Order parameter, symmetry breaking, and phase transitions in the description of multifractal sets

Tomas Bohr and Mogens H. Jensen
The Niels Bohr Institute and Nordisk Institut for Teoretisk Atomfysik (NORDITA), Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
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A new method is proposed to locate and analyze phase transitions in a thermodynamic formalism for the description of fractal sets. By studying order parameters appropriate to the transitions we get both an efficient numerical tool for locating phase transitions and an understanding of the structure of the ordered phase. With this method, we examine fractal sets generated by a class of maps of the interval close to the map $x \to 4x(1-x)$. We show that the existence of phase transitions is a persistent phenomenon and remains when the map is perturbed although the structure of the entropy function changes drastically. For strong perturbations the transition disappears and the entropy function becomes nonsingular. The phase transitions describe transitions in the distribution of the characteristic Lyapunov exponents.

I. INTRODUCTION

It is now widely recognized that fractal objects, i.e., objects with a similar structure at all length scales, play a fundamental role in many branches of physics. Recently much progress has been made in finding an appropriate description, in terms of "multifractals; or "thermodynamical formalism" for these structures.[1-11] With this formalism we hope to be able to get a better understanding of the relation between the invariants characterizing a fractal set (e.g., a strange attractor) and the dynamics that generates it. In the present work we have studied "thermodynamical" quantities describing how characteristic, or Lyapunov, exponents are distributed on a fractal set generated by a one-dimensional map.[7] We have chosen a family of maps whose attractors all have the same "$f(\alpha)$ spectrum"[5] but we shall find that the "entropies" describing the distribution of Lyapunov exponents can be very different. In particular we have focused on whether the thermodynamical quantities have phase transitions.[7,12-15] The existence and types of such transitions might turn out to be a worthwhile characterization of universality classes for such structures.

So far, these transitions have been found by evaluating "free energies" or entropies and locating singular behavior. It is, in general, hard to determine such behavior unambiguously by numerical methods if one does not have further arguments or exact solutions to depend on. Furthermore, singular behavior in the free energy does not tell anything about what kind of change the state of the system undergoes at the transition. The method proposed here remedies both of these deficiencies: It shows the phase transition already for quite small systems and it tells us what happens at the transition. Motivated by analogs to usual phase transitions, we introduce an order parameter which will differ in the two phases. Calculating the order parameter as a function of the temperature clearly shows where the transition will occur.

As mentioned already we shall look at Cantor sets generated by one-dimensional maps of an interval. The dynamics naturally defines a partition of the interval into "cylinders" such that all points within a cylinder move according to the same string of symbolic dynamics. This string can, via the thermodynamic formalism, be mapped onto a system of up and down spins; an Ising model.[9] Each configuration possesses an "energy" and from a partition function all relevant thermodynamic information can be derived. Furthermore, the mapping to an Ising model suggests that the natural choices for the order parameter can be found by looking at local averages of the spin configuration, i.e., the local magnetization.

In the following we shall focus on a specific class of one-dimensional maps. They should be unimodal (one hump) and map an interval I *onto* itself, which is the case of "crisis" or "fully developed chaos." The attractor is then the whole interval and specifically always has Hausdorff dimension 1. If f is made slightly "higher" the maximal value of f (the critical point) will be outside of I and the invariant set will be a repelling Cantor set in I: Almost all points escape from I. As shown in Ref. 7 this precludes any phase transitions in the thermodynamical formalism so we shall stick to the attractive case.

The particular map $f(x)=4x(1-x)$ ("logistic map") has been studied from this point of view.[7,13,14] Because of the conjugacy to the symmetric tent map $f(x)=1-|2x-1|$ all thermodynamical quantities can be found analytically.[7,16] They differ from the trivial results for the tent map because the conjugating function is ill behaved at the ends of the interval, and, in fact, display a first-order phase transition. The question that is asked in the following is whether the phase transition will persist if the map is perturbed maintaining the crisis condition and the second-order maximum. One should bear in mind that the conjugacy implies an enormous de-

generacy in the scaling spectrum; the perturbation will at least partly lift that degeneracy, and possibly that is enough to destroy the phase transition.

II. THERMODYNAMIC FORMALISM AND THE TRANSFER OPERATOR TECHNIQUE

For a given one-hump crisis map we want to describe the scaling properties of the attracting set, which in this case is the entire interval, I. For this we need a natural partition and that is provided by the cylinders (we follow here Ref. 7). For an index n, I is partitioned into 2^n intervals or n-cylinders, these being the segments where $f^{(n)}$ is monotonic or, equivalently, the sets of points having identical symbolic-dynamics sequences of length n with respect to the critical point c. The inverse function $h=f^{-1}$ has two branches, h_{-1} and h_1, as shown in Fig. 1(a). The n-cylinders are all the nth-order preimages of I and can therefore be labeled by binary strings

$$I_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}=h_{\varepsilon_1}\circ h_{\varepsilon_2}\circ\cdots\circ h_{\varepsilon_n}(I)\ , \tag{1}$$

where $\varepsilon_i\in\{-1,1\}$. It is easily seen that (a) $f(I_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n})=I_{\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_n}$, (b) for any $m\le n$: $I_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}\subseteq I_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m}$ and specifically $I_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}=I_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n,0}\cup I_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n,1}$. The length of cylinder $I_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}$ is denoted by $l_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}$.

The partition function $Z(\beta)$ is defined as

$$Z_n(\beta)=\sum_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} l^{\beta}_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}$$

$$=\sum_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} e^{-\beta|\ln l_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}|}\ , \tag{2}$$

where we identify the energies as $E_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}=|\ln l_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}|$.[9,13,17] In the limit $n\to\infty$ the sum behaves as

$$Z_n(\beta)=e^{-nF(\beta)}\ , \tag{3}$$

which defines the free energy, $F(\beta)$.[18] The entropy $S(\lambda)$ is the Legendre transform

$$S(\lambda)=-F(\beta)+\lambda\beta\ , \tag{4}$$

where the relation between λ and β is found from

$$\lambda=\frac{d}{d\beta}F(\beta)\ , \tag{5}$$

and this has the following meaning: In the limit $n\to\infty$, $e^{nS(\lambda)}$ is the number of cylinders with length $l=e^{-n\lambda}$ or, equivalently, with Lyapunov exponent λ. The dimension of the set of points in I having Lyapunov exponent λ is $S(\lambda)/\lambda$. (Some care is required here since there are exceptional points with arbitrary large negative Lyapunov exponents—the ones close to preimages of the critical point, i.e., the edges of the cylinders.)

