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Two-body-operator matrix elements in SU(n) for application to complex spectroscopy

R. D. Kent
School of Computer Science, University of Windsor, Windsor, Ontario, Canada N9B 3P4

M. Schlesinger
Department of Physics, University of Windsor, Windsor, Ontario, Canada N9B 3P4
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We present a derivation of closed-form algebraic expressions for two-body operators in the context of the special-unitary-group approach to complex spectroscopy. The method involves the use of a graphical representation of the permutational symmetry-adapted irreducible-representation basis which facilitates the decomposition of the matrix elements into simple factors. The results afford significant reductions of computational complexity in cases pertaining to higher-than-spin-$\frac{1}{2}$ N-particle systems.

I. INTRODUCTION

The purpose of this paper is to present closed-form expressions for the matrix elements of two-body operators in permutation symmetry (S_N)-adapted unitary bases [bases of SU(n)$\downarrow S_N$]. The derivation of the matrix-element formulas is demonstrated with use of properties of a recently developed[1-3] graphical realization of the irreducible representations (irreps). We originally presented this method, a generalization of an SU(2)-based method of Shavitt,[4] as an alternative to approaches using either Gel-fand arrays or Weyl-Young tableaux and applied it to the evaluation of one-body operator matrix elements. In this latter regard the present paper can be viewed as a sequel to Ref. 1.

The domain of application of this theory is in the study of systems of N particles exhibiting unitary symmetry, either U(n) or SU(n) (there is no essential difference). Fundamental to such studies are the construction of the irrep bases and their tensor operators. Our approach is a variant of the unitary-group approach (UGA) and for the purposes of this work we assume that the reader is familiar with Refs. 1–3. Nevertheless, we have attempted to incorporate an outline of the theory in the paper.

In Sec. II we present a review of the basic aspects of the irrep construction, graphical representation, and one-body-operator matrix element factorization technique. We also describe the two-body-operator problem. In Secs. III and IV we focus on the cases of so-called raising-raising and raising-lowering two-body operators. In Sec. V we illustrate the use of the general expressions by treating actual cases.

II. BACKGROUND THEORY

In Refs. 1–3 we presented a method for describing the irrep bases of N particle systems with unitary symmetry SU(n) [or U(n)]. In our approach to the unitary-group decomposition we incorporate explicitly the symmetry group labels, $\{p\}_m$, applied at each SU(m) subgroup level, hence,

$$\mathrm{SU}_{\{p\}_n}(n)\supset \mathrm{SU}_{\{p\}_{n-1}}(n-1)\supset \cdots \supset \mathrm{SU}_{\{p\}_m}(m)\supset \cdots \supset \mathrm{SU}_{\{p\}_1}(1)\ . \tag{2.1}$$

The $N+1$ labels $\{p\}_m \equiv \{p_{ml};\ l=0,1,\ldots,N;\ p_{ml}\geq 0\}$ are the integer partitions of N_m and m simultaneously, satisfying the relations

$$\sum_{l=0}^{N} l p_{ml} = N_m,\quad N_n = N \tag{2.2}$$

and

$$\sum_{l=0}^{N} p_{ml} = m,\quad 0\leq m\leq n \tag{2.3}$$

where N_m is the number of particles in the SU(m) subgroup (with index $m\leq n$).

A complete description of an irrep $\{p\}_n$ of SU(n)$\downarrow S_N$ requires a listing of all irreps $\{p\}_m$ of SU(m)$\downarrow S_{N_m}$ contained in the decomposition (2.1) organized according to a lexical ordering which is defined by

$$\sum_{l=q}^{N} (p_{mlj}-p_{mlk})\geq 0,\quad j<k,\quad q=0,1,\ldots,N\ . \tag{2.4}$$

Here we have introduced an additional label (j,k) in order to distinguish the various irreps at level m. Further, the irrep $\{p\}_{m\pm 1,k}$ is contained in irrep $\{p\}_{mj}$ (expressible using Clebsch-Gordan decomposition) if and only if the two $\{p\}$'s satisfy

$$S_t^{\pm}\{p\}_{m,j}=\{p\}_{m\pm 1,k_{mjt}^{\pm}},\quad t=0,\ldots,2^N-1\ . \tag{2.5}$$

The $k_{mjt}^{\pm}$ are linkage indices. The step operators $S_t^{\pm}$ are defined in terms of difference labels $\{d^{\pm}\}_t$ which, in turn, satisfy

$$\{p\}_{m\pm 1k^{\pm}_{mkt}} = \{p\}_{m,j} + \{d^{\pm}\}_t \ ,$$

$$|d^{\pm}_{tl}| \le 1, \quad l=0,1,\ldots,N \qquad (2.6a)$$

$$0 \le \sum_{l=0}^{q} d^{-}_{tl} \le 1, \quad -1 \le \sum_{l=0}^{q} d^{+}_{tl} \le 0, \quad q=0,1,\ldots,N \qquad (2.6b)$$

$$t = \sum_{l=0}^{N} [\delta(d_{tl},1) - \delta(d_{tl},-1)]2^l - 1 \ . \qquad (2.7)$$

In other words, in order that $\{p\}_{m-1}$ be contained within $\{p\}_m$ the sets of labels must be related by some $\{d\}_t$ satisfying (2.6) and (2.7).

A single basis function (basis tensor) is represented, alternatively, by a set of a set of labels, $\{\{p\}\}$, or, more compactly, by the set of linkage types, $[T]$, defined in terms of a specific product sequence of step operators applied to the SU(n) irrep labels, $(\prod_{m=1}^{n} S^{-}_{t_m})\{p\}_{nl}$ with t_m in $[T]$.

The collection of all labels $\{p\}_{mj}$ ($m=0,1,\ldots,n$, $j=1,2,\ldots,J_m$, the number of irreps at level m), are ordered within each subgroup level m according to (2.4), as well as the accompanying set of linkage indices (k) and weights for each $\{p\}_{mj}$. This constitutes the distinct row table (DRT). This idea was presented originally by Shavitt[4] for SU(2). The same set of labels and linkages can be represented graphically as nodes and links in an hierarchical, two-rooted, multiply linked digraph (see, for instance, Fig. 2 of Ref. 1).

Each node in the graph represents one set of labels $\{p\}_{mj}$ while lines joining nodes represent linkages $\{d^{\pm}\}_t$. Thus the graph and the DRT are isomorphic constructions. In particular, a single basis function is represented by a traversal from the node labeled $\{p\}_n$ (head node) to the node labeled $\{0\}$ (tail node) along the existing links, proceeding directly from level m to $m-1$ in each case. Furthermore, the graph topology can be manipulated so that it conveniently conveys the same ordering principle established for the DRT with respect to (2.4), namely the node for $\{p\}_{mj}$ is to the left of the node $\{p\}_{mk}$ for $j<k$ in a two-dimensional planar projection. It is also possible to construct the graph in such a way that the linkages between nodes at successive levels have definite and unique geometric properties, thereby making them immediately useful as identifiers of specific linkage types.

In Ref. 1 we deduced factorized expressions for the matrix elements of the elementary SU(n) generators, $E_{\mu,\nu}$. For the case $\mu<\nu$ (raising operator),

$$\langle [T'] \,|\, E_{\mu\nu} \,|\, [T] \rangle = \Delta_0^{\mu-1} \Delta_\nu^n A_\mu \left[\prod_{\tau=\mu+1}^{\nu-1} T_\tau \right] B_\nu \ , \qquad (2.8)$$

where the Δ_α^β, A_μ, B_ν and T_τ factors were defined in Ref. 1. The graphical representations of these are shown in Figs. 1 and 2. The definition of each of these factors can be derived in an algebraic manner, as shown originally by Beidenharn [5]*et al.* The use of a graph for deriving the expression (2.8) and its constituent factors

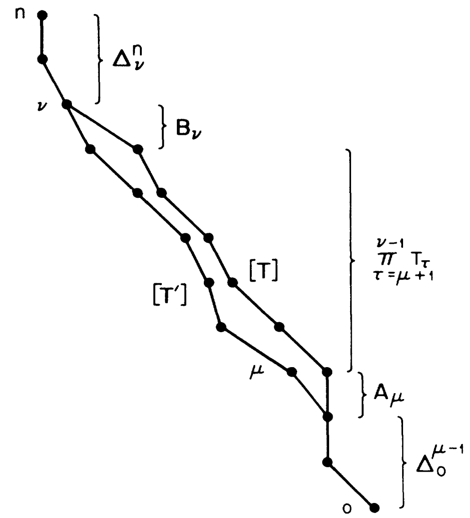


FIG. 1. Graphical representation of an SU(n) generator matrix element. For $\mu<\nu$ all nodes in $[T']$ are identical or to the left of those in $[T]$. Subgraph factors arising in the decomposition of Eq. (2.8) are indicated.

simplifies the process, however.

