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Wave equation for a dissipative force quadratic in velocity

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Some recent works on the path-integral formulation of nonconservative forces quadratic in velocity are examined critically. It is argued that the ambiguity resulting from ordering the operators in the Lagrangian and the Hamiltonian for this force is more serious than it is usually believed, and that one can construct a number of Hermitian Hamiltonians all satisfying Ehrenfest's theorem, but forming a set of noncommuting operators. A similar problem is encountered in the ordering of the Lagrangian for the path-integral approach. This is in addition to a number of well-known difficulties such as violation of the position-velocity uncertainty, and the differences between the invariances of the equation of motion, and the Lagrangian. An alternative way is suggested in which the damping force is replaced by a purely time-dependent term which produces the same classical motion, thus avoiding the above-mentioned difficulties. Using this formulation, wave functions and time-dependent energies are found for a quadratically damped harmonic oscillator, and for a particle moving in a dissipative medium when the friction is proportional to an arbitrary power of velocity.

I. INTRODUCTION

There is a wealth of literature on the subject of canonical formulation of frictional forces,[1–4] and the difficulties associated with their quantization.[5–9] At the classical level, it is well known that (a) the Lagrangian and the Hamiltonian are not unique,[1–4] (b) that the Hamiltonian is, in general, not the energy of the system,[4–6] and (c) the symmetries of the equation of motion are not necessarily shared by either the Lagrangian or Hamiltonian.[10] In the quantum description of dissipative motions, one is faced with the nonuniqueness of the canonical formalism which in this case causes a major difficulty, since equivalent classical Hamiltonians give rise to entirely different quantal systems.[11] It is questionable whether the energy or the Hamiltonian operator should be regarded as the generator of the unfolding of the system in time,[5,12] and whether the Schrödinger equation should reflect the symmetries of the actual motion or those of the Hamiltonian or Lagrangian. In addition to these, there are other objections raised against some of the approaches in quantum friction, for instance, violation of the velocity-position uncertainty,[9] incompatibility of the equations of motion and the canonical commutation relations,[5,6] and the nonlinearity of the wave equation.[9] Nearly all of these questions are studied using either Heisenberg's equations of motion or the Schrödinger equation. If the method of path integration is utilized, then it is not easy to discover the inconsistencies or investigate whether or not the symmetries and invariances of the resulting solutions are compatible with those of the classical motion. In this paper we present a critical study of some of the recent works[13–15] done on the quantization of dissipative forces quadratic in velocity in Sec. II. Section III deals with an alternative formulation of the quantum-mechanical problems which hopefully bypasses these difficulties and is patterned after the solution of the Schrödinger-Langevin equation. In Sec. IV, we first show that for a frictional force proportional to any power of velocity the conventional canonical quantization is not satisfactory, and then proceed to show that by the method proposed in this paper one can find the Schrödinger equation for such a damped motion.

II. PROBLEMS WITH CANONICAL QUANTIZATION

The equation of motion for a one-dimensional motion of a particle in a potential field $V(x)$ and subject to a quadratic dissipative force $(\frac{1}{2})\gamma\dot{x}^2$ is given by

$$m\ddot{x}+(1/2)m\gamma\dot{x}^2+(dV/dx)=0\ . \tag{2.1}$$

The Lagrangian[1,13]

$$L=(\tfrac{1}{2})m\dot{x}^2\exp(\gamma x)-\int^x dy\,\exp(\gamma y)dV(y)/dy \tag{2.2}$$

is one of a two-parameter family of Lagrangians which generate Eq. (2.1).[13] First we note that classical equivalent Lagrangians (Hamiltonians) are not equivalent in quantum mechanics and they describe different dynamical systems.[16] Also we observe that the invariances of the equation of motion (2.1) are different from the invariances of L. This can readily be seen when $V=0$, i.e., for $\ddot{x}+(\frac{1}{2})\gamma\dot{x}^2=0$, and $L=[(\frac{1}{2})m\dot{x}^2\exp(\gamma x)]$. Both of these equations are invariant under time translation, i.e., $t\rightarrow t+t_0$, but whereas the equation of motion remains invariant under the displacement of coordinate $x\rightarrow x+x_0$, L does not. Since the Hamiltonian corresponding to L is[1]