For *hyperbolic* systems $S(\lambda)$ has a further meaning. Hyperbolicity for 1D maps means simply "everywhere expanding," or equivalently that $1<f'<\infty$ in all points. In this case the natural measure is the Sinai, Bowen, Ruelle measure which is a Gibbs measure that assigns

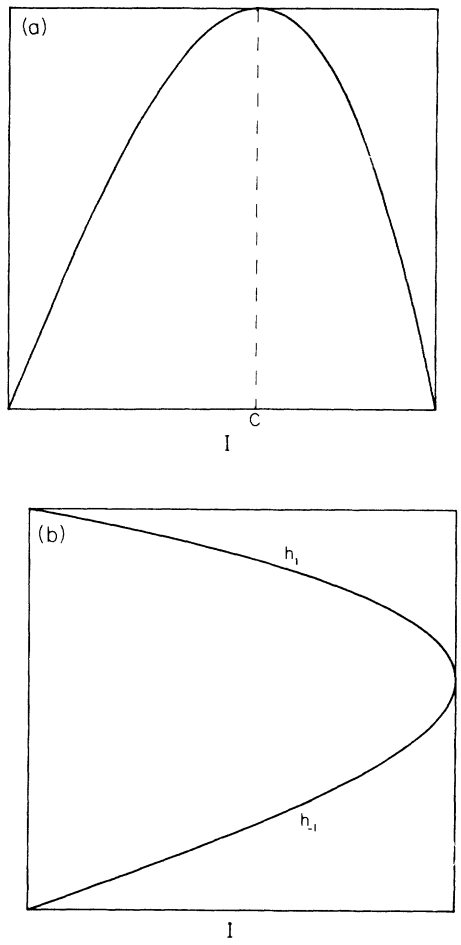

FIG. 1. (a) An asymmetrical one-hump map which maps the interval I onto itself. c denotes the critical point. (b) The inverse of the map. The two branches are denoted h_{-1} and h_1.

each cylinder a weight proportional to its length and therefore $S(\lambda)$ is the metric entropy for a family of measures.[2,7] In particular, the value of $S(\lambda)$ evaluated at the λ for which $\beta(\lambda)=S'(\lambda)=1$, is the Kol'mogorov entropy. For the nonhyperbolic cases studied here that relation breaks down. The case $f(x)=4x(1-x)$ shows this clearly: The natural measure, which is carried over from the tent map, assigns all cylinders the same weight, although those at the edges become very small.

For the hyperbolic case there is a simple relation between the entropy function $S(\lambda)$ and the spectrum of scaling indices introduced in Ref. 5, the $f(\alpha)$ spectrum. In our case again this relation breaks down for the above-mentioned reason and indeed, since the natural measure for the whole class of maps studied here is smooth except for square-root singularities at the edges, the $f(\alpha)$ spectrum should be the same for them all:[5] It should consist of the two points $(\frac{1}{2},0)$ and $(1,1)$. The entropy function $S(\lambda)$ should, as is the $f(\alpha)$ spectrum, be accessible from an experimental time series. Let us say we consider level n in the construction. Then for a point x_i in the time series, we start in x_i and its nearest neigh-

bor, follow the series n times, and obtain a Lyapunov exponent. If we do this for a number of points x_i which are uniformly distributed on the interval we obtain a distribution of Lyapunov exponents and may calculate $S(\lambda)$. If the time series defines a one-hump return map we may also obtain the nth-order symbolic dynamics with respect to the critical point to calculate the order parameter introduced in the following.

In complete analogy with statistical mechanics we now define the average value over the ensemble consisting of all cylinders on a given level, n:

$$\langle A \rangle = \frac{1}{Z_n} \sum_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} A(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n) l^{\beta}_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} . \tag{6}$$

To relate one level in the hierarchy to the preceding level we introduce the Feigenbaum scaling function[19]

$$\sigma_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n+1}} = \frac{l_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n+1}}}{l_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}} . \tag{7}$$

The scaling function can be considered as a function on the unit interval by setting[13,19]

$$\sigma_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} = \sigma\left[\sum_{i=1}^{n} \frac{(\varepsilon_i+1)}{2} 2^{-(n+1-i)}\right] . \tag{8}$$

In general σ will look quite complicated but the important point is that a unique limiting function exists as $n \to \infty$.

As the scaling function takes us from one level to the next it acts as a transfer operator[9] and all the thermodynamic quantities can be computed from eigenvalues and eigenvectors for the matrix defined by[10]

$$\langle \varepsilon_{n+1},\ldots,\varepsilon_3,\varepsilon_2 \mid T(\beta) \mid \varepsilon_n',\ldots,\varepsilon_2',\varepsilon_1 \rangle = \sigma^{\beta}_{\varepsilon_1,\ldots,\varepsilon_2,\varepsilon_{n+1}} \delta_{\varepsilon_2\varepsilon_2} \cdots \delta_{\varepsilon_n\varepsilon_n'} . \tag{9}$$

The largest eigenvalue $\lambda(\beta)$ is related to the free energy as $-F(\beta)=\ln\lambda$ and averages like Eq. (6) are replaced by the average value in the state determined by the corresponding eigenvector $|\Psi\rangle$, i.e.,

$$\langle A \rangle = \langle \Psi \mid A \mid \Psi \rangle . \tag{10}$$

III. INTRODUCTION OF SPIN VARIABLES

As noted by Feigenbaum[9] the thermodynamics of cylinders is closely analogous to 1D Ising-like systems: Strings of -1's and 1's can be interpreted as strings of spins being up or down. Now it is well known that 1D systems with short-range interactions do not exhibit phase transitions at finite temperature. The reason that these system after all have transitions is the existence of infinite-range interactions[13] which is intimately related to the behavior around the critical point, c, where $f'=0$.

To make the analogy even closer consider the asymmetrical "tent" map and its inverse (Fig. 2) with slopes $a, -b$. The energy $E = |\ln l_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}|$ of a cylinder is simply $-n_0 \ln a - n_1 \ln b$, where n_0 is the number of -1's and n_1 is the number of 1's so that $n = n_0 + n_1$. If $a > b$

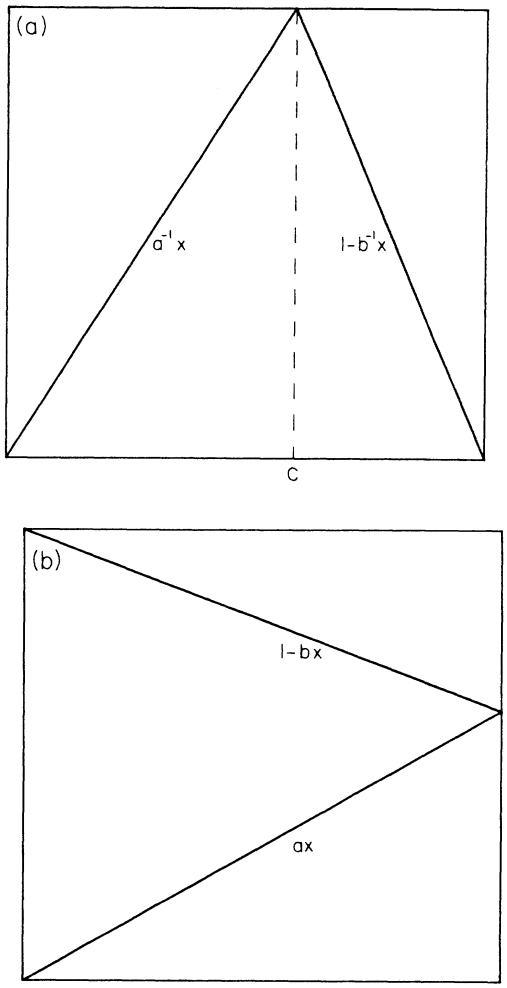

FIG. 2. (a) An asymmetrical tent map. (b) The inverse of the tent map. The two branches have slopes a and $-b$, respectively.

$$E_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} = -n \ln a + n_1 \ln\frac{a}{b} . \tag{11}$$

This corresponds to a nearest-neighbor Ising model with n_1 domain walls (spin flips) of an energy $2J = \ln(a/b)$ over the ground state $-n \ln a$. From this we learn that the symbolic-dynamic sequences $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ correspond not to spins but to *spin flips*, and in order to use Ising-like order parameters, i.e., the magnetization, we must reorder the sequences. If the spin configuration corresponding to the sequence $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ is $s_1,s_2,\ldots,s_n$, then

$$\varepsilon_k = -s_{k-1}s_k , \tag{12}$$

and so $\varepsilon_k = 1$ if s_{k-1} and s_k differ. If the end point is fixed,

$$\varepsilon_1 = s_1 , \tag{13}$$

and if the number of 1's in the string to the left of and including ε_k is denoted $n_1(k)$, then Eqs. (12) and (13) read

$$-s_k=\prod_{i=1}^{k}(-\varepsilon_i)=(-1)^{n_1(k)} . \tag{14}$$

The new ordering is actually of fundamental importance in the "kneading theory" of unimodal maps, since the ordering in the s_i's corresponds to monotonic ordering of the cylinders on the interval.[20] In the s_i variables the scaling function has striking symmetry properties. If all s_i's in a string are inverted all the ε_i's remain unchanged except ε_1, which is inverted: The domain walls remain fixed if the chain is inverted. If we (as we shall do in the following) use spin variables in (8) (i.e., substitute s_i for ε_i) we see that this implies the symmetry $\sigma(1-x)=\sigma(x)$ in the limit $n\to\infty$. Furthermore, property (b) of Sec. II means that $\sigma(x+\frac{1}{2})=1-\sigma(x)$ which is an exact symmetry valid even for finite n.

