Continuous-time dynamics of asymmetrically diluted neural networks

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(Received 25 June 1987)

We study the continuous-time dynamics of a strongly diluted Hopfield model with asymmetric synaptic connections. The model is exactly soluble for static as well as dynamic properties. The time evolution of the autocorrelation, the susceptibility, and the overlap function of two configurations are given in explicit form.

I. INTRODUCTION

A model of content-addressable, associative memory has been proposed by Hopfield[1] and Little[2] in an attempt to explain properties of the human brain in terms of the dynamics of a network of two-state neurons. If the synaptic couplings J_{ij} between neuron i and neuron j are symmetric ($J_{ij}=J_{ji}$), the static (long-time) properties of the network can be studied as a problem in equilibrium statistical mechanics.[3] The assumption of symmetric synapses is, however, not supported by neurophysiology.[4] For $J_{ij}\neq J_{ji}$ the equilibrium approach is no longer possible and the behavior of the network can only be obtained from its dynamics. In fact, even the attractors of the network dynamics are, in general, time dependent (e.g., cycles), so that static properties may only exist as averaged long-time limits of dynamical quantities. Another unrealistic feature of the Hopfield-Little model is its assumption of a completely connected network (i.e., every neuron is connected with every other neuron by a synaptic bond). In the human brain,[4] the average connectivity is fairly low.

In the present work we consider an asymmetric, diluted neural network in the limit of strong dilution, where each neuron is no longer connected with a finite fraction of the other neurons in a macroscopic network. The model becomes soluble if the average coordination number K of a neuron and the number p of learned patterns are both large, such that the ratio $\alpha=p/K$ remains finite. This model has recently been analyzed by Derrida *et al.*[5] for a discrete-time dynamics. One of the main results of Ref. 5 is the existence of a critical line $T_c(\alpha)$, such that for temperatures $T<T_c(\alpha)$ the network can retrieve p stored patterns, whereas for $T>T_c(\alpha)$ it cannot. Here we recover and generalize the results of Ref. 5 to either a continuous-time Glauber dynamics or to relaxational dynamics of soft spins. We calculate various time-dependent correlation and response functions exactly. These provide information on the averaged dynamic properties of the attractors, which are reached by the network in the limit of long times. Some insight into the structure of the attractors can be obtained by studying the time evolution of two different configurations which start out in the vicinity of one stored pattern. We find that the overlap of the two configurations becomes identical to the time-persistent part of the spin autocorrelation, irrespective of its initial value. This shows that there is only one characteristic distance between configurations on the attractor. We furthermore investigate the possibility of a spin-glass transition, as signaled by the spontaneous appearance of a time-persistent part of the spin autocorrelation or a divergence of the relaxation time of the spin autocorrelation. No such transition can take place for finite T and α.

II. THE MODEL AND ITS REDUCTION TO A SINGLE-SPIN PROBLEM

In the following we consider a Hopfield-Little model of neural networks.[1,2] The N neurons are represented by Ising spins $s_i=\pm 1$ which interact via synaptic couplings $J_{ij}c_{ij}$. The J_{ij} depend on p stored, uncorrelated patterns $\xi_i^\nu=\pm 1$ according to Hebb's learning rule[1,2]

$$J_{ij}=J\sum_{\nu=1}^{p}\xi_i^\nu\xi_j^\nu \; . \tag{1}$$

Each ξ_i^ν takes its values ± 1 with equal probabilities. The factors c_{ij} represent *asymmetric* synaptic dilution. They are assumed to be statistically independent random variables for each pair (i,j) with distribution

$$\rho(\{c_{ij}\})=\prod_{\substack{i,j=1\\(i\neq j)}}^{N}\rho(c_{ij}) =\prod_{\substack{i,j=1\\(i\neq j)}}^{N}[c\delta(c_{ij}-1)+(1-c)\delta(c_{ij})] \; . \tag{2}$$

For convenience we choose a soft spin version of the model for studying its dynamics.[6] In the Appendix we show, however, that our results may also be achieved from Glauber dynamics[7] using a method proposed recently by Sommers.[8]

The model is defined by the equations of motion for spin variables s_i:

$$\Gamma^{-1}\partial_t s_i(t)=-U_\lambda(s_i(t))+\sum_j J_{ij}c_{ij}s_j(t)+f_i(t) \; . \tag{3}$$

Γ^{-1} sets the microscopic spin-flip time scale and $U_\lambda(s_i(t))$ restricts the fluctuations of spin length, so that for $\lambda \to \infty$ the Ising limit is reached. We have added an additional white noise term

$$\langle f_i(t) f_j(t') \rangle = (2/\beta\Gamma)\delta_{ij}\delta(t-t')$$

representing external disturbances of the neurons. Here $T = 1/\beta$ will be called "temperature," even though it does not correspond to the thermodynamic variable. For symmetric synaptic bonds this noise leads to a canonical equilibrium distribution which governs the static properties of the network.[3] For asymmetric synapses the stationary distribution is not known *a priori* and static quantitities have to be calculated from the long-time dynamics.