It will prove useful to describe the properties of the fundamental subgraphs A_μ and B_ν in some detail. The subgraphs B_μ and A_μ [see Figs. 2(a) and 2(b)] are represented by three nodes and the lines joining them. As the subgraphs are essentially inverses of each other we shall deal specifically only with B_μ. Considering states labeled $[T]$ and $[T']$, in order that B_μ be a nonzero matrix element subgraph the nodes $\{p\}_\mu$ and $\{p\}_{\mu=1}$ and $\{p\}_\mu$ and $\{p'\}_{\mu-1}$ must be linked by valid step operations according to (2.5). Further, the nodes $\{p\}_{\mu-1}$ and $\{p'\}_{\mu-1}$ must satisfy the following conditions, namely, for a unique $\lambda_{\mu-1}$, referred to as pivot index,

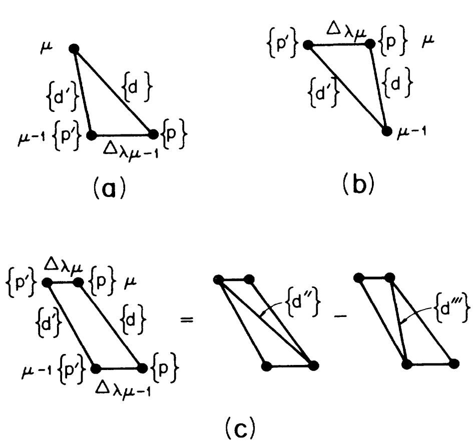


FIG. 2. Subgraph representation of nonzero factors (a) $B_\mu(\lambda_{\mu-1})$, (b) $A_\mu(\lambda_\mu)$, and (c) $T_\mu(\lambda_\mu,\lambda_{\mu-1})$ showing their graphical decompositions. Nodes and links are labeled indicating that for solid lines it is necessary that $\{p\}_{\mu-1}=\{p\}_\mu+\{d^-\}_t$, $\{p'\}_\mu=\{p\}_\mu-\Delta_{\lambda_\mu}$, and so on.

$$p'_{\mu-1,1}=p_{\mu-1,1}-\Delta_{\lambda_{\mu-1},l}\ , \tag{2.9a}$$

$$\Delta_{\lambda_{\mu-1},l}=\delta_{\lambda_{\mu-1}-1,l}-\delta_{\lambda_{\mu-1},l},\quad l=0,1,\ldots,N\ . \tag{2.9b}$$

The verification of (2.9) is achieved referring to Weyl-Young tableaux (see Fig. 5 in Ref. 1). A subgraph which satisfies these nodal relationships, and hence is nonzero, has all links drawn as solid lines, otherwise no connecting line is drawn. The subgraph B_μ will also be written in the form $B_\mu(\lambda_{\mu=1})$ by which we indicate that the labels required to evaluate the subgraph are $\{p\}_\mu$, $\{p\}_{\mu-1}$, and $\{p'\}_{\mu-1}=\{p\}_{\mu-1}-\Delta_{\lambda_{\mu-1}}$. Similarly, for $A_\mu(\lambda_\mu)$ we use labels $\{p\}_\mu$, $\{p'\}_\mu=\{p\}_\mu-\Delta_{\lambda_\mu}$ and $\{p\}_{\mu-1}$.

The subgraph T_τ is comprised of both A_τ and B_τ subgraphs. This is shown in Fig. 2(c). For the T_τ subgraphs valid links must exist between nodes $\{p\}_\tau$ and $\{p\}_{\tau-1}$ and their primed counterparts. Relations (2.9) and (2.10) must apply for both $\mu=\tau$ and $\tau-1$. These subgraphs can be written in expanded form

$$T_\tau(\lambda_\tau,\lambda_{\tau-1})=B_\tau^2(\lambda_{\tau-1})A_\tau^1(\lambda_\tau)-A_\tau^2(\lambda_\tau)B_\tau^1(\lambda_{\tau-1})\ . \tag{2.10}$$

The superscript notation in (2.10) refers to the order of evaluation of the subgraphs and implies an accumulative effect on the $\{p\}$ labels. In the first term in (2.10) $A_\tau^1(\lambda_\tau)$ is evaluated (first) using labels $\{p\}_\tau$, $\{p'\}_\tau=\{p\}_\tau-\Delta_{\lambda_\tau}$ and $\{p\}_{\tau-1}$; $B_\tau^2(\lambda_{\tau-1})$ is then evaluated (second) using labels $\{p'\}_\tau$, $\{p\}_{\tau-1}$, and $\{p'\}_{\tau-1}=\{p\}_{\tau-1}-\Delta_{\lambda_{\tau-1}}$. For the second term in (2.10) the label sets are $\{p\}_\tau$, $\{p'\}_{\tau-1}$ and $\{p'\}_{\tau-1}$ for $B_\tau^1(\lambda_{\tau-1})$ and $\{p\}_\tau$, $\{p'\}_\tau$ and $\{p'\}_{\tau-1}$ for $A_\tau^2(\lambda_\tau)$.

Relations (2.9) serve as zero or non-zero selection rules in the evaluation of matrix elements (2.8). Expressed in terms of the DRT each basis function is a list of $\{p\}_m$ labels, from $m=n$ down to 0. A matrix element is nonzero if and only if all labels at all m levels in both basis functions are identical except for adjacent labels p_{λ_m} and $p_{\lambda_{m-1}}$, $\mu\le m\le\nu-1$, which are related through (2.9).

An example involving the matrix element of the generator $E_{2,7}$ between two such related basis functions of the irrep $\{1\,2\,1\,3\,2\}$ of SU(9) for a system of 15 particles is shown in Fig. 3. At each affected level the adjacent pairs of p labels satisfying (2.10) has been surrounded by a box. For additional clarification the corresponding Weyl-Young tableaux have been included.

A two-body operator ϕ in SU(n) can be expressed in terms of one-body operators (group generators) as a sum of products, hence

$$\phi=\sum_{\mu,\nu}\sum_{\alpha,\beta}\phi(\mu,\alpha;\nu,\beta)E_{\mu,\nu}E_{\alpha,\beta}\ . \tag{2.11}$$

The generators satisfy the Lie bracket relations

$$E_{\mu,\nu}E_{\alpha,\beta}-E_{\alpha,\beta}E_{\mu,\nu}=\delta_{\alpha,\nu}E_{\mu,\beta}-\delta_{\mu,\beta}E_{\alpha,\nu} \tag{2.12}$$

so (2.11) reduces to the form

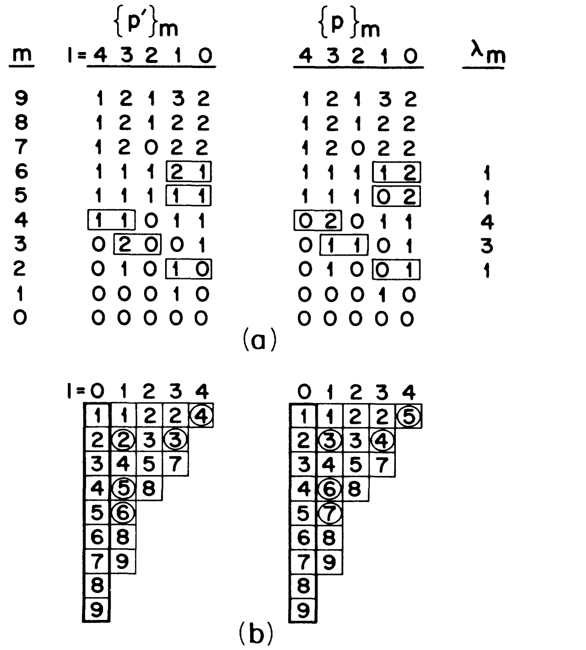


FIG. 3. Distinct-row-table (DRT) and Weyl-Young tableau (WYT) representations of basis states of the $\{1\,2\,1\,3\,2\}$ irrep of SU(9), $N=15$. The states shown produce nonzero matrix elements of $E_{2,7}$. In (a) the DRT labels which undergo change are contained in boxes with the λ_m index given. In (b) the corresponding WYT labels are contained in circles.