$$H=(1/2m)p^2\exp(-\gamma x)+\int^x dy\,\exp(\gamma y)dV(y)/dy \tag{2.3}$$

and this Hamiltonian, with $V(x)=0$, does change when x is replaced by $x+x_0$, the Schrödinger equation and its solution will not have the same symmetry as the classical equation of motion. We note that in classical mechanics the Hamiltonian (2.3) cannot represent the energy of the system, the proof follows immediately from Hamilton's canonical equations, and the fact that for (2.1) the energy is not conserved. For quantum-mechanical systems this point will be considered in some detail. The first step in quantizing the motion defined by (2.3) is to find a Hermitian Hamiltonian, and there are a number of rules of associating a Hermitian operator with $H(x,p)$, i.e., ordering the factors of p and x in (2.3). In some recent works[13–15] the validity of the Ehrenfest theorem for the Hermitian operator constructed from (2.3) is used to justify the correctness of the quantum-mechanical Hamiltonian. Let us examine both the question of ordering the factors and the validity of Ehrenfest's theorem for the Hamiltonian (2.3). Since the second term in (2.3) is a function of x alone, it is sufficient to consider the ordering of the simpler Hamiltonian

$$H=(1/2m)p^2\exp(-\gamma x) . \tag{2.4}$$

Using the Dirac rule of association,[17,18] and noting that $p^3, \exp(-\gamma x)$, $\exp[(-\frac{1}{3})\gamma x]p\exp[(-\frac{1}{3})\gamma x]$, and $p\exp[(-\frac{1}{3})\gamma x]p$ are all Hermitian, we can write

$$H_1=O_1[(p^2/2m)\exp(-\gamma x)]=(6im\gamma\hbar)^{-1}[p^3,\exp(-\gamma x)]=(1/2m)\exp(-\gamma x)[p^2+i\hbar\gamma p-(\tfrac{1}{3})\hbar^2\gamma^2] \tag{2.5}$$

where O denotes a Hermitian ordering of the argument. But H_1 is not the only possible ordering of H, since:

$$\begin{aligned}H_2=O_2[(1/2m)p^2\exp(-\gamma x)]&=i(2\gamma\hbar m)^{-1}[\exp[(-\tfrac{1}{3})\gamma x]p\exp[(-\tfrac{1}{3})\gamma x],p\exp[(-\tfrac{1}{3})\gamma x]p]\\&=(1/2m)\exp(-\gamma x)[p^2+i\hbar\gamma p-(\tfrac{2}{27})\hbar^2\gamma^2] .\end{aligned} \tag{2.6}$$

Thus

$$H_2=H_1+(7/2m)\hbar^2\gamma^2\exp(-\gamma x) \tag{2.7}$$

and

$$[H_2,H_1]=(63/2m^2)(\hbar\gamma)^3\exp(-2\gamma x)(\hbar\gamma-ip) . \tag{2.8}$$

One can construct an infinite number of these noncommuting Hamiltonian operators. If we assume that one of them, say H_1, is a constant of motion, as the classical H [Eq. (2.4)] is, then others do change with time as (2.8) indicates.

Using either H_1 or H_2, we find the Heisenberg equations of motion:

$$\dot p=(1/i\hbar)[p,H_j]=\gamma H_j \tag{2.9}$$

and

$$m\dot x=(m/i\hbar)[x,H_j]=(\tfrac{1}{2})[\exp(-\gamma x)p+p\exp(-\gamma x)],\quad j=1,2, . \tag{2.10}$$

The expectation value of these operators amounts to what can be called Ehrenfest's theorem, and thus $H=aH_1+(1-a)H_2$, with $0\le a\le 1$ is an infinite set of noncommuting Hamiltonians all satisfying Ehrenfest's theorem.