One should note a peculiarity in the mapping of our "partition function" (2) to statistical mechanics. In (2) β can take any value between $-\infty$ and ∞: Each value of β singles out an important length scale. In thermodynamics $\beta=1/T>0$ so the analog is more precisely that (2) is equivalent to two statistical mechanics problems: One with positive β and one with negative β which can be reformulated as positive temperature but inverted energies. For negative β the energies of the tent map (11) should be changed to

$$E_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n}=-n\ln b+n_0\ln(a/b) . \tag{15}$$

which shows that the ground state of the chain has all $\varepsilon_j=1$. In spin language this means that the ground state has become *antiferromagnetic:* $s_j=(-1)^j$. Thus as β changes from ∞ to $-\infty$ we start from $T=0$ in a ferromagnetic state and go to a disordered state at $\beta=0$, which means either $T=\infty$ or $T=-\infty$. From there we cool the inverted system to $\beta=-\infty$, i.e., the antiferromagnetic state at $T=0$. (For $b>a$ the sequence of states would of course be opposite.) In the following we shall call $T=1/\beta$ the temperature allowing it to be both positive and negative.

IV. THE ORDER PARAMETER

To calculate order parameters we shall apply the formulas of Sec. II [(6)–(10)] using always the reordered chains (spin chains) $s_1,s_2,\ldots,s_n$ instead of the original $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$. The most natural order parameter for a spin system is the magnetization which can be calculated either for the lattice as a whole or at a specific site, i, in the lattice. We shall, in general, calculate the latter quantity, $\langle s_i\rangle$. The particular class of dynamical systems we consider is defined by the maps

$$f(x)=Ax(1-x)(1+\gamma x) . \tag{16}$$

For a specific value of γ, a corresponding value of A is found such that the function maps the unit interval onto itself. For $\gamma=0$ this value is $A=4$ and f is the logistic map at its "fully" chaotic point, which as already mentioned has a first-order phase transition at $\beta_c=-1$. For γ values in an interval around zero we look at what happens when the map is made slightly asymmetrical, but still maps the interval onto itself. Is the phase transition robust against this perturbation? How will the entropy curve change? What is the ordered state? These are the questions we shall focus on.

To obtain good convergence of the order parameter $\langle s_i\rangle$ we use the transfer-matrix technique. Since the cylinders are found by inverse iterates the construction follows precisely the construction of Julia sets by inverse iterates[21] and the transfer matrix Eq. (9) will, at the fourth stage of the hierarchy, be on the form

$$\begin{array}{c|cccccccc}
 & -1-1-1 & -1-1+1 & -1+1-1 & -1+1+1 & +1-1-1 & +1-1+1 & +1+1-1 & +1+1+1 \\
-1-1-1 & \sigma^{\beta}_{-1-1-1-1} & \sigma^{\beta}_{-1-1-1+1} & 0 & 0 & 0 & 0 & 0 & 0 \\
-1-1+1 & 0 & 0 & \sigma^{\beta}_{-1-1+1-1} & \sigma^{\beta}_{-1-1+1+1} & 0 & 0 & 0 & 0 \\
-1+1-1 & 0 & 0 & 0 & 0 & \sigma^{\beta}_{-1+1-1-1} & \sigma^{\beta}_{-1+1-1+1} & 0 & 0 \\
-1+1+1 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^{\beta}_{-1+1+1-1} & \sigma^{\beta}_{-1+1+1+1} \\
+1-1-1 & \sigma^{\beta}_{+1-1-1-1} & \sigma^{\beta}_{+1-1-1+1} & 0 & 0 & 0 & 0 & 0 & 0 \\
+1-1+1 & 0 & 0 & \sigma^{\beta}_{+1-1+1-1} & \sigma^{\beta}_{+1-1+1+1} & 0 & 0 & 0 & 0 \\
+1+1-1 & 0 & 0 & 0 & 0 & \sigma^{\beta}_{+1+1-1-1} & \sigma^{\beta}_{+1+1-1+1} & 0 & 0 \\
+1+1+1 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^{\beta}_{+1+1+1-1} & \sigma^{\beta}_{+1+1+1+1}
\end{array} . \tag{17}$$

Higher-order matrices will be on a "similar" form. At the nth stage of the construction there are a total of 2^n cylinders so the maximal size of the transfer matrix is $2^{n-1}\times 2^{n-1}$. To obtain the highest precision we shall, in general, apply the largest possible matrix but one can also use lower-order approximations to the transfer matrix which work very well for Julia sets.[21] Since the matrix (17) is very sparse and has a simple construction one simply keeps all nonzero elements in a string and multiply as a vector. It is then quite easy to obtain the largest eigenvalue and the corresponding eigenvector; start with a random vector and apply the matrix, renormalize the vector to a unit vector, apply the matrix again, renormalize, etc. This is continued until convergence; the re-

normalizing factor is then the largest eigenvalue λ_1 and the converged vector is the eigenvector $|\Psi\rangle$ of Eq. (10). Let this eigenvector be on the form

$$|\Psi\rangle = |a_1, a_2, \ldots, a_n\rangle \ , \tag{18}$$

and normalized. As indicated on the matrix Eq. (17) for the case $n=4$, the first element a_1 corresponds to a spin state $-1-1-1$, a_2 corresponds to the spin state $-1-1+1$, etc., and a_8 corresponds to the spin state $+1+1+1$. So in order to calculate $\langle s_2 \rangle$, for instance, we use Eq. (10), weight the value of the second spin in the ith state with a_i^2, and sum over all possible states.

Furthermore, we may calculate the second largest eigenvalue λ_2 of the transfer matrix Eq. (17). This is of interest because the correlation length ξ diverges when the difference between the two leading eigenvalues approaches zero:

$$\xi \sim \frac{1}{\lambda_1 - \lambda_2} \ . \tag{19}$$

Again, we can obtain the second largest eigenvalue just from iteration of a random vector by operating with the transfer matrix. Straight iteration, of course, results in the largest eigenvalue and its eigenvector, as described above. If we, on the other hand, in each step subtract the contribution from the dominant eigendirection, the contribution from the next-to-dominant eigendirection is obtained. Again this procedure converges very rapidly. At a first-order transition ξ does not necessarily diverge. Here we observe, however, that the two eigenvalues merge continuously and remain equal below the transition. This procedure defines a way to locate the transition temperature.