All response and correlation functions can be obtained from a generating functional

$$Z[l,\tilde{l} \mid \{c_{ij},\xi_i^\nu\}] = \int D(\tilde{s},s) e^{-L} , \tag{4}$$

which is constructed from the equation of motion of spin variables in the standard way.[9,6] The weight in the functional integral (4) is determined by

$$L = -\int dt \sum_{\substack{i,j=1\\(i\neq j)}}^{N} c_{ij} J_{ij} \tilde{s}_i(t) s_j(t) + L_0 \tag{5}$$

and

$$L_0 = \sum_{i=1}^{N} \int dt \left[\tilde{s}_i(t) \left[\Gamma^{-1}\partial_t s_i(t) + U_\lambda(s_i) - \frac{T}{\Gamma}\tilde{s}_i(t) \right] - \tilde{l}_i(t)\tilde{s}_i(t) - l_i(t) s_i(t) \right] . \tag{6}$$

The $\tilde{s}$ integrations extend over the imaginary axis. Averaging Z over the distribution of the c_{ij} we get the generating functional for the asymmetrically diluted system,

$$\langle Z \rangle_{c_{ij}} = \int D(\tilde{s},s) e^{-L_0} \prod_{\substack{i,j\\(i\neq j)}} \left[1 - c + c \exp\left[J_{ij} \int dt\, \tilde{s}_i(t) s_j(t) \right] \right] . \tag{7}$$

Let us point out here two distinct ways to achieve a nontrivial limit for a macroscopic network ($N \to \infty$). Either we choose $c = O(1)$ and $J = 1/N$ ("weak dilution") (Refs. 10 and 11), or we choose $c = K/N \to 0$ for $N \to \infty$ and $J = 1/K$ ("strong dilution"). Most of our discussion concerns the latter case, for which we can expand (7) in powers of c,

$$\langle Z \rangle_{c_{ij}} = \int D(\tilde{s},s) e^{-\bar{L} - L_0} \tag{8}$$

with

$$\bar{L} = \frac{K}{N} \sum_{\substack{i,j=1\\(i\neq j)}}^{N} \left[1 - \exp\left[\int dt\, \tilde{s}_i(t) s_j(t) \frac{1}{K} \sum_{\nu=1}^{p} \xi_i^\nu \xi_j^\nu \right] \right] + O(K^2/N^2) . \tag{9}$$

For a finite number of patterns all nonlinearities in (9) are of importance. The problem resembles that of a diluted ferromagnet or diluted spin glass, and its treatment requires the introduction of an infinity of order parameters.[12] Note, however, that (9) does not correspond to a diluted equilibrium system. Here we want to treat the limit of large p and K. We choose $p = \alpha K$ [$\alpha = O(1)$] and expand in powers of $1/K$. Then we let K go to infinity (after $N \to \infty$). In this way, only terms up to $O(J_{ij}^2)$ need to be taken into account in (7) and J_{ij}^2 may be replaced by its average $\langle J_{ij}^2 \rangle_\xi = \alpha/K$. $\bar{L}$ takes on the form

$$\bar{L} = -(1/N) \sum_{\substack{i,j=1\\(i\neq j)}}^{N} \int dt \sum_{\nu=1}^{p} \xi_i^\nu \xi_j^\nu \tilde{s}_i(t) s_j(t) - (\alpha/2N) \sum_{\substack{i,j=1\\(i\neq j)}}^{N} \int dt \int dt'\, \tilde{s}_i(t) s_j(t) \tilde{s}_i(t') s_j(t') . \tag{10}$$

To handle the remaining average over the patterns we introduce the overlaps $m^\nu = N^{-1}\sum_{i=1}^{N} s_i \xi_i^\nu$ by

$$1 = \int D(\tilde{m},m) \exp\left[-N \int dt \sum_{\nu=1}^{p} \tilde{m}_\nu \left[m_\nu - \frac{1}{N}\sum_i s_i \xi_i^\nu \right] \right] , \tag{11}$$

and proceed analogously to the treatment of the symmetric network.[3] In the self-consistent solution we find $m^\nu = O(1/\sqrt{N})$ for most patterns, except for a finite number $m^1, m^2, \ldots, m^s$, which are of order 1. Thus the contributions from random overlaps vanish as p/N for a macroscopic network.