The question of nonuniqueness of ordering the Lagrangian in the path-integral approach has been discussed by a number of authors (see Schulman's book[17] for references to the original papers). A careful analysis of the method of path integration indicates that in the Lagrangian $L=(\frac{1}{2})m\dot x^2\exp(\gamma x)$, the two $\dot x$'s and $\exp(\gamma x)$ cannot be taken at the same time,[19] thus L can be written as the limit of a number of different quantities such as

$$L=\lim_{\epsilon\to 0}(1/\epsilon^2)[(x_{n+1}-x_n)\exp(\gamma x_n)(x_n-x_{n-1})] \tag{2.11}$$

or

$$\begin{aligned}L=\lim_{\epsilon\to 0}&(1/\epsilon^2)\exp[(\tfrac{1}{3})\gamma x_{n+1}](x_{n+1}-x_n)\\&\times\exp[(\tfrac{1}{3})\gamma x_n](x_n-x_{n-1})\exp[(\tfrac{1}{3})\gamma x_{n-1}]\end{aligned} \tag{2.12}$$

and these different Lagrangians do not commute with each other.

III. QUADRATIC DAMPING AS AN EXTERNALLY APPLIED FORCE

As an alternative approach to the methods discussed earlier, let us consider a particle of mass m moving in a potential $V(x)$ and subject to an external field which causes loss of energy. Here we do not try to find a specific model which produces a damping proportional to $\dot x^2$, but we assume as in the case of linear dissipation that there are a number of models which provide us with the mechanism for damping.[20–22] Since the damping is assumed to be external, we write the Hamiltonian as

$$H=(1/2m)p^2+V(x)+(\tfrac{1}{2})m\gamma\dot\xi^2[x-\xi(t)] , \tag{3.1}$$

where x and p are the coordinate and the momentum of the particle, respectively, and $\xi(t)$ is a solution of the equation

$$m\ddot\xi+(\tfrac{1}{2})m\gamma\dot\xi^2+\partial V(\xi)/\partial\xi=0 . \tag{3.2}$$

The Hamiltonian (3.1) describes a conservative system driven by an external damping force.

A comparison of the equation of motion

$$m\ddot x+dV/dx+(\tfrac{1}{2})m\gamma\dot\xi^2=0 \tag{3.3}$$

with (3.2) indicates that $x=\xi(t)$ is the particular solution of (3.3). Therefore if $x(0)=\xi(0)$ and $\dot{x}(0)=\dot{\xi}(0)$ are the initial conditions of (3.3), then $x(t)=\xi(t)$ will be the unique solution of the problem for all times. No ambiguity is encountered when we quantize (3.1) with the resulting Schrödinger equation

$$(-\hbar^2/2m)\partial^2\psi/\partial x^2 + [V(x)+(\gamma/2)m\dot{\xi}^2(x-\xi)]\psi = i\hbar\partial\psi/\partial t \ . \tag{3.4}$$

This equation for $V(x)=0$ can be solved exactly with the result that

$$\psi(x,t)=\exp\{i/\hbar[m\dot{\xi}(x-\xi)+(\tfrac{1}{2})\int^t m\dot{\xi}^2 dt]\} \ . \tag{3.5}$$

Since there is no potential energy the energy of the particle is kinetic and is given by

$$(-\hbar^2/2m)\partial^2\psi/\partial x^2=(\tfrac{1}{2})m\dot{\xi}^2(t)\psi \tag{3.6}$$

and is a decreasing function of time. The classical motion

$$\ddot{\xi}+(\tfrac{1}{2})\gamma\dot{\xi}^2=0 \tag{3.7}$$

is invariant under the displacement of the coordinate $\xi(t)\to\xi(t)+x_0$, $x\to x+x_0$. This latter invariance does not appear in the Lagrangian or the Hamiltonian Eq. (2.4). Therefore any method of quantization based on the canonical formulation[14,15] will not yield a wave function having the symmetries of (3.7). But (3.5) has this invariance of the classical equation of motion. Next let us study the quantum damped motion of a harmonic oscillator which satisfies the classical equation

$$\ddot{\xi}+(\tfrac{1}{2})\gamma\dot{\xi}^2+\Omega^2\xi=0 \ . \tag{3.8}$$

The time-dependent Schrödinger equation for this problem is

$$(-\hbar^2/2m)\partial^2\psi/\partial x^2+[(\tfrac{1}{2})m\gamma\dot{\xi}^2(x-\xi) + (\tfrac{1}{2})m\Omega^2x^2]\psi=i\hbar\partial\psi/\partial t \ . \tag{3.9}$$