V. THE SYMMETRIC CASE

Let us now return to the logistic map Eq. (16) with $\gamma=0$ as shown in Fig. 3(a). Figure 3(b) shows the corresponding scaling function Eq. (8).[13] As proved in Sec. III for any unimodal map, the scaling function is symmetric around 0.5 so, for instance, the energies of the two ferromagnetic states $-1-1-1-1-1\ldots$ and $+1+1+1+1+1\ldots$ are equal in the limit. Therefore a straightforward calculation of $\langle s_i \rangle$ will always give zero. This is in complete analogy with the behavior of the Ising model: The up/down symmetry implies that the *ensemble* average of any spin is zero. The spontaneous symmetry breaking is related to a break in ergodicity: Temporal averages (which we measure in the laboratory) do no longer equal ensemble averages. To see the transition in an ensemble average we need a weak anisotropy to trigger the transition. One might consider applying a weak magnetic field which favors one of the ferromagnetic states. However, even the tiniest field will destroy the transition. Instead we can apply a "surface" field, or even simpler keep one spin in the lattice fixed, up or down depending on which ferromagnetic state should be favored. Then the partition sum is formed only over half of the states—the ones with this spin fixed. To avoid surface effects, which are indeed present and observable, we always calculate the magnetization of

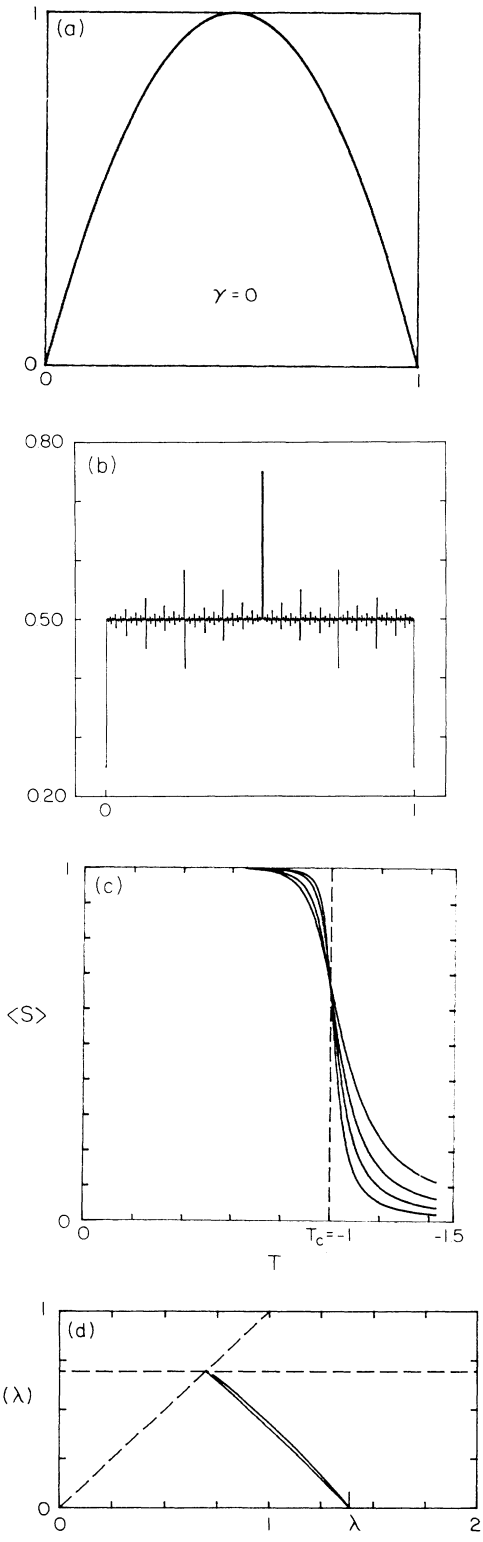

FIG. 3. (a) The map $x \to 4x(1-x)$. (b) The corresponding scaling function (Ref. 13). (c) The order parameter giving a first-order transition at $T_c=-1$ between the disordered and the ferromagnetic state. Shown are curves corresponding to $n=9$–12. (d) The entropy function for $n=9,12$. Here and in the following the two dotted lines indicate $S(\lambda)=\lambda$ and $S(\lambda)=\ln 2$. The mark on the λ axis is at $\ln 4$.

the spin in the center of the system, $\langle s_{i_c} \rangle$, where i_c is int$\{(n-1)/2\}$. Figure 3(c) shows this order parameter at negative temperatures calculated for different sizes of the system, corresponding to $n=9-12$. The curves converge towards the dotted line and result in a first-order transition at $T_c=-1$ as we already know.

How can the transition be determined accurately? There are basically two approaches to follow. The first is a well-known technique used to locate T_c in spin glasses. For two consecutive sizes of the system we follow the crossings of the order parameter curves.[22] These crossings converge geometrically towards T_c and result in this case in $T_c=-1.00$. The second method is based on finite-size scaling procedures. For first-order transitions a typical finite-size scaling ansatz is on the form[23]

$$\frac{\Delta T_c}{T_c} \sim \frac{1}{L^{\zeta}} . \tag{20}$$

Here L is the size of the spin system. The relation $\zeta=d$ is often found in usual transitions, where d is the dimension of the system.[23] ΔT_c is a measure for the smearing of the transition and we can obtain this smearing at a specific value of the order parameter, $\langle s_i \rangle=0.4$, say. We obtain a very good fit to the finite-size scaling law Eq. (18) with an exponent $\zeta=4.6$, which however deviates from the dimension $d=1$. In summary, both methods work very well and give compatible results.

Figure 3(d) shows the entropy function $S(\lambda)$. This curve can be calculated analytically[7] and is a straight line between the points (ln2, ln2) and (ln4,0). We see that the transfer-matrix calculation for $n=12$ has converged very well. In the disordered state cylinders with Lyapunov exponent ln2 dominate and the dimension of the set of cylinders with this exponent is $S(\lambda)/\lambda=\ln 2/\ln 2=1$ which means that basically all points have the Lyapunov exponent ln2. At the transition the system jumps to cylinders with Lyapunov exponent ln4. The corresponding dimension is 0: Only the two cylinders containing the points $x=0$ and $x=1$ will contribute and these characterize the two ferromagnetic states. Since the slope of the $S(\lambda)$ curve is equal to β we can also determine the transition temperature from the entropy curve which indeed in this case results in $\beta_c=1/T_c=-1$. A unique feature of this highly degenerate system is that it never orders at positive temperatures. It remains completely disordered all the way down to $T=0$.

VI. THE ASYMMETRIC CASE

To start our investigation of asymmetric maps, let us first discuss the general appearance of the entropy function. First of all it should be positive on some interval $[\lambda_{\min},\lambda_{\max}]$. The value $\lambda=\ln 2$ must belong to that interval, which follows from the fact that the sum of the lengths of all cylinders on a given level is 1. Secondly it is often found that the values of $\lambda_{\min}$ and $\lambda_{\max}$ are given by the logarithms, a_1 and a_0, of the slopes at the unstable fixed points on the right-hand side of I and at the origin.[7] Note that for the logistic map $a_1=\lambda_{\min}=\ln 2$ so when we study maps with nonzero γ in Eq. (16) we might guess that $\gamma<0$ and $\gamma>0$ provide two fundamentally different cases. Increasing γ above 0 increases a_1 whereas decreasing γ decreases a_1. Thus for the case $\gamma>0$ we do not even have a good understanding of the value of $\lambda_{\min}$.

We start with the simplest case $\gamma<0$. Here $a_0>2\ln 2>\ln 2>a_1$. We have specifically studied $\gamma=-0.5$ where $A=5.1961$, $a_0=\ln 5.1961=1.6479$ and $a_1=\ln 1.9124=0.6483$. The entropy curve, Fig. 4(e), does seem to cover the interval $[a_1,a_0]$ and the main difference from the case $\gamma=0$ is that the degeneracy has been broken so the entropy comes back down at a_1 on the left-hand side.