$$\phi=\sum_{\mu,\nu}\sum_{\alpha,\beta}\phi^+(\mu,\alpha;\nu,\beta)\{E_{\mu,\nu},E_{\alpha,\beta}\}$$
$$+\sum_{\mu,\nu}\sum_{\alpha}\{\phi^-(\mu,\alpha;\alpha,\nu)-\phi^-(\alpha,\mu;\nu,\alpha)\}E_{\mu,\nu}\ , \tag{2.13}$$

where $\{U,V\}=UV+VU$ is the (symmetric) anticommutator and the superscripts denote the symmetric (+) and antisymmetric (−) parts of ϕ. It is seen that ϕ decomposes into terms proportional to a generator product and a single generator. The latter can be evaluated using (2.8).

For the two-body operator (group-generator product) matrix elements there will be terms, in general, involving the change of labels around two pivot points, λ_m^1 and λ_m^2. Choose $\lambda_m^1<\lambda_m^2$. It follows that

$$\{p'\}_m=\{p\}_m-\Delta_{\lambda^1}-\Delta_{\lambda^2}<\{p\}_m-\Delta_{\lambda^2}<\{p\}_m-\Delta_{\lambda^1}<\{p\}_m\ .$$

As in Refs. 6 and 7 it is possible to consider a number of subcases. We consider two general cases, (a) raising-raising (RR), $\mu<\nu$ and $\alpha<\beta$ and (b) raising-lowering (RL), $\mu<\nu$ and $\alpha>\beta$. A reversal of relations (LL and LR) is obtained by applying Hermitian conjugation to (2.13). We shall not treat cases such as $\mu=\nu$ and $\mu=\alpha$ as these can be easily formulated in terms of number operators which count the number of particles in each SU(m) subgroup. As will be shown below each of these cases can be further broken down into subcases.

The matrix element of a symmetric commutator can be written as

$$\langle [T'] \,|\, (E_{\mu,\nu}E_{\alpha,\beta}) \,|\, [T] \rangle$$
$$= \sum_{[T'']} \langle [T'] \,|\, E_{\mu,\nu} \,|\, [T''] \rangle \langle [T''] \,|\, E_{\alpha,\beta} \,|\, [T] \rangle$$
$$+ \sum_{[T'']} \langle [T'] \,|\, E_{\alpha,\beta} \,|\, [T''] \rangle \langle [T''] \,|\, E_{\mu,\nu} \,|\, [T] \rangle \; . \tag{2.14}$$

Each matrix element on the right side is evaluated with use of (2.8). The problem with this approach is the possibly large number of intermediate states in the summation. What we show below is that the matrix element expressions for two-body operators decompose into multiplicative factors, each dependent only on nodes ($\{p\}_m$ labels) at adjacent subgroup levels.

Finally, concerning phase convention, we assume that the matrix elements of $E_{\mu-1,\mu}$ are positive. All phases are derived from this basic assumption and the Lie bracket relations (2.12).

III. RAISING-RAISING OPERATORS

The matrix elements of the symmetric commutator $\{E_{\mu,\nu},E_{\alpha,\beta}\}$ for the cases $\mu<\nu$ and $\alpha<\beta$ can be treated in three subcases: (i) the disjoint case, $\nu\le\alpha$; (ii) the partial overlap case, $\mu<\alpha<\nu<\beta$; and (iii) the completely overlapping case, $\mu\le\alpha<\beta\le\nu$. Subcases can be derived from these either by changing labels or by using the hermiticity property of the generators.

The simplest case is (i). Due to the fact that the $E_{\mu,\nu}$ and $E_{\alpha,\beta}$ operate on distinct parts of the basis function the operators commute. Thus the matrix element can be written as

$$\langle [T'] \,|\, \{E_{\mu,\nu},E_{\alpha,\beta}\} \,|\, [T] \rangle$$
$$=2\langle [T'] \,|\, E_{\mu,\nu} \,|\, [T] \rangle \,|_0^\nu \langle [T'] \,|\, E_{\alpha,\beta} \,|\, [T] \rangle \,|_\nu^n \; , \tag{3.1}$$

where the ranges 0 to ν and ν to n indicate that only those parts of $[T]$ and $[T']$ within those ranges contribute to the evaluation of each product term in (3.1), the remaining parts are ignored. Such cases are graphically represented in Fig. 4(a).

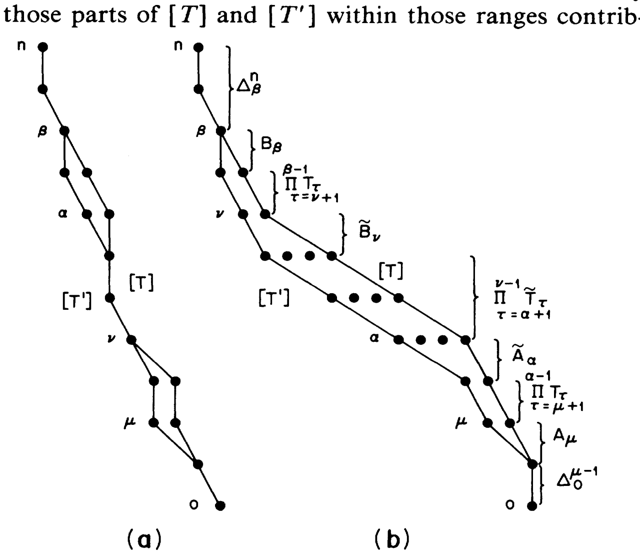


FIG. 4. Representation of (a) disjoint and (b) partial overlap RR-type matrix element graphs for $\mu<\nu$, $\alpha<\beta$ cases. Subgraph terms appearing in the decomposition of (3.2) are identified.

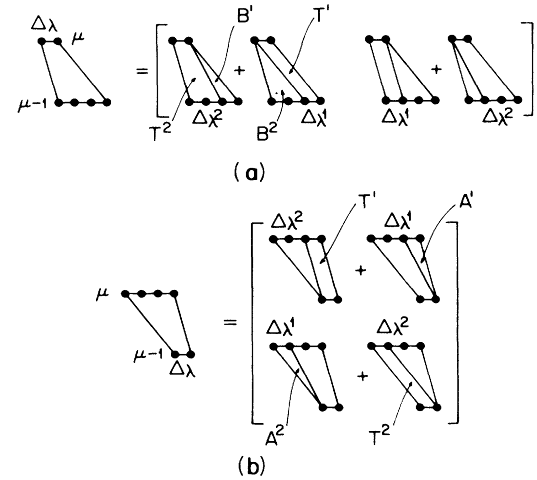


FIG. 5. Graphical representation of $\tilde{B}_\mu$ and $\tilde{A}_\mu$ subgraphs showing node and linkage labels and subgraphs used in matrix-element decompositions. Intermediate nodes at level $\mu=1$ in (a) which are not linked to a node at level μ do not participate in the subgraph evaluation; similarly for level μ in (b).