We expect the solution to be that of a harmonic oscillator whose energy is dissipated and asymptotically decays to one of its eigenstates. Therefore let us consider the following solution of (3.9) in terms of a wave packet ϕ_n,

$$\psi=[\exp(i/\hbar\{m\dot{\xi}(x-\xi)-\epsilon_n t + \int^t[(\tfrac{1}{2})m\dot{\xi}^2-(\tfrac{1}{2})m\Omega^2\xi^2]dt\})]\{\phi_n[x-\xi(t)]\} \ , \tag{3.10}$$

where

$$\epsilon_n=[n+(\tfrac{1}{2})]\hbar\Omega, \quad n=0,1,2\ldots \tag{3.11}$$

are the harmonic oscillator eigenvalues.

By substituting Eq. (3.10) in (3.9) we find the differential equation satisfied by ϕ_n. This can be conveniently expressed in terms of the "coherent state" coordinate

$$y=x-\xi(t)$$

as the Schrödinger equation

$$(-\hbar^2/2m)d^2\phi_n/dy^2+(\tfrac{1}{2})m\Omega^2y^2\phi_n=\epsilon_n\phi_n \ . \tag{3.12}$$

Thus (3.10) represents a wave packet whose center moves according to the classical equation (3.8). From Eq. (3.10) we can calculate the mean value of some of the dynamical quantities. Thus the mean energy of the system in its nth state is given by

$$E_n=\int_{-\infty}^{+\infty} dx\,\psi_n^*[(-\hbar^2/2m)\partial^2/\partial x^2+(\tfrac{1}{2})m\Omega^2x^2]\psi_n \ , \tag{3.13}$$

and this can be found by substituting for ψ_n from (3.10) and carrying out the integration over x;

$$E_n(t)=\epsilon_n+(\tfrac{1}{2})m(\dot{\xi}^2+\Omega^2\xi^2) \ . \tag{3.14}$$

Next let us consider the expectation values of the position and the momentum of the particle.

$$\langle x\rangle=\int_{-\infty}^{+\infty} dx\,\psi_n^* x\psi_n=\xi(t) \tag{3.15}$$

and

$$\langle p\rangle=\int_{-\infty}^{+\infty} dx\,\psi_n^*(-i\hbar\partial/\partial x)\psi_n=m\dot{\xi}(t) \ . \tag{3.16}$$

The velocity of the particle can also be obtained from the ratio (j/ρ), where j and ρ are the probability current and probability density, respectively,

$$j/\rho=\dot{\xi}(t) \ . \tag{3.17}$$

The quantities j and ρ are determined from (3.10), and are related to ϕ_n by $\dot{\xi}\,|\phi_n|^2$ and $|\phi_n|^2$.

IV. PATH-INTEGRAL FORMULATION

The two wave functions that we obtained in the last section are also derivable from the path-integral formalism. Let us discuss the details for the motion of a damped oscillator. To this end from the Hamiltonian (3.1) we find the Hamilton-Jacobi equation.

$$\partial S/\partial t+(1/2m)(\partial S/\partial x)^2 + V(x)+(\tfrac{1}{2})m\gamma\dot{\xi}^2(t)[x-\xi(t)]=0 \ . \tag{4.1}$$

For the quadratically damped oscillator, $V(x)=(\tfrac{1}{2})m\Omega^2x^2$ and the solution of (4.1) turns out to be

$$S(x,t)=(\tfrac{1}{2})m\Omega[x-\xi(t)]^2\cot(\Omega t)+mx\dot{\xi}+g(t) \ , \tag{4.2}$$

where

$$g(t)=-m\int^t[\xi\ddot{\xi}+(\tfrac{1}{2})\dot{\xi}^2-(\tfrac{1}{2})\Omega^2\xi^2]dt \ . \tag{4.3}$$