When we look at the magnetization ($\langle s_i \rangle$), always keeping one spin at the edge fixed, we find that for $T\to 0^-$ the system settles in a ferromagnetic state whereas for $T\to 0^+$ an antiferromagnetic state appears. The cylinder containing the unstable fixed point within I has the symbolic dynamics $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n=1,1,\ldots,1$, which in the spin language is $s_1,s_2,\ldots,s_n=1,-1,1,-1,1,\ldots$, i.e., the antiferromagnetic state.

The scaling function is shown in Fig. 4(b) and the order parameter $\langle s_{i_c} \rangle$ in Fig. 4(c). We see a clear indication of a phase transition on the ferromagnetic ($T<0$) side. The transition temperature is found quite accurately to $T_c=-1.376$ by looking at the crossings of the curves, and the finite-size scaling ansatz Eq. (19) with $\langle s_{i_c} \rangle=0.4$ gives the same result with $\zeta=3.9$. In the positive temperature domain no phase transition seems to take place. Figure 4(d) shows $\langle s_{i_c} \rangle$ which for large n seems to converge to a smooth curve but the ordered phase is in this case antiferromagnetic. The absence of a phase transition is further supported by the fact that the two largest eigenvalues never cross.

The precise form of the entropy function is, as mentioned in the introduction, not easy to assess with great accuracy. The existence of a first-order phase transition implies that there should be a straight line segment in $S(\lambda)$. Combining the value of T_c ($=1/\beta_c$) with the end point a_0 of the function $S(\lambda)$ we find that this straight segment (of slope β_c) should go all the way from $(a_0,0)$ to (ln2,ln2). In the latter point we believe that the slope is discontinuous and on the left-hand side the entropy goes down to $(a_1,0)$. This scenario is seen in Fig. 4(e), but to check it in detail we have enlarged the region around (ln2,ln2) as shown in Fig. 4(f). The dotted line is the asymptote of the entropy function which we get from the value of the transition temperature, $T_c=-1.376$. The curve in the figure correspond to the sizes $n=9,12$. Clearly, we see that the maximum moves towards (ln2,ln2). Of course, with finite-size data, it is impossible to rule out other types of behavior in an interval of a few thousandths around ln2.

The case $\gamma>0$ is more complicated. Now, $2\ln 2>a_0>a_1>\ln 2$. We have specifically studied $\gamma=0.5$ where $A=3.1689$, $a_0=1.1533$, $a_1=0.7486$. The entropy function, Fig. 5(d), shows that the support is the interval $[\lambda_{\min},a_0]$ where $\lambda_{\min}<\ln 2$. The scaling function and the order parameter for negative temperatures is shown in Figs. 5(b) and 5(c). We see again a clear indication of a phase transition at $T_c=-0.649$—the mag-

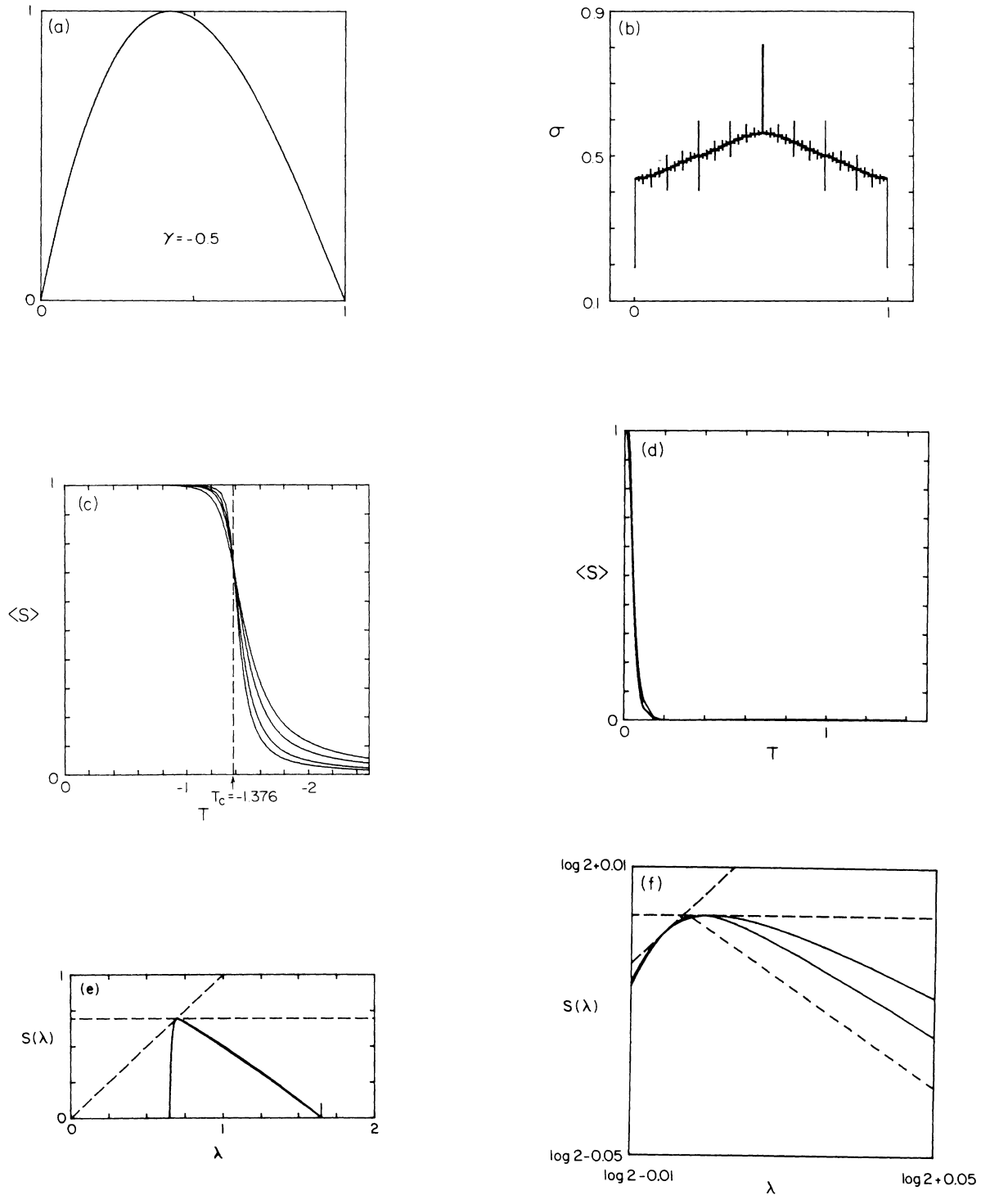

FIG. 4. (a) The function Eq. (16) with $\gamma=-0.5$. (b) The corresponding scaling function. Note that the symmetry around the center is still there. (c) The order parameter giving a transition at $T_c=-1.376$. (d) The order parameter calculated at positive temperatures. There is no sign of a transition. (e) The entropy function $S(\lambda)$ for $n=9,12$. The mark on the λ axis is at $\ln 5.1961=1.6479$. Also, there is a mark at $\ln 1.9124=0.6483$ which is hidden behind the entropy function. (f) An enlargement around the point (ln2,ln2). The lowest lying dotted line is the straight line part corresponding to $\beta=\beta_c$. The line appears to end in the point (ln2,ln2).