As we shall show below case (ii) [Fig. 4(b)] is expressed in the form

$$\langle [T'] \,|\, \{E_{\mu,\nu},E_{\alpha,\beta}\} \,|\, [T] \rangle$$
$$=\Delta_0^{\mu-1}\Delta_\beta^n A_\mu \left[\prod_{\tau=\mu+1}^{\alpha-1} T_\tau \right]$$
$$\times \tilde{A}_\alpha \left[\prod_{\tau=\alpha}^{\nu-1} \tilde{T}_\tau \right] \tilde{B}_\nu \left[\prod_{\tau=\nu+1}^{\beta-1} T_\tau \right] B_\beta \; , \tag{3.2}$$

where the subgraphs $\tilde{A}_\alpha$, $\tilde{B}_\nu$, and $\tilde{T}_\tau$ are matrices consisting of graphical elements (Figs. 5 and 6). Note that in the case of SU(2) expression (3.2) reduces to a form analogous to one described by Drake and Schlesinger[6] [expression (31) of Ref. 6].

The matrix subgraphs can be expressed in terms of A_ν, B_ν, and T_ν as

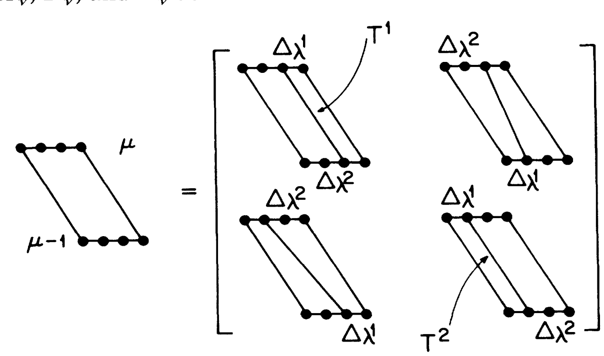


FIG. 6. Graphical representation of $\tilde{T}_\mu$ subgraph showing node and linkage labels and subgraphs used in matrix-element decompositions. Intermediate nodes at levels μ and $\mu-1$ which are not linked vertically to another node do not participate in the subgraph evaluation.

$$\widetilde{B}_\nu^T = \begin{bmatrix} T_\nu^2(\lambda_\nu,\lambda_{\nu-1}^2)B_\nu^1(\lambda_{\nu-1}^1)+B_\nu^2(\lambda_{\nu-1}^2)T_\nu^1(\lambda_\nu,\lambda_{\nu-1}^1) \\ T_\nu^2(\lambda_\nu,\lambda_{\nu-1}^1)B_\nu^1(\lambda_{\nu-1}^2)+B_\nu^2(\lambda_{\nu-1}^1)T_\nu^1(\lambda_\nu,\lambda_{\nu-1}^2) \end{bmatrix} \tag{3.3}$$

and

$$\widetilde{A}_\nu = \begin{bmatrix} A_\nu^2(\lambda_\nu^2)T_\nu^1(\lambda_\nu^1,\lambda_{\nu-1}+T_\nu^2(\lambda_\nu^2,\lambda_{\nu-1})A_\nu^1(\lambda_\nu^1) \\ A_\nu^2(\lambda_\nu^1)T_\nu^1(\lambda_\nu^2,\lambda_{\nu-1})+T_\nu^2(\lambda_\nu^1\lambda_{\nu-1})A_\nu^1(\lambda_\nu^2) \end{bmatrix} \tag{3.4}$$

corresponding to Figs. 5(a) and 5(b), and

$$\widetilde{T}_\nu = \begin{bmatrix} T_\nu^2(\lambda_\nu^2,\lambda_{\nu-1}^2)T_\nu^1(\lambda_\nu^1,\lambda_{\nu-1}^1) & T_\nu^2(\lambda_\nu^2,\lambda_{\nu-1}^1)T_\nu^1(\lambda_\nu^1,\lambda_{\nu-1}^2) \\ T_\nu^2(\lambda_\nu^1,\lambda_{\nu-1}^2)T_\nu^1(\lambda_\nu^2,\lambda_{\nu-1}^1) & T_\nu^2(\lambda_\nu^1,\lambda_{\nu-1}^1)T_\nu^1(\lambda_\nu^2,\lambda_{\nu-1}^2) \end{bmatrix} \tag{3.5}$$

corresponding to Fig. 6. Here, (3.3) is the transpose of $\widetilde{B}_\nu$. We added a superscript (T_τ^i, $i=1,2$) to indicate the order of evaluation of the subgraph products, and so the order of the particular $\{p\}$ labels considered.

Each matrix element of (3.3) has two terms. The first terms apply to the generator product $E_{\alpha,\beta}E_{\mu,\nu}$ the second to $E_{\mu,\nu}E_{\alpha,\beta}$. The opposite is the case in (3.4), however.

Finally, case (iii) can be derived immediately from (3.2) by interchanging ν and β, a point which is verified by referring to Fig. 4(b).

Proof of (3.2) can be obtained in a number of ways, one of which is a direct substitution of the complete algebraic expressions. Another method uses proof by induction, based on (2.14), in the graphical decomposition of a matrix-element expression.

In order to present the essential details of the proof by induction consider the special case of the generator product $E_{\mu,\nu}\,E_{\mu,\nu}$. Using as base cases $\nu=\mu+1$ (one step) and $\nu=\mu+2$ (two step) we find that

$$\langle [T'] \,|\, E_{\mu,\mu+1}E_{\mu,\mu+1} \,|\, [T] \rangle = \Delta_0^{\mu-1}\Delta_{\mu+1}^n [B_{\mu+1}^2(\lambda_\mu^2)B_{\mu+1}^1(\lambda_\mu^1) \;\; B_{\mu+1}^2(\lambda_\mu^1)B_{\mu+1}^1(\lambda_\mu^2)] \begin{bmatrix} A_\mu^2(\lambda_\mu^2)A_\mu^1(\lambda_\mu^1) \\ A_\mu^2(\lambda_\mu^1)A_\mu^1(\lambda_\mu^2) \end{bmatrix} , \tag{3.6}$$

where attention has been paid to the ordering of the subgraph factors. Referring to the row and column matrices as $\widetilde{B}_{\mu+1}$ and $\widetilde{A}_\mu$, respectively, we have demonstrated the base assertion as shown in Fig. 7(a). Note that some terms in (3.6) may be zero if the linkages between nodes are invalid. This case is also included in Fig. 7(a).

Next consider

$$\langle [T'] \,|\, E_{\mu,\mu+2}E_{\mu,\mu+2} \,|\, [T] \rangle = \Delta_0^{\mu-1}\Delta_{\mu+2}^n \widetilde{B}_{\mu+2} \begin{bmatrix} T_{\mu+i}^2(\lambda_{\mu+1}^2,\lambda_\mu^2)T_{\mu+1}^1(\lambda_{\mu+1}^1,\lambda_\mu^1) & T_{\mu+1}^2(\lambda_{\mu+1}^2,\lambda_\mu^1)T_{\mu+1}^1(\lambda_{\mu+1}^1,\lambda_\mu^2) \\ T_{\mu+1}^2(\lambda_{\mu+1}^1,\lambda_\mu^2)T_{\mu+1}^1(\lambda_{\mu+1}^2,\lambda_\mu^1) & T_{\mu+1}^2(\lambda_{\mu+1}^1,\lambda_\mu^1)T_{\mu+1}^1(\lambda_{\mu+1}^2,\lambda_\mu^2) \end{bmatrix} \widetilde{A}_\mu , \tag{3.7}$$

which we arrive at by inspection of Fig. 7(b) or, alternatively, by algebraic manipulation using the fact that $E_{\mu,\mu+2}=E_{\mu,\mu+1}E_{\mu+1,\mu+2}-E_{\mu+1,\mu+2}E_{\mu,\mu+1}$, by (2.12), and considering all possible terms (subgraphs) which arise. Identifying the 2×2 matrix as $\widetilde{T}_{\mu+1}$ demonstrates our second base assertion.