The action S for two times t_2 and t_1 and the corresponding positions x_2 and x_1 takes the following form:

$$S(2,1)=(m\Omega)/2\sin(\Omega T)\{[(x_1-\xi_1)^2+(x_2-\xi_2)^2]\cos(\Omega T)-2(x_1-\xi_1)(x_2-\xi_2)\}+m(x_2\dot{\xi}_2-x_1\dot{\xi}_1)+g(t_2)-g(t_1) \tag{4.4}$$

where $T=t_2-t_1$, $\xi_1=\xi(t_1)$, and $\xi_2=\xi(t_2)$. The kernel $K(2,1)$ for a Lagrangian quadratic in x, and $\dot{x}$ is given by[19]

$$K(2,1)=F(t_2,t_1)\exp[(i/\hbar)S(2,1)] , \tag{4.5}$$

where

$$F(t_2,t_1)=\int_0^0 \exp[(i/\hbar)S(2,1)]Dx(t) . \tag{4.6}$$

Now consider the initial wave packet at the time $t_1=0$

$$\psi_n(x_1,0)=(2^n n!)^{(-1/2)}(\alpha^2/\pi)^{(1/4)}H_n[\alpha(x_1-\xi_1)]\exp[-(\tfrac{1}{2})\alpha^2(x_1-\xi_1)^2+(im/\hbar)x_1\dot{\xi}_1] , \tag{4.7}$$

where $\alpha^2=m\Omega/\hbar$, $\xi_1=\xi(0)$, and $\dot{\xi}_1=\dot{\xi}(0)$. The wave function at t is given by

$$\psi_n(x_2,t_2)=\int_{-\infty}^{+\infty} K(x_2,t_2;x_1,0)\psi_n(x_1,0)dx_1 . \tag{4.8}$$

By substituting (4.7) in (4.8) and carrying out the integration over x_1 we get

$$\psi_n(x_2,t_2)=\alpha^{1/2}[(2^n n!)^{1/2}\pi^{1/4}]^{-1}H_n[\alpha(x_2-\xi_2)] \times\exp\{-(\tfrac{1}{2})\alpha^2(x_2-\xi_2)^2-i\Omega[n+(\tfrac{1}{2})]t_2+(i/\hbar)[mx_2\dot{\xi}_2+g(t_2)-g(0)]\} , \tag{4.9}$$

provided that

$$F(t)=1/[2\pi i\sin(\Omega t)]^{(1/2)} . \tag{4.10}$$

The wave function (4.9) is identical to (3.10) apart from a constant phase.

V. MOTION IN A VISCOUS FORCE PROPORTIONAL TO AN ARBITRARY POWER OF VELOCITY

In this section we first consider the nonuniqueness of the Hamiltonian when the damping force is proportional to $\dot{x}^a$, $a\neq 1,2$, and then study the canonical quantization of this system to show some of the difficulties mentioned in Sec. II. For this problem the Hamiltonian is an implicit function of p and x, and is defined by the relation[4]

$$p=\int dH[C(H)-\beta(2-a)x]^{1/(2-a)} , \tag{5.1}$$

where C is an arbitrary function of H. In particular if we choose C to be independent of H we find

$$H=p[C-\beta(2-a)x]^{1/(2-a)}=pf(x) . \tag{5.2}$$

Using (5.2), the Hamilton canonical equations yield

$$\dot{x}=H/p \text{ and } \dot{p}=\beta p\dot{x}^{(a-1)} . \tag{5.3}$$

Remembering that H is a constant of motion, and eliminating p between this pair we find the equation of motion to be

$$\ddot{x}+\beta\dot{x}^a=0 . \tag{5.4}$$

A classical Hamiltonian equivalent to (5.2) can be found by choosing $C(H)$ to be linear in H, i.e.,

$$C(H)=DH . \tag{5.5}$$

This time we find the following Hamiltonian

$$H=-r\beta x/[D(1-r)]+D^{(r-1)}(1/r)^r p^r \tag{5.6}$$

where $r=(2-a)/(1-a)$. Again (5.6) leads to the equation of motion (5.4).