In a further step we reduce (8) and (9) to a single-spin problem by decoupling the quartic term in (8).[6] All integrations over the auxiliary fields may be performed by the saddle-point method and we are left with

$$\bar{L} = -\sum_i \int dt \left[\sum_{\nu=1}^{s} m^\nu \xi_i^\nu \tilde{s}_i(t) + \tfrac{1}{2}\alpha \tilde{s}_i(t) \times C(t-t') \tilde{s}_i(t') \right] . \tag{12}$$

The quantities $C(t-t')$ and m have to be determined self-consistently from (12),

$$m^\nu = N^{-1} \sum_{i=1}^{N} \xi_i^\nu \langle s_i \rangle_{\bar{L}} , \tag{13}$$

$$C(t-t') = N^{-1} \sum_{i=1}^{N} \langle s_i(t) s_i(t') \rangle_{\bar{L}} . \tag{14}$$

The last term in (12) corresponds to additional noise generated by static dilution. For a fixed realization $y(t)$ of this internal noise, the single-spin problem can be stated in the form

$$\bar{L}_s = -\int dt \left[\sum_{\nu=1}^{s} (m^\nu \xi^\nu) + \sqrt{\alpha} y(t) \right] \tilde{s}(t) . \tag{15}$$

The random field $y(t)$ has a Gaussian distribution with zero mean and variance

$$\langle y(t) y(t') \rangle_y = C(t-t') . \tag{16}$$

Equations (12) and (15) show that the effects of strong asymmetric dilution on the neural network are the following.

(i) The random dilution creates additional noise, but does not modify the spin response function. This is due to the fact that the synaptic dilution factors c_{ij} and c_{ji} are completely uncorrelated in our model. For other models, however, which contain correlations of the synapses between neutrons i and j (like, e.g., the symmetrically diluted network[13]) there are modifications of the response functions which may lead to an anomalous behavior of the spin susceptibilities for long times.

(ii) Static noise due to random overlaps is suppressed, because the total number of stored patterns is not extensive in the limit $N \to \infty$. For the case of weak dilution $[c = O(1)]$, contributions from random overlaps cannot be neglected. They modify the internal noise as well as the response and may also lead to anomalous long-time behavior of the susceptibility.

At this point we would like to comment on a recent paper by Feigelman and Ioffe.[11] They obtain the results of Eqs. (12)–(16) as an *approximate* solution to the Hopfield model with unidirectional bonds, i.e., $c_{ij} = 1 - c_{ji}$ and $c = \frac{1}{2}$. However, in this case, correlations of the c_{ij} give rise to a modification of the response, as in the symmetric model. Hence the above solution cannot be exact, as it is in our model with strong dilution and uncorrelated c_{ij}.

III. SELF-CONSISTENT SINGLE-SPIN DYNAMICS

In the thermodynamic limit we are left with the problem of a single spin in a time-dependent field. We choose the limit of Glauber dynamics for an Ising spin $s = \pm 1$ because various time-dependent correlation functions can be calculated exactly in this case. The dynamics of Glauber[7] is defined by the equation of motion for the probability $p(s,t)$ of finding the spin in state s at time t,

$$\partial_t p(s,t) = -w(s,t) p(s,t) + w(-s,t) p(-s,t) . \tag{17}$$

The flip rate

$$w(s,t) = \tfrac{1}{2} \Gamma \{ 1 - s \tan[\beta h(t)] \}$$

is determined by the local field, which—for a single spin—is just the time-dependent field $h(t)$.

In the strongly diluted network the time-dependent field $h(t) = \boldsymbol{\xi} \cdot \mathbf{m} + \sqrt{\alpha} y(t)$ has to be calculated *self-consistently*. Its systematic part $h(t) = \boldsymbol{\xi} \cdot \mathbf{m}$ is related to the overlap with pattern ξ^ν,

$$m^\nu = \langle \xi^\nu \langle s(t) \rangle_s \rangle_{y,\xi} \tag{18}$$

and its fluctuations are the time-delayed spin autocorrelations

$$\langle y(t) y(t') \rangle_{y,\xi} = C(t-t') = \langle \langle s(t) s(t') \rangle_s \rangle_{y,\xi} . \tag{19}$$

Here $\langle \ \rangle_s$ denotes the trace over s with $p(s,t)$ and $\langle \ \rangle_\xi$ and $\langle \ \rangle_y$ denote the averages over ξ and $y(t)$, respectively. We remark that this self-consistent single-spin dynamics of the Glauber type will be obtained directly, without the detour of introducing soft spins in the Appendix.