The product of $\widetilde{T}_\tau$ subgraphs in (3.2) is valid for several $\tau=\mu+1$ to $\tau=\mu+2,\mu+3$, and so on; thus we consider it valid for $\mu+1\le\tau\le\nu-1$, namely,

$$\langle [T'] \,|\, E_{\mu,\nu}E_{\mu,\nu} \,|\, [T] \rangle = \Delta_\nu^n\Delta_0^{\mu-1}\widetilde{B}_\nu \left[\prod_{\tau=\mu+1}^{\nu-1} \widetilde{T}_\tau \right] \widetilde{A}_\mu . \tag{3.2$'$}$$

An inductive proof is achieved if this case can be shown to extend to μ and ν product ranges. For the latter this follows directly by using (2.12) to yield $E_{\mu,\nu+1}=E_{\mu,\nu}E_{\nu,\nu+1}-E_{\nu,\nu+1}E_{\mu,\nu}$. Next apply these to find the relevant subgraphs [Fig. 8(a)]. These graphs are then decomposed [Fig. 8(b)] and reassembled into factors [Figs. 8(c)–8(e)] in the order given for the operators.

The graphs shown pertain to one element only of $\widetilde{T}_\tau$ and $\widetilde{B}_{\nu+1}$. The remaining elements are obtained by

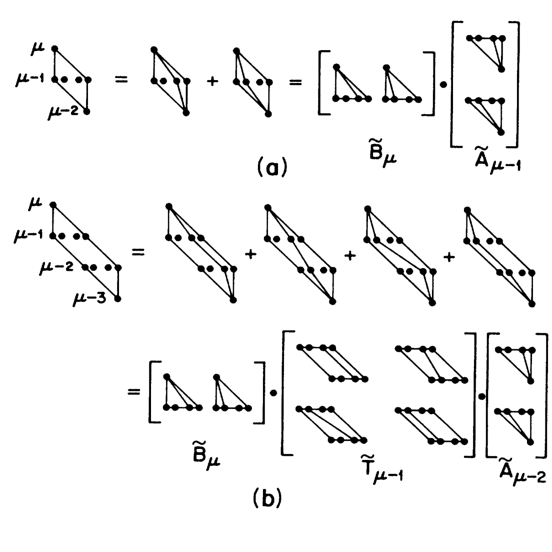


FIG. 7. Proof by induction of (3.2) base cases representing matrix-element decompositions of (a) $E_{\mu-1,\mu}E_{\mu-1,\mu}$ and (b) $E_{\mu-2,\mu}E_{\mu-2,\mu}$.

shifting the links between intermediate nodes and applying the same assembly algorithm as shown in Fig. 8. A similar process is applied to extending the lower end of the graph, μ to $\mu-1$, in order to verify the $\tilde{A}_{\mu-1}$ factor.

In Ref. 1 we showed that the T_τ factors can be expressed in all cases as a product of A_τ and B_τ subgraphs and a factor which is the square root of a simple rational number [see Fig. 2(c) and Eqs. (3.18′) and (3.19) of Ref. 1]. Due to this fact one need not compute A_τ and B_τ repeatedly.

For the derivation of $\tilde{A}_\mu$, $\tilde{T}_\tau$, and $\tilde{B}_\nu$ Weyl-Young tableaux prove to be useful. In the case of $E_{\mu,\nu+1}E_{\mu,\nu+1}$ matrix-element expressions we find

$$\tilde{B}_{\nu+1}=B^1_{\nu+1}(\lambda^1_\nu)B^1_{\nu+1}(\lambda^2_\nu)\times\left[\left[\frac{\tilde{h}^{\nu}_{\lambda^1_\nu-1}-1}{\tilde{h}^{\nu}_{\lambda^1_\nu-1}-2}\right]^{1/2}\left[\frac{\tilde{h}^{\nu}_{\lambda^2_\nu-1}+1}{\tilde{h}^{\nu}_{\lambda^2_\nu-1}+2}\right]^{1/2}\right] \tag{3.8}$$

and

$$\tilde{A}_\mu{}^T=A^1_\mu(\lambda^1_\mu)A^1_\mu(\lambda^2_\mu)\times\left[\left[\frac{\tilde{h}^{\mu}_{\lambda^1_\mu-1}}{\tilde{h}^{\mu}_{\lambda^1_\mu-1}-1}\right]^{1/2}\left[\frac{\tilde{h}^{\mu}_{\lambda^2_\mu-1}}{\tilde{h}^{\mu}_{\lambda^2_\mu-1}1}\right]^{1/2}\right], \tag{3.9}$$

where the hook lengths (h and $\tilde{h}$) were defined in Ref. 1 [expressions (3.10)–(3.14)]. It is to be understood in (3.8) and (3.9) that the irrep labels used are the $\{p\}$ labels (no prime), or bra-state labels. It is also to be understood that the hook lengths are evaluated using λ^2 (λ^1) for the pivot index in the first (second) term of each matrix.

In the case of $\tilde{T}_\tau$ we find

$$\tilde{T}_\tau=A^1_\tau(\lambda^1_\tau)A^1_\tau(\lambda^2_\tau)B^1_\tau(\lambda^1_{\tau-1})B^1(\lambda^2_{\tau-1})\times\tilde{M}_\tau(\lambda^1_\tau,\lambda^2_\tau,\lambda^1_{\tau-1},\lambda^2_{\tau-1}), \tag{3.10}$$

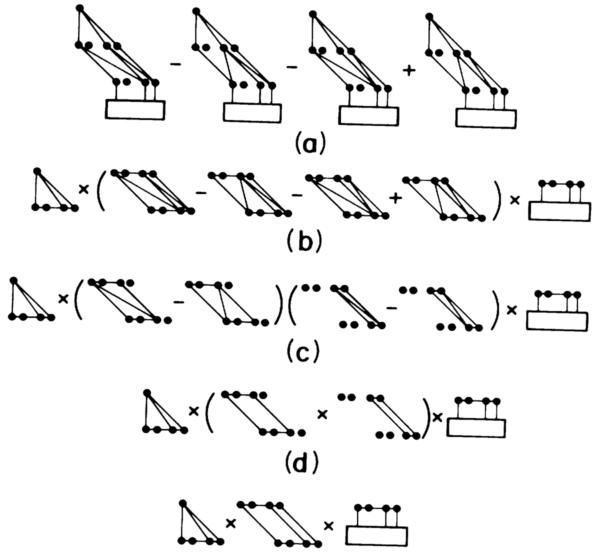


FIG. 8. Steps involved in (a) a single term, (b) a decomposition, and (c)–(e) reassembly in the induction step leading to expression (3.2).

where the matrix $\tilde{M}_\tau(\lambda^1_\tau,\lambda^2_\tau,\lambda^1_{\tau-1},\lambda^2_{\tau-1})$ is comprised of simple factors analogous to those in (3.8) and (3.9). The general form of $\tilde{M}_\tau$ is cumbersome to express since a slightly different form is required to meet each of the following subcases, namely,

$$\text{(a)}\lambda^1_{\tau-1}<\lambda^2_{\tau-1}\le\lambda^1_\tau\le\lambda^2_\tau,\quad \text{(b)}\lambda^1_{\tau-1}\le\lambda^1_\tau<\lambda^2_{\tau-1}\le\lambda^2_\tau\,,$$
$$\text{(c)}\lambda^1_{\tau-1}\le\lambda^1_\tau<\lambda^2_\tau<\lambda^2_{\tau-1},\quad \text{(d)}\lambda^1_\tau<\lambda^1_{\tau-1}<\lambda^2_{\tau-1}\le\lambda^2_\tau\,,$$
$$\text{(e)}\lambda^1_\tau<\lambda^1_{\tau-1}\le\lambda^2_\tau<\lambda^2_{\tau-1},\quad \text{(f)}\lambda^1_\tau<\lambda^2_\tau<\lambda^1_{\tau-1}<\lambda^2_{\tau-1}\,.$$