Before considering "wave equations" for these equivalent Hamiltonians we observe that the commutator $[x,\dot{x}]$ is different for the two Hamiltonians, and is not given by the inequality $(\overline{\Delta\dot{x}})^2(\overline{\Delta x})^2\geq\hbar^2/4m^2$, but is state dependent. Following the usual procedure for "quantizing" (5.2), we first symmetrize H and construct a Hermitian Hamiltonian operator for the classical motion (5.4)

$$H=-i\hbar f(x)\partial/\partial x-(i/2)\hbar(df/dx) . \tag{5.7}$$

The "Schrödinger equation" obtained from (5.7) is

$$d\psi/dx+[(\tfrac{1}{2})(df/fdx)-i\epsilon/\hbar f]\psi=0 \tag{5.8}$$

which can be solved to yield $\psi(x)$;

$$\psi(x)=f^{-(1/2)}\exp\left[i\epsilon/\hbar\int dx/f(x)\right] . \tag{5.9}$$

The corresponding "wave function" for (5.6) in momentum space is given by

$$\psi(p)=\exp\{[iD(1-r)/\beta r] \times[\epsilon' p-D^{(r-1)}(1/r)^r p^{(r+1)}/r+1]\} . \tag{5.10}$$

Clearly (5.8) and (5.10) are wave functions for different systems, and ϵ and ϵ' in these equations have no simple relation to the energy of the particle. Noting that the canonical quantization is unsatisfactory, let us consider a formulation of this problem using the method discussed in Sec. III. The Schrödinger equation for this case is similar to (3.4). In fact the wave equation for a general damping force $F(\xi,\dot{\xi},t)$, according to the present formulation is given by

$$\{(-\hbar^2/2m)\partial^2/\partial x^2+V(x) +F(\xi,\dot{\xi},t)[x-\xi(t)]\}\psi=i\hbar\partial\psi/\partial t , \tag{5.11}$$

where ξ is a solution of

$$m\ddot{\xi}+dV(\xi)/d\xi+F(\xi,\dot{\xi},t)=0 . \tag{5.12}$$

If the potential $V(x)$ in (5.11) and (5.12) is zero, then the wave function has the same form as (3.5) except now ξ is a solution of (5.12) with $V=0$.

VI. DISCUSSION

In the last section, we showed that for a dissipative force proportional to $\dot{x}^{a}$, we can construct different classical Hamiltonians and by canonical quantization we obtain different wave functions. However, this procedure is not satisfactory since the energy of the system is not given by the expectation value of the Hamiltonian, and there is also a violation of the velocity-position uncertainty relation. The case of quadratic damping has the additional difficulty due to the ambiguity of ordering of the factors. When the damping force is proportional to $\dot{x}$, it has been shown by a number of arguments that the Schrödinger-Langevin equation is a suitable candidate for the wave equation.[23,24] This wave equation which is nonlinear admits an interesting solution for the damped motion of oscillator which is similar to (3.10).[25] The solution describes the wave function of a forced harmonic oscillator, where the forcing term is damped. By starting from this wave function as a model, we found that we can apply the canonical quantization to Hamiltonian (3.1) or to a similar Hamiltonian for $\dot{x}^{a}$ and $F(x,\dot{x},t)$ forces to obtain the wave equation. The resulting Schrödinger equation is linear and time dependent, and has a number of desirable properties that we mentioned in Sec. III. The similarity between the present formulation for a general damping force, $F(x,\dot{x},t)$, and the Schrödinger-Langevin equation becomes more apparent when we study the classical limit of the wave equation obtained from Van Vleck's determinant.[26] This gives us a probability distribution, but this distribution is not proportional to $|\dot{x}(x)|^{-1}$ of the classical motion. Let us consider the limit of $|\psi|^2$ as $\hbar\to 0$. In this limit, according to Van Vleck, we have

$$|\psi|^2 \underset{\hbar\to 0}{\longrightarrow} |\partial H_c/\partial p|^{-1} = |\dot{x}(x)|^{-1} \tag{6.1}$$

where H_c is the classical Hamiltonian. The simplest example to study is that of a particle moving in a viscous field proportional to $\dot{x}^2$. In this case the wave function is given by (3.5), and thus $|\psi|^2=1$, but as the classical equation (3.7) shows:

$$|\dot{x}(x)|^{-1} = \exp[(\tfrac{1}{2})\gamma x] \; . \tag{6.2}$$

This is to be expected, since here we have a "free" particle driven by a damping force. Exactly the same picture emerges from the solution of the Schrödinger-Langevin equation for a free particle,[23] thus in all these models, when $V=0$, the actual form of damping has no effect on $|\psi|^2$.

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