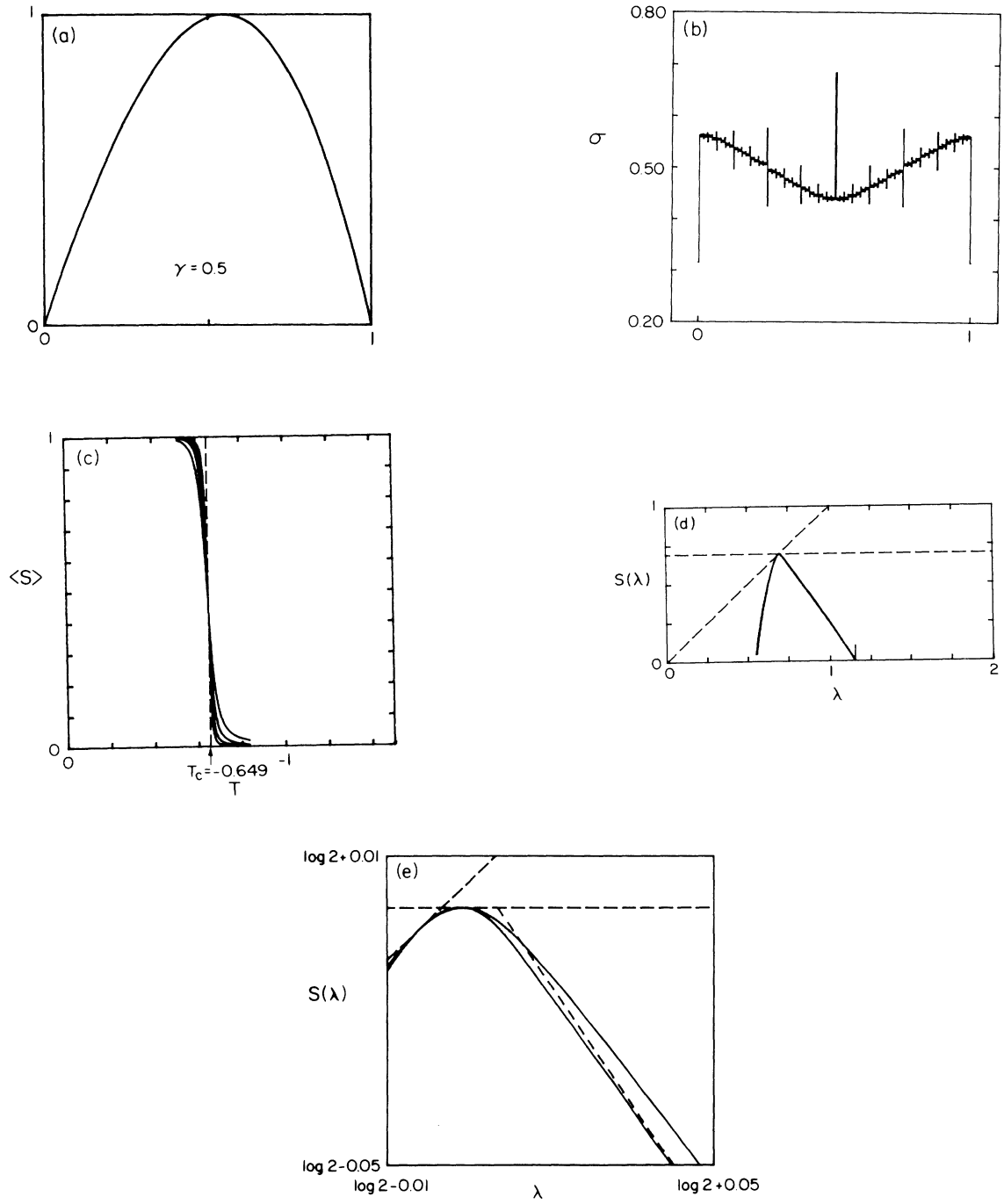

FIG. 5. (a) The function Eq. (16) with $\gamma=0.5$. (b) The corresponding scaling function. (c) The order parameter giving a first-order transition at $T_c=-0.649$. The transition has become quite sharp. (d) The corresponding entropy function. The mark on the axis is at ln3.1689=1.1533. (e) An enlargement around the point (ln2,ln2). The dotted line of the transition does not go to the point (ln2,ln2) and the entropy function beds off before reaching that point.

netization is even sharper than at $\gamma=0$. Again the ordered state here is ferromagnetic, but, as one might have suspected, the order for $T\rightarrow 0^+$ is no longer antiferromagnetic. In fact, it is numerically even harder to make the system order at positive temperatures. It finally seems to order very close to $T=0$ but there is no phase transition and the order is again *ferromagnetic*. Actually the cylinders which are largest and therefore determine the order at $\beta\rightarrow+\infty$ are the ones closest to the critical point. They have the symbolic dynamics $\varepsilon_1,\varepsilon_2,\ldots=\varepsilon_1, 1, -1, -1, -1, -1, \ldots$ where $\varepsilon_1=-1$ or 1 corresponding to the spin chains $s_1,s_2,\ldots=1,-1,-1,-1,-1,\ldots$ or $-1, 1, 1, 1, 1, \ldots$. They differ from the pure ferromagnet only in the first digit

and thus clearly show the existence of long-range interactions induced by the critical point. Since the critical point is quadratic these cylinders scale as the square root of the ones at the origin and thus we expect that $\lambda_{\min}=\frac{1}{2}a_0$, which agrees within 1% of our data.

The structure of the entropy function, Fig. 5(d), seems also to be more complicated in this case. The maximum of $S(\lambda)$ does not appear to fall at $\lambda=\ln 2$ and consequently the straight segment will most likely not go all the way up to ln2. We see an indication of this in the enlargement around the maximum, Fig. 5(e). Here the dotted line corresponds to the asymptote and the entropy curve appears to bend off before it reaches the value ln2. To stress this point even further we have studied the case $\gamma=1$ where $A=2.5980$, $a_0=0.9547$, and $a_1=0.7856$. In this case the transition is extremely sharp as indicated on the order parameter Fig. 6(c). The transition takes place at $T_c=-0.331$ and only one level