For case (a) above we have included the elements of $\tilde{M}_\tau$ as follows:

$$\tilde{M}^{(a)}_{\tau,11}=Q^{(a)}_1\left[\frac{\tilde{h}^{\tau,2}_{\lambda^1_\tau-1}h^{\tau,1}_{\lambda^2_\tau-1}(h^{\tau,2}_{\lambda^1_\tau-1}+1)(\tilde{h}^{\tau-1,1}_{\lambda^2_{\tau-1}-1}+1)}{(\tilde{h}^{\tau,2}_{\lambda^1_\tau-1}-1)(h^{\tau,1}_{\lambda^2_\tau-1}+1)h^{\tau,2}_{\lambda^1_\tau-1}\tilde{h}^{\tau-1,1}_{\lambda^2_{\tau-1}-1}}\right]^{1/2}, \tag{3.11a}$$

$$\tilde{M}^{(a)}_{\tau,21}=Q^{(a)}_2\left[\frac{\tilde{h}^{\tau,1}_{\lambda^2_\tau-1}h^{\tau,1}_{\lambda^1_\tau-1}(h^{\tau,2}_{\lambda^2_\tau-1}-1)(\tilde{h}^{\tau-1,1}_{\lambda^2_{\tau-1}-1}+1)}{(\tilde{h}^{\tau,1}_{\lambda^2_\tau-1}-1)(h^{\tau,1}_{\lambda^1_\tau-1}+1)h^{\tau,1}_{\lambda^1_\tau-1}\tilde{h}^{\tau-1,2}_{\lambda^1_{\tau-1}-1}}\right]^{1/2}, \tag{3.11b}$$

$$\tilde{M}^{(a)}_{\tau,12}=Q^{(a)}_2\left[\frac{\tilde{h}^{\tau,2}_{\lambda^1_\tau-1}h^{\tau,2}_{\lambda^2_\tau-1}(h^{\tau,1}_{\lambda^1_\tau-1}+1)(\tilde{h}^{\tau-1,2}_{\lambda^1_{\tau-1}-1}+1)}{(\tilde{h}^{\tau,2}_{\lambda^1_\tau-1}-1)(h^{\tau,2}_{\lambda^2_\tau-1}+1)h^{\tau,1}_{\lambda^1_\tau-1}\tilde{h}^{\tau-1,2}_{\lambda^1_{\tau-1}-1}}\right]^{1/2}, \tag{3.11c}$$

$$\tilde{M}^{(a)}_{\tau,22}=Q^{(a)}_1\left[\frac{\tilde{h}^{\tau,1}_{\lambda^2_\tau-1}h^{\tau,2}_{\lambda^1_\tau-1}(h^{\tau,1}_{\lambda^2_\tau-1}+1)(\tilde{h}^{\tau-1,2}_{\lambda^1_{\tau-1}-1}-1)}{(\tilde{h}^{\tau,1}_{\lambda^2_\tau-1}+1)(h^{\tau,2}_{\lambda^1_\tau-1}+1)h^{\tau,1}_{\lambda^2_\tau-1}\tilde{h}^{\tau-1,2}_{\lambda^1_{\tau-1}-1}}\right]^{1/2}, \tag{3.11d}$$

where we define the factors $Q_i^{(a)}$ as

$$Q_1^{(a)}=[h^{\tau,1}_{\lambda_\tau^1-1}h^{\tau,2}_{\lambda_\tau^2-1}(h^{\tau,1}_{\lambda_\tau^1-1}+1)(h^{\tau,2}_{\lambda_\tau^2-1}+1)]^{-1/2}\ , \quad (3.12a)$$

$$Q_2^{(a)}=[h^{\tau,1}_{\lambda_\tau^2-1}h^{\tau,2}_{\lambda_\tau^1-1}(h^{\tau,1}_{\lambda_\tau^2-1}+1)(h^{\tau,2}_{\lambda_\tau^1-1}+1)]^{-1/2}\ . \quad (3.12b)$$

Note that an additional superscript has been added to the hook lengths. $h^{\tau,i}$ $(i=1,2)$ indicating the use of λ_τ^i as pivot index.

What is achieved in using an expression such as (3.10) is an effective factorization of the fundamental A and B subgraphs. That is, instead of calculating the A and B subgraphs repeatedly for each element in the matrix (3.5), one calculates them only once and multiplies their product by the $\tilde{M}_\tau$ matrix.

Note that in the case of SU(2) the off-diagonal terms in these matrices vanish, a fact most easily seen by inspecting two-column Weyl-Young tableaux. As shown by Drake and Schlesinger[6] the remaining diagonal terms can be categorized as arising from singlet and triplet intermediate couplings. When such matrices are multiplied the couplings do not mix; therefore it becomes possible to write the matrix-element expressions as the sum of two terms, each of which is a product of scalar factors. In the case of SU(n), however, it is no longer possible to categorize the matrix elements in a similar fashion. Further, these terms do mix on multiplication of the matrices. Therefore, the 2×2 matrix-factorization technique which we have employed is the simplest form into which these expressions can be cast.

IV. RAISING-LOWERING OPERATORS

The treatment of raising-lowering two-body operators parallels that of raising-raising products except for one important difference and that pertains to the diagonal matrix elements. We shall deal with these issues below.

Considering the commutator $\{E_{\mu,\nu},E_{\alpha,\beta}\}$ for $\mu<\nu$ and $\alpha>\beta$ we have the cases (a) disjoint, $\nu\leq\beta$; (b) partial overlap, $\mu\leq\beta\leq\nu\leq\alpha$; and (c) complete overlap, $\mu\leq\beta<\alpha\leq\nu$.

The disjoint case is derivable from (3.1) by exchanging primes between the primed and unprimed labels and subsequently taking the Hermitian conjugate of the matrix element of $E_{\alpha,\beta}$. See Fig. 4(a) also.

It is worthwhile noting that each application of an elementary generator $E_{\mu-1,\mu}$ on a bra state results in shifting a node (at level $\mu=1$ of the graph) to the left ($E_{\mu,\mu-1}$ shifts a node to the right.) For any overlapping region contained in products $E_{\mu,\nu}E_{\alpha,\beta}$ there must be, in general, products $E_{\tau,\tau\pm1}E_{\tau\pm1,\tau}$ (for τ in the overlap region). Thus, three instances can arise at each level, namely (i) $\{p'\}<\{p\}$ [Fig. 9(a)], (ii) $\{p'\}>\{p\}$ [Fig. 9(a), with interchange of primes and no primes] and (iii) $\{p'\}=\{p\}$ [Fig. 9(b)]. Whenever (iii) holds in the overlap region we refer to it as a diagonal subgraph. Further, for each set of nodes in the overlap region there are two associated pivot indices when $\{p'\}\neq\{p\}$. We refer to these as λ^+ (right shifting) and λ^- (left shifting) and we find $\{p\}-\Delta_{\lambda^-}<\{p\}<\{p\}+\Delta_{\lambda^-}$ (similarly for $\{p'\}$.) For the diagonal subgraphs it is also to be noted that the number of pivot indices may be greater than two.

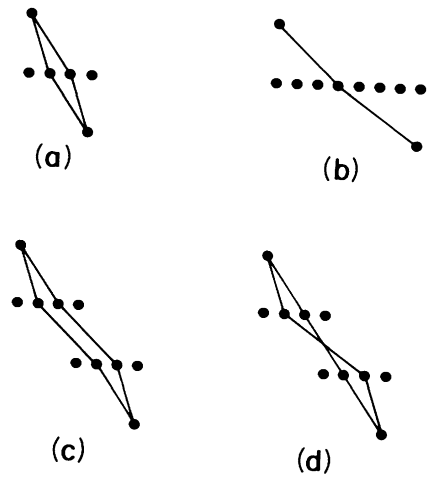


FIG. 9. One-step RL-type matrix-element subgraphs for (a) $\{p'\}_{\mu-1}<\{p\}_{\mu-1}$ and (b) the diagonal case, $\{p'\}_{\mu-1}=\{p\}_{\mu-1}$. Cases (c) and (d) represent subgraphs occurring in an overlap region for two-step matrix elements.

For the case $\mu<\beta<\nu<\alpha$ the matrix element expression, as in (3.2), is given as

$$\langle [T'] \mid \{E_{\mu,\nu},E_{\alpha,\beta}\} \mid [T]\rangle = \Delta_0^{\mu-1}\Delta_\alpha^n A_\mu \left[\prod_{\tau=\mu+1}^{\beta-1} T_\tau\right] \times \hat{A}_\beta \left[\prod_{\tau=\beta}^{\nu-1}\hat{T}_\tau\right]\hat{B}_\nu \left[\prod_{\tau=\nu+1}^{\alpha-1} T'_\tau\right] B'_\alpha\ . \quad (4.1)$$

In this notation the prime on T and B factors indicates interchanging primes and no primes on the $\{p\}$ labels. The factors $\hat{A}_\beta$, $\hat{T}_\tau$, and $\hat{B}_\nu$ are represented in Figs. 9 and 10 for cases ($\mu=\beta$, $\alpha=\nu$), and Fig. 11. Note that the $\hat{T}_\tau$ matrix has two possible representations differentiated by the lexical ordering of labels (nodes) $\{p'\}_m$ and $\{p\}_m$ at levels $m=\mu$ and $\mu-1$.

