omega)}{|1 - J_0 G(\omega)|^2}, \quad (\text{B24})$$

$$G_{\text{FM}}(\omega) = \frac{G(\omega)}{1 - J_0 G(\omega)}, \quad (\text{B25})$$

where $C_{\text{FM}}(\omega)$ and $G_{\text{FM}}(\omega)$ are the Fourier components of the average FM correlation and response functions. If we are in the FM phase, Eqs. (B24) and (B25) are still valid, but with the connected correlation functions. If $C(\omega)$ and $G(\omega)$ are related by the FDT, then one has also

$$C_{\text{FM}}(\omega) = \frac{2T}{\omega} \text{Im} G_{\text{FM}}(\omega). \quad (\text{B26})$$

APPENDIX C: AN ASYMMETRIC DILUTED NETWORK

In this appendix we show how asymmetric diluted models can be related to the asymmetric Gaussian model we have studied in this paper. Consider a system of N fully connected spins σ_i interacting via an interaction exchange of the form

$$J_{ij} = J_{ij}^s \epsilon_{ij}, \quad i \neq j \quad (\text{C1})$$

where J_{ij}^s is a random symmetric matrix, e.g., the SK matrix¹² or the Hopfield matrix, Eq. (5.3), and ϵ_{ij} is a random asymmetric matrix which takes the value 0 or 1. For definiteness let us assume that the probability distribution of ϵ_{ij} is given by

$$\begin{aligned} P(\epsilon_{ij}; \epsilon_{ji}) &= (1-c) \delta(\epsilon_{ij}-1) \delta(\epsilon_{ji}-1) \\ &\quad + \frac{c}{2} \delta(\epsilon_{ij}) \delta(\epsilon_{ji}-1) + \frac{c}{2} \delta(\epsilon_{ij}-1) \delta(\epsilon_{ji}). \end{aligned} \quad (\text{C2})$$

The parameter c ($0 \leq c \leq 1$) gives the strength of the dilution. From Eqs. (C1) and (C2) one readily sees that each pair of spins i and j is connected by a symmetric bond ($J_{ij} = J_{ij}^s; J_{ji} = J_{ji}^s$) with a probability $1-c$ and either by the unidirectional bond ($J_{ij} = J_{ij}^s; J_{ji} = 0$) or by the unidirectional bond ($J_{ij} = 0; J_{ji} = J_{ji}^s$) with a probability $c/2$. In other words, the concentration of symmetric bonds is $1-c$ and that of unidirectional bonds is c . If $c=0$ all the bonds are symmetric; if $c=1$ all the bonds are unidirectional with random direction.

We can divide the matrix J_{ij} into the symmetric and the antisymmetric parts:

$$\begin{aligned} J_{ij} &= \frac{1}{2}(J_{ij} + J_{ji}) + \frac{1}{2}(J_{ij} - J_{ji}) \\ &= J_{ij}^s \frac{1}{2}(\epsilon_{ij} + \epsilon_{ji}) + J_{ij}^s \frac{1}{2}(\epsilon_{ij} - \epsilon_{ji}) \\ &= J_{ij}^s \epsilon_{ij}^s + J_{ij}^s \epsilon_{ij}^{as}. \end{aligned} \quad (\text{C3})$$

From Eq. (C2) one readily sees that

$$P(\epsilon_{ij}^s) = (1-c) \delta(\epsilon_{ij}^s - 1) + c \delta(\epsilon_{ij}^s - \frac{1}{2}), \quad (\text{C4})$$

$$P(\epsilon_{ij}^{as}) = (1-c)\delta(\epsilon_{ij}^{as}) + \frac{c}{2}\delta(\epsilon_{ij}^{as} - \frac{1}{2}) + \frac{c}{2}\delta(\epsilon_{ij}^{as} + \frac{1}{2}). \quad (C5)$$

By definition we have [see Eqs. (2.1) and (2.3)]

$$K^2 = \frac{\text{var}[(J_{ij})_{\text{antisym}}]}{\text{var}[(J_{ij})_{\text{sym}}]}. \quad (C6)$$

Substituting Eq. (C3) in Eq. (C6) we have

$$k^2 = \frac{\text{var}(J_{ij}^s) \left[\frac{1}{4} \frac{c}{2} + \frac{1}{4} \frac{c}{2} \right]}{\text{var}(J_{ij}^s) \left[(1-c) + \frac{1}{4}c \right]} = \frac{c}{4-3c}, \quad (C7)$$

so that

$$k = \left[\frac{c}{4-3c} \right]^{1/2}. \quad (C8)$$

Note that, as expected,

$$\begin{aligned} k=0 &\iff c=0, \\ k=1 &\iff c=1. \end{aligned} \quad (C9)$$

The generalization to other probability distributions of ϵ_{ij} is straightforward.

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