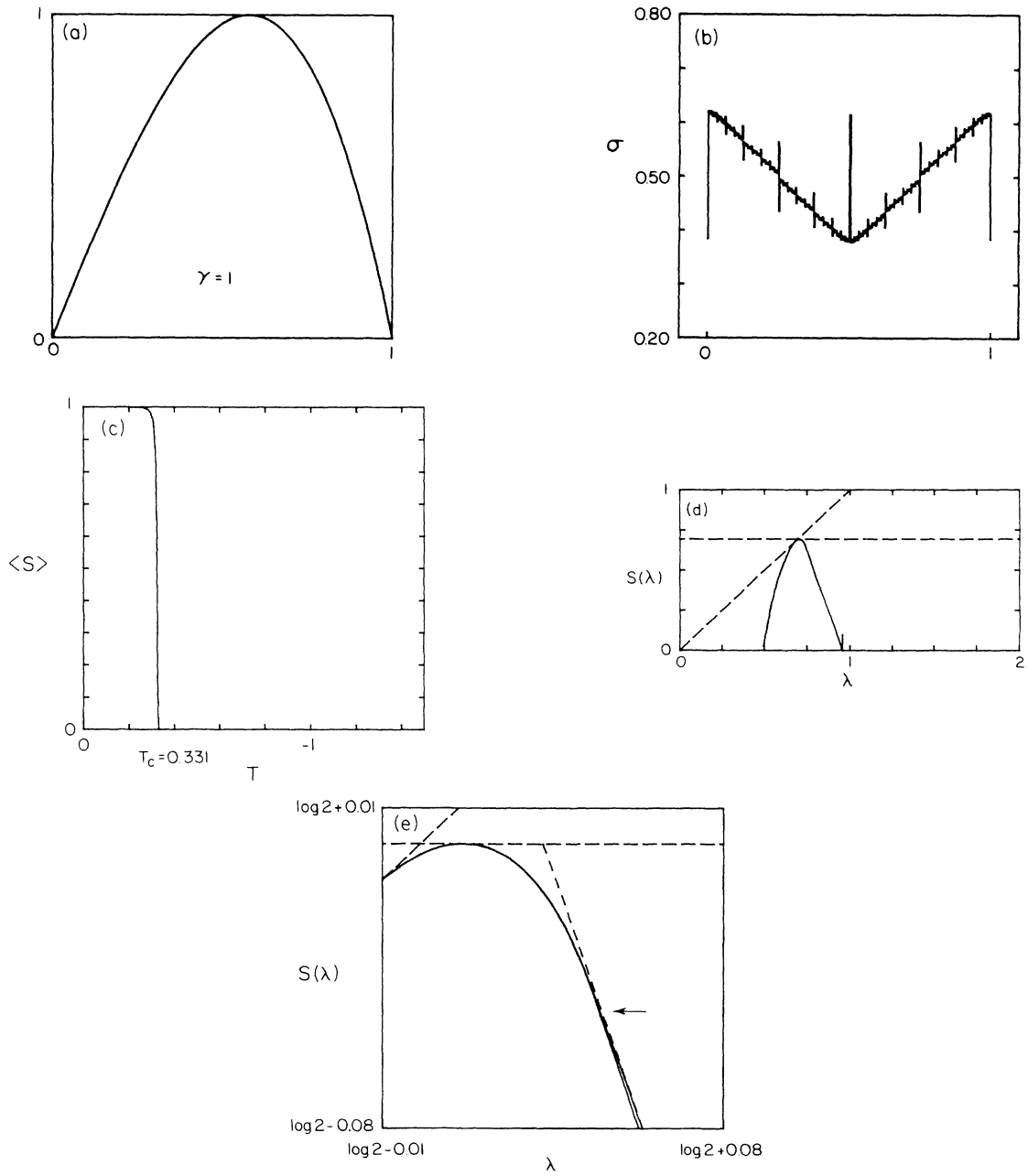

FIG. 6. (a) The function Eq. (16) with $\gamma=1$. (b) The corresponding scaling function. Note that the minimum value around the center is nearly degenerate with the values at the borders. (c) The order parameter giving a transition at $T_c=-0.331$. Only one level is shown because the transition is very sharp. (d) The corresponding entropy function. The mark on the λ axis is at $\ln(2.5980)=0.955$. (e) The enlargement around (ln2,ln2). The curve follows the straight line up to the arrow and then bends off and is nonsingular otherwise.

is drawn here because of the sharpness of the transition. Figure 6(d) is the corresponding entropy curve and we focus on the enlargement around the maximum, Fig. 6(e). Here we clearly see that the entropy bends as indicated by the arrow around $\lambda=0.738$. After this the curve is nonsingular and is tangent to the two dotted lines.

For larger γ completely new behavior occurs. Take as an example $\gamma=1.5$ for which the corresponding curves are shown in Fig. 7. Here, $A=2.1927$, $a_0=\ln(2.1927)=0.7851$, and $a_1=\ln(2.2532)=0.8123$, so for the first time the slope at the fixed point away from zero is *larger* than the slope at the origin and we would expect to find quite different behavior than in the previous cases. First of all the state at $T\to 0^-$ is now antiferromagnetic, since the smallest cylinders now come from the fixed point *inside I*. If we look at the order parameter curve [Fig. 7(c)] the situation has changed a lot.

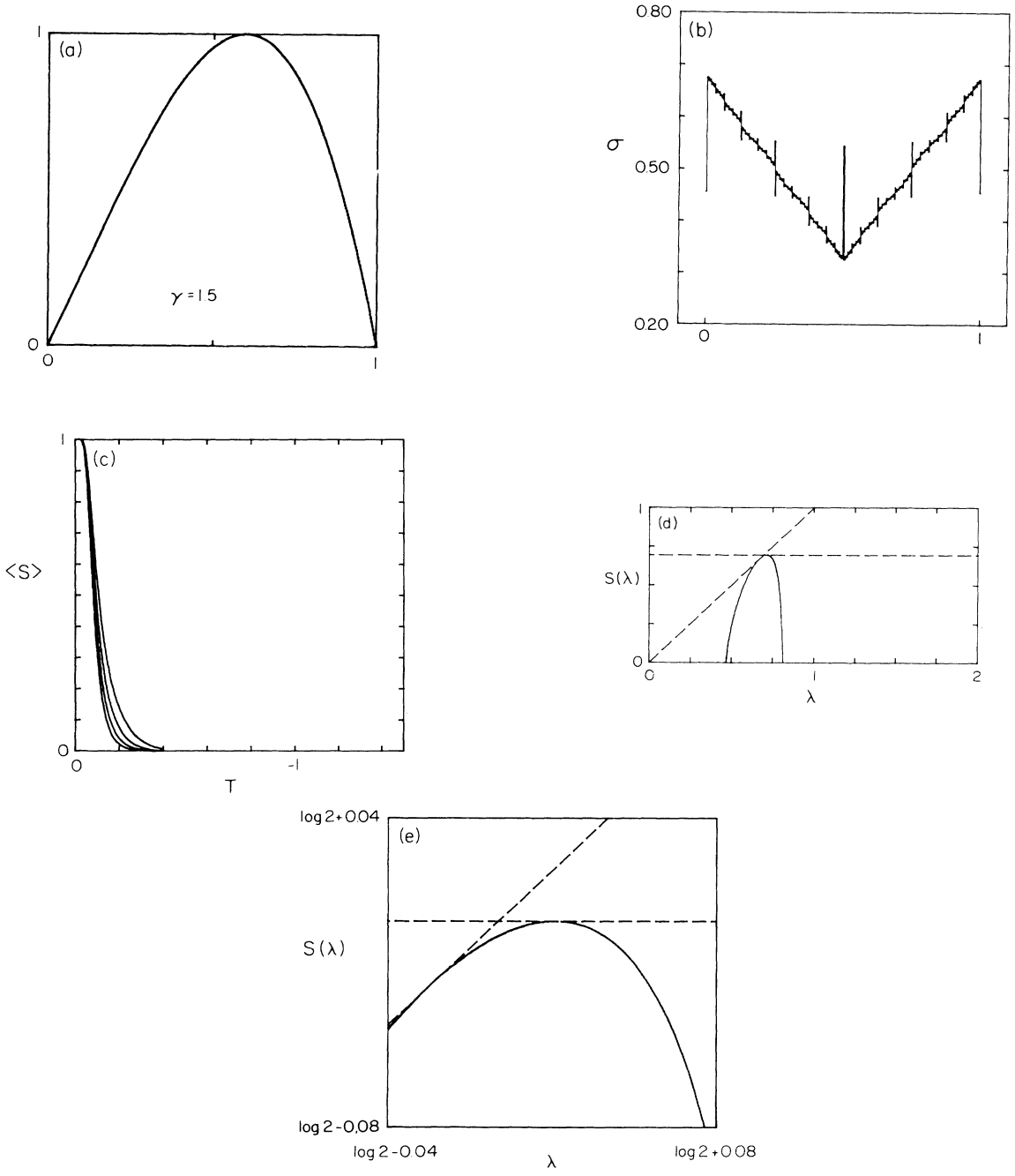

FIG. 7. (a) The function Eq. (16) with $\gamma=1.5$. (b) The corresponding scaling function. (c) The order parameter. The type of the ordering has completely changed as compared to the cases shown in the previous figures and there is no sign of a transition. The ordered state is antiferromagnetic. (d) The corresponding entropy function appears nonsingular which is supported by the enlargement (e).