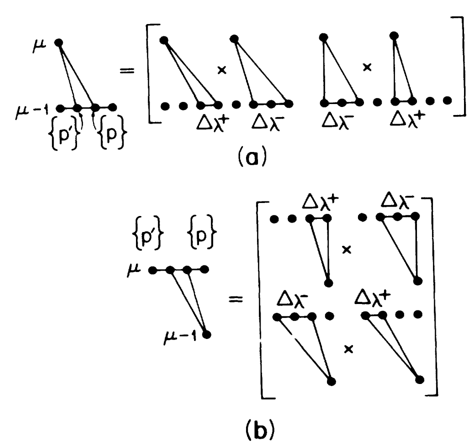


FIG. 10. Representation of RL-type matrix-element subgraphs. Case (a) is a $\hat{B}$ subgraph. Case (b) is an $\hat{A}$ subgraph.

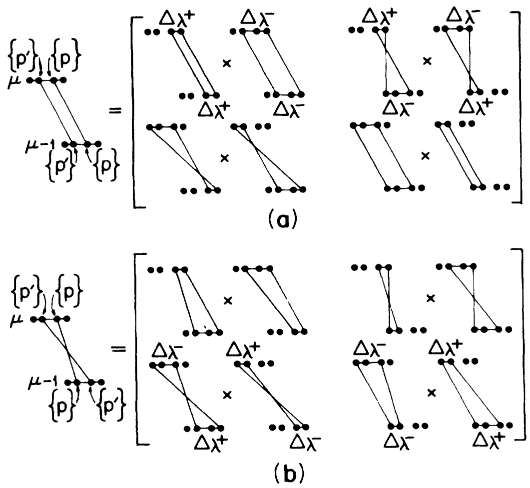


FIG. 11. Representations of $\hat{T}$ subgraphs. Case (a) $\{p'\}_m < \{p\}_m$, $m=\mu$, $\mu-1$ and (b) $\{p'\}_\mu < \{p\}_\mu$, $\{p'\}_{\mu-1} > \{p\}_{\mu-1}$.

Once again, by virtue of there being two pivot indices (except for diagonal cases) at each level in the overlap region these coefficients are matrices (at most 2×2). The proof of (4.1) can be obtained using induction in parallel to the lowering-lowering case. The base cases for one-step and two-step matrix elements are shown in Fig. 9. [The SU(2) analogue of (4.1) is expression (26) of Ref. 6.]

The remaining cases involve diagonal matrix elements of operators such as $\{E_{\mu,\nu}, E_{\nu,\mu}\}$ as well as cases where labels $\{p'\}_m = \{p\}_m$ at one or more levels in the overlap region. For the former it follows that $\lambda_\tau^+ = \lambda_\tau^- = \lambda_\tau$ for all $\mu < \tau < \nu - 1$. Further, as seen from Fig. 9(b), 12, and 13, there will be more than two intermediate nodes, in general, which arise in the matrix-element decomposition. Thus, in order to deal with these cases, a different approach is necessary.

For simple cases involving pure diagonal matrix elements ($[T'] = [T]$) one can adapt (2.14),

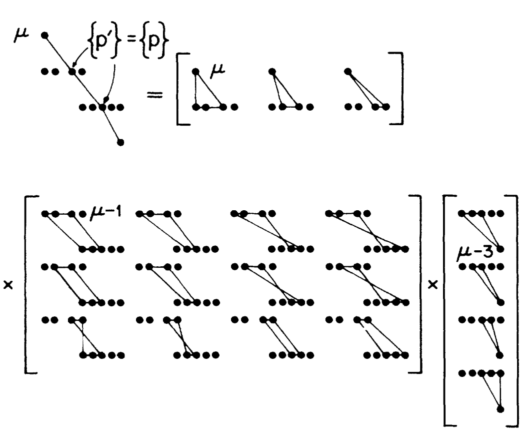


FIG. 12. RL-type matrix-element decomposition for pure diagonal case. Though not shown explicitly, all matrix elements are understood to be squared.

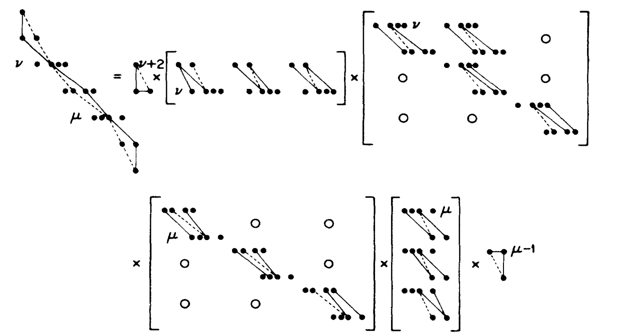


FIG. 13. RL-type matrix-element graph showing partial decomposition of mixed diagonal and/or nondiagonal case.

$$\langle [T] | \{E_{\mu,\nu}, E_{\mu,\nu}\} | [T] \rangle = \sum_{[T']} (\langle [T] | E_{\mu,\nu} | [T'] \rangle^2 + \langle [T'] | E_{\mu,\nu} | [T] \rangle^2) . \tag{4.2}$$

Now, using (2.6), (2.7), and (2.9), one checks for all valid Δ_λ links between $\{p\}$ and $\{p'\}$ labels at each level. Using (2.8) and the fact that each term in (4.2) is the square of a rational root it can be quickly computed.

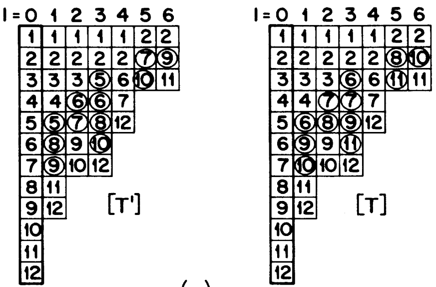
For complicated cases (large N and n) the search for and verification of all intermediate states using (4.2) is too time consuming. Using Fig. 12 as a prototype of such cases, however, it can be noted that if R intermediate nodes occur at level τ and C nodes at level $\tau - 1$ in the overlap region then the subgraph factor for that level can be expressed in the form of an $R \times C$ matrix. Additional simplification of these matrices can be achieved through factorization of the fundamental A and B subgraphs so as to achieve increased computational efficiency. It is to be noted that the $\hat{A}_\beta$ and $\hat{B}_\nu$ terms are no longer two-element matrices. Similar results hold for cases involving mixed diagonal and/or non-diagonal matrix elements as shown in Fig. 13 where $\{p'\}_m = \{p\}_m$, $m = \mu, \nu$.

The properties of the diagonal matrix-element factors deriving from the involvement of many intermediate nodes were previously noted in SU(2) as well [see examples (3.5)-(3.7) of Kent and Schlesinger[7]].

V. SAMPLE CALCULATIONS

We shall now demonstrate the use of the decomposition techniques which we have derived by considering several cases of generator products for two different irreps. Our purpose in examining these cases is solely didactic and intended to focus attention on a variety of subcases in addition to the use of the formulas.

The first example is shown in Fig. 14. Two states of the irrep $\{3\,0\,2\,2\,0\,2\,3\}$ of SU(12), $N=34$, are given. Our objective is to use these to demonstrate the evaluation of the matrix element of $E_{5,11}E_{5,11}$. It is noted first that $\{p\}_\tau = \{p'\}_\tau$ for $0 \le \tau \le 4$ and $11 \le \tau \le 12$. Further, from Fig. 14(b), it is seen that all labels agree up to changes around the pivot positions; two boxes, one for each pivot

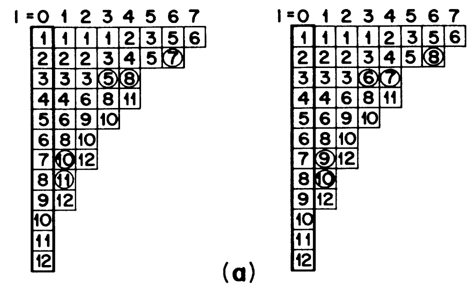


(a)

m	$\{p'\}_m$ l=6 5 4 3 2 1 0	$\{p\}_m$ l=6 5 4 3 2 1 0	λ^1_m	λ^2_m
12	3 0 2 2 0 2 3	3 0 2 2 0 2 3		
11	3 0 1 2 1 1 3	3 0 1 2 1 1 3		
10	2 1 1 2 1 0 3	2 0 2 1 2 0 3	3	5
9	2 0 2 1 1 1 2	1 1 2 1 1 0 3	1	6
8	1 1 2 1 0 1 2	1 1 2 0 1 0 3	1	3
7	1 1 2 0 1 0 2	1 0 3 0 0 1 2	2	5
6	1 0 2 1 0 1 1	1 0 2 0 0 2 1	2	3
5	1 0 1 1 0 2 0	1 0 1 0 1 1 1	1	3
4	1 0 1 0 1 1 0	1 0 1 0 1 1 0		
3	1 0 1 0 1 0 0	1 0 1 0 1 0 0		
2	1 0 1 0 0 0 0	1 0 1 0 0 0 0		
1	0 0 1 0 0 0 0	0 0 1 0 0 0 0		
0	0 0 0 0 0 0 0	0 0 0 0 0 0 0		

(b)

FIG. 14. States used to calculate the matrix element of $E_{5,11}E_{5,11}$ in irrep $\{3022023\}$ of $SU(12)\downarrow S_{34}$ using (a) WYT and (b) DRT notation with pivot indices included.



(a)

m	$\{p'\}_m$ l=7 6 5 4 3 2 1 0	$\{p\}_m$ l=7 6 5 4 3 2 1 0	λ^+_m	λ^-_m
12	1 1 0 2 1 2 2 3	1 1 0 2 1 2 2 3		
11	1 1 0 2 1 1 2 3	1 1 0 2 1 1 2 3		
10	1 1 0 1 2 1 1 3	1 1 0 1 2 1 2 2		1
9	1 1 0 1 1 1 1 3	1 1 0 1 1 1 2 2		1
8	1 1 0 1 1 0 2 2	1 1 0 1 1 0 2 2		
7	1 1 0 0 1 1 1 2	1 0 1 1 0 1 1 2	6	4
6	1 0 1 0 1 1 1 1	1 0 1 1 0 1 1 2		
5	0 1 1 0 1 0 1 1	0 1 1 0 0 1 1 1	3	
4	0 0 1 1 0 1 1 0	0 0 1 1 0 1 1 0		
3	0 0 1 0 1 1 0 0	0 0 1 0 1 1 0 0		
2	0 0 0 1 0 1 0 0	0 0 0 1 0 1 0 0		
1	0 0 0 0 1 0 0 0	0 0 0 0 1 0 0 0		
0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0		

(b)

FIG. 15. States used to calculate the matrix element of $\{E_{5,9}E_{11,6}\}$ in irrep $\{11021223\}$ of $SU(12)\downarrow S_{30}$ using (a) WYT and (b) DRT notation with pivot indices included.

position, surround the pairs of changing labels at each level. At level $\tau=6$ the boxes overlap ($\lambda_6^2=\lambda_6^1+1$); this special case is the only case which applies in SU(2) for RR-type matrix elements.

From (3.8) and (3.9) we calculate the results

$$\tilde{B}_{11}=\left[\frac{7\times7\times8\times8}{5\times5}\right]^{1/2}[(5/4)^{1/2}\ \ (5/6)^{1/2}]\ , \qquad (5.1)$$

$$\tilde{A}_5=(1/5)^{1/2}\begin{bmatrix}(5/4)^{1/2}\\(3/4)^{1/3}\end{bmatrix}\ . \qquad (5.2)$$

The remaining matrix-element terms refer to $\tilde{T}_\tau$ for $\tau=6,7,\ldots,10$. These are represented by RR cases (b) $\tau=6$, 7, and 9, (c) $\tau=10$, and (e) $\tau=8$, as seen by comparison of the pivot indices in Fig. 14(b).

For the $\tilde{T}_\tau$ matrix terms we find

$$\tilde{T}_{10}=-\frac{(26)^{1/2}}{980}\begin{bmatrix}(16\times88)^{1/2} & (32)^{1/2}\\(33)^{1/2} & (49\times27)^{1/2}\end{bmatrix}\ , \qquad (5.3a)$$

$$\tilde{T}_9=-(256\times65)^{-1/2}\begin{bmatrix}(99)^{1/2} & 0\\7/3 & (400/3)^{1/2}\end{bmatrix}\ , \qquad (5.3b)$$

$$\tilde{T}_8=\left[\frac{5\times49}{9\times36\times40}\right]^{1/2}\begin{bmatrix}(7/5)^{1/2} & 0\\0 & 8/7\end{bmatrix}\ , \qquad (5.3c)$$

$$\tilde{T}_7=1/12\begin{bmatrix}-(7)^{1/2} & 0\\(125/3)^{1/2} & 0\end{bmatrix}\ , \qquad (5.3d)$$

$$\tilde{T}_6=-(800)^{-1/2}\begin{bmatrix}5 & (15)^{1/2}\\0 & 0\end{bmatrix}\ . \qquad (5.3e)$$

After multiplying the terms in (5.1)–(5.3) together, paying particular attention to the ordering of the matrices according to (3.2′), we finally obtain the matrix-element value $(456\,296/50\,625)(195/2)^{1/2}$.

As examples of RL-type matrix elements we consider two cases involving the states described in Fig. 15. The first example concerns the diagonal matrix element of the operator $(E_{8,7},E_{7,8})$ using the primed state $\{p'\}_m$. This case can be calculated either using (4.2) or by decomposing a graph such as Fig. 9(b). Taking the latter approach we find

$$\hat{B}_8=(35/12\ \ 27/4\ \ 10/3\ \ 704/135)\ , \qquad (5.4)$$

$$\hat{A}_7^T=(4/9\ \ 2/15\ \ 1/12\ \ 1/11)\ . \qquad (5.5)$$

Multiplying the two matrices gives the result (398/135). Note the property that each of the elements are the square of fundamental subgraphs.

For the final example we consider the operator $\{E_{11,6},E_{5,9}\}$ using the states in Fig. 15 to determine the matrix element. The matrix-element graph corresponding to this case resembles Fig. 13. Note that the labels $\{p'\}_m=\{p\}_m$ for $m=6,8$ in the overlap region, indicating the occurrence of diagonal subgraphs. Since it is not our purpose to compute any specific model we shall not actually calculate the matrix-element value.

We find $B'_{11}=(24/7)^{1/2}$, $T'_{10}=-1$, and

$A_5=(8/75)^{1/2}$. Remaining terms are expressed as matrices. In the case of $\hat{B}_9$ one determines a 1×5 row matrix while $\hat{A}_6$ is a 4×1 matrix. The remaining $\hat{T}_\tau$ matrices are all greater than 2×2, each containing zero elements in some positions.

VI. CONCLUSIONS

We have presented a method for deriving closed-form expressions for arbitrary two-body operator matrix elements in $SU(n)\downarrow S_N$ irrep bases. The method is expected to be of use in cases where the number of particles in a system is very large or when the spin symmetry is large or both. The resulting formulas are remarkably similar to those derived in SU(2) using a variety of approaches.[6,7] The essential features of the method are a significant saving in computer storage for the irrep representation, a means for displaying and manipulating matrix elements graphically and a significant gain in computational efficiency for matrix-element calculations through the use of formulas which are factorized in terms of fundamental subgraph products and, at most, matrices comprised of simple elements.

ACKNOWLEDGMENT

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