Even though the systems order there seems to be *no phase transition* at positive nor at negative temperatures. This conclusion is supported by the fact that the correlation length always remains finite. Further evidence is provided by the entropy function Figs. 7(d) and 7(e) which shows no sign of singular behavior. Apparently there exists a critical value of γ where the series of first-order transitions terminates. At this point the transition becomes very sharp and then suddenly disappears.

We can give some intuition for when and why the transition disappears. The existence of a transition has something to do with the lack of uniformity by which the scalings are distributed. In the logistic map almost all cylinders scale with $\lambda = \ln 2$ and the transition comes about when suddenly the very sparsely populated region around ln4 is being felt. The spectral density between these two points is not enough to secure a smooth changeover. For γ around 1 [Fig. 6(b)] the situation is

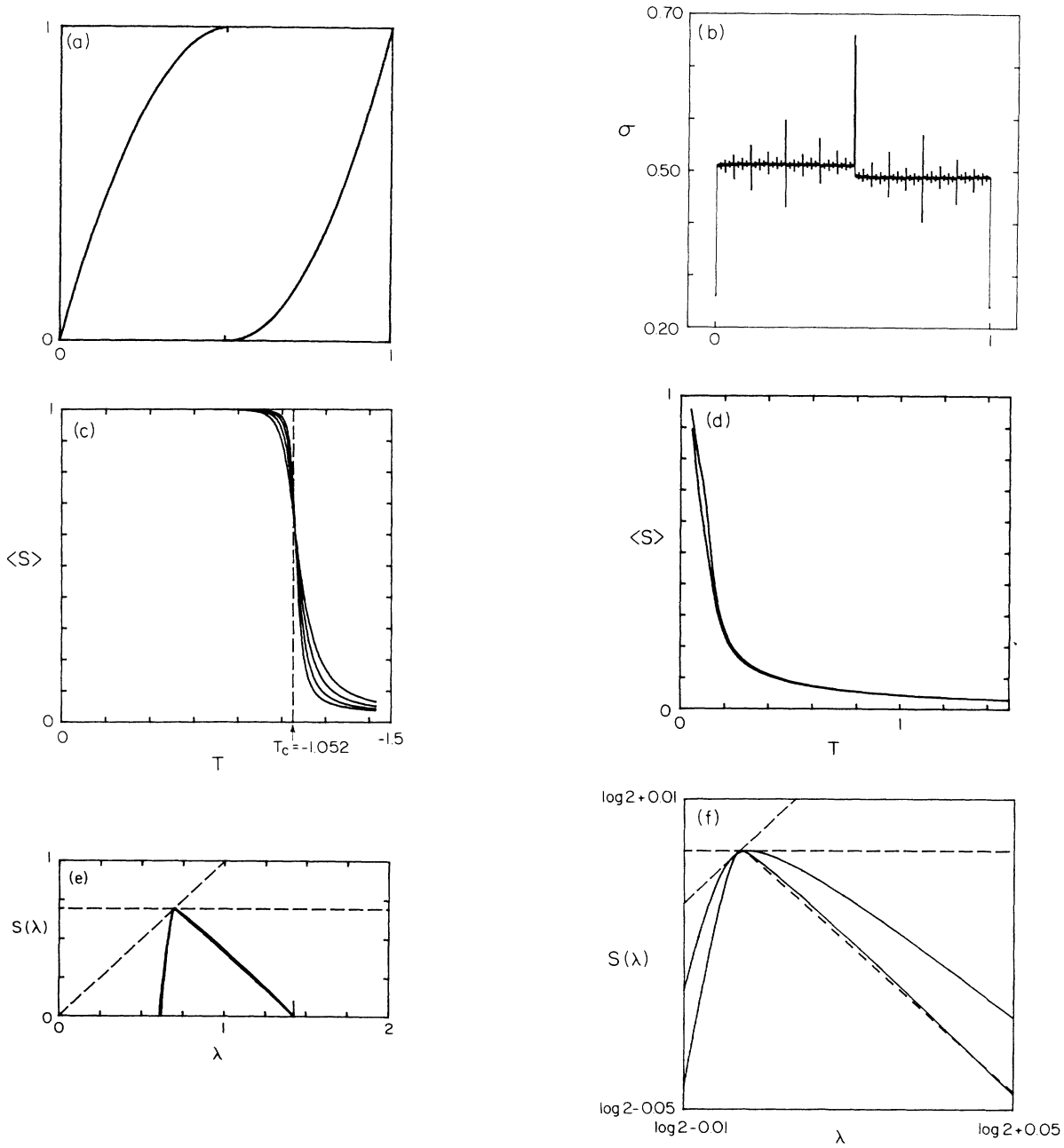

FIG. 8. (a) The function Eq. (21) with $\eta = 0.1$. (b) The scaling function. Now the symmetry around its center is broken. (c) The order parameter showing a first-order transition at $T_c = -1.052$. (d) The order parameter calculated for positive temperatures. There is no sign of a transition and the parameter is numerically hard to saturate at $\langle s \rangle = 1$. (e) The entropy function. The mark at the λ axis is at $\ln 3.835 = 1.344$. (f) The enlargement around (ln2,ln2) shown for $n = 9, 12$. The dotted line representing the transition appears to go to the point (ln2,ln2).

very different. Here the value of the scaling function in the center is comparable with its value at the ends which means that a lot of spectral density is building up at the ends. For $\gamma=1.5$ [Fig. 7(b)] the effect is even stronger. We have not pursued these questions further here but it would clearly be of interest to study the disappearance of the phase transitions in more detail. Supposedly the series of first-order transitions terminate at a critical point although very likely the transition temperature will approach zero.

VII. TRANSITIONS FOR LORENTZ-TYPE RETURN MAPS

In some flows, like the Lorentz system, one may obtain discontinuous return maps similar to the one shown in Fig. 8(a).[24] Consider the map

$$f(x)=\begin{cases} Bx(1-x-\eta x^2), & x\le x_m \\ -Bx(1-x-\eta x^2)+1, & x>x_m \end{cases}, \tag{21}$$

where x_m is the location of the discontinuity. For $\eta=0$ this discontinuos map has precisely the same cylinders as the logistic map $x\to 4x(1-x)$ only they appear naturally "spin ordered." For this type of maps (monotonic) the symbolic dynamics by itself creates monotonically ordered symbol sequences[20] and we do not have to reorder them.

We study the case $\eta=0.1$ and generate again the cylinders from backwards iterates. The scaling function is shown in Fig. 8(b) and shows an interesting new feature: Now the symmetry $x\to 1-x$ is *broken*. To calculate the order parameter we do not have to put in extra anisotropy, it is already there. Figure 8(c) shows the order parameter for various sizes and again we find a very sharp transition at the temperature $T_c=-1.052$ with finite-size scaling exponent $\zeta=4.7$. One might have expected that the asymmetry is related to a magnetic field in the corresponding spin system and that the transition therefore should have been destroyed. It is easy to check if that is the case: We calculate energy differences from the unperturbed system ($\eta=0$) and check if these differences are proportional to the total magnetization of the state, the proportionality factor being a magnetic field. This is however not the case, presumably the effect is that of a surface field.

The entropy curve $S(\lambda)$, Fig. 8(e), seems again to be a straight line at negative temperatures and the Lyapunov exponent jumps from ln2 to ln4.159=1.4125 at the transition. The value 4.159 is the slope in the upper fixed point of the map in Fig. 8(a) (which is larger than the slope at the origin). If we magnify the entropy function around its maximum we surprisingly find that the straight line branch of the transition (negative temperatures) continues directly up to the point (ln2,ln2). This is consistent with the straight line (dotted) calculated from the transition temperature.

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