The magnetization

$$\mu(t) = \sum_{s=\pm 1} s p(s,t) \tag{20}$$

obeys the equation of motion

$$\Gamma^{-1} \partial_t \mu(t) = -\mu(t) + \tanh[\beta h(t)] , \tag{21}$$

which is solved by

$$\mu(t) = \mu(t_0) e^{-\Gamma(t-t_0)} + \Gamma \int_{t_0}^{t} dt' \, e^{-\Gamma(t-t')} \tanh[\beta h(t')] . \tag{22}$$

In the following we shall always consider the limit $t_0 \to -\infty$ such that the initial value $\mu(t_0)$ has been forgotten. To obtain the macroscopic overlap m^ν $(\nu = 1, \ldots, s)$, we average Eq. (12) over the internal noise $y(t)$,

$$m^\nu = \langle \xi^\nu \mu(t) \rangle_{\xi,y} = \left\langle \xi^\nu \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} e^{-y^2/2} \tanh \left[\beta \left[\sum_{\nu=1}^{s} m^\nu \xi^\nu + \sqrt{\alpha} y \right] \right] \right\rangle_\xi . \tag{23}$$

Since $\mu(t)$ only depends on one time t, the Gaussian functional integration is reduced to a single Gaussian integral with variance $\langle y(t) y(t) \rangle_y = C(t=0) = 1$. Equation (23) generalizes the results of Ref. 5 to an arbitrary but finite number of patterns. Of particular interest are retrieval states $m^\nu = m \delta_{\nu 1}$ which exist for $T < T_c(\alpha)$ with

$$T_c(\alpha) = \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} e^{-y^2/2} \cosh^{-2}[\sqrt{\alpha} y / T_c] \to 1 - \alpha + O(\alpha^2) \quad \text{as } \alpha \to 0 . \tag{24}$$

The transition is continuous and disappears beyond $\alpha_c = 2/\pi$, as has been discussed in Ref. 5.

The time-delayed autocorrelation

$$\langle s(t')s(t+t')\rangle_s = \sum_{s=\pm 1} sp(s,t') \sum_{s'=\pm 1} s'p(s',t+t' \mid s,t') \tag{25}$$

is defined in terms of $p(s,t)$, and the conditional probability $p(s',t'+t \mid s,t')$ of finding state s' at time $t'+t$ given state s at time t'. The conditional probability obeys the same equation of motion as $p(s,t)$, which implies an equation for the time-delayed autocorrelation

$$\Gamma^{-1}\partial_t - C(t',t+t') = -C(t',t+t') + \mu(t')\tanh[\beta h(t+t')] \; . \tag{26}$$

This is solved in explicit form by

$$C(t',t+t') = \Theta(t)\left[e^{-\Gamma t} + \Gamma^2 \int_{-\infty}^{t'} dt_1 \int_{t'}^{t+t'} dt_2 \tanh[\beta h(t_1)]\tanh[\beta h(t_2)] e^{-\Gamma(t'-t_2)} e^{-\Gamma(t+t'-t_1)}\right] + \Theta(-t)(t \to -t) \; . \tag{27}$$

The averaged autocorrelation

$$C(t) = \langle \langle s(t')s(t+t')\rangle_s \rangle_{\xi,y}$$

has to be calculated self-consistently with the internal noise related to

$$C(t-t') = \langle y(t)y(t')\rangle_y \; .$$

The existence of retrieval states ($m \neq 0$) implies a time-persistent part of the spin autocorrelation

$$q = \lim_{t\to\infty} C(t) \; . \tag{28}$$

It is then convenient to decompose the noise $y(t) = \eta(t) + z$ into a static (z) and a dynamic component (η) (Ref. 6) with

$$\langle \eta(t_1)\eta(t_2)\rangle = \tilde{C}(t_1 - t_2) = C(t_1 - t_2) - q \; , \tag{29}$$

$$\langle z^2 \rangle = q \; . \tag{30}$$

To calculate the time-persistent part of the autocorrelation, we note that for infinite time separation t, the average of

$$\langle s(t')s(t+t')\rangle_{s,\eta} \to \langle s(t')\rangle_{s,\eta} \langle s(t+t')\rangle_{s,\eta}$$

factorizes, since the correlation time of η and the external noise are finite. We can then apply the same argument as used above for the magnetization to reduce the functional integral over $\eta(t)$ to a single Gaussian integral with variance $\langle \eta(t)\eta(t)\rangle = 1 - q$. The resulting self-consistent equation for q reads

$$q = F(q,m) = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \left[\int_{-\infty}^{+\infty} \frac{d\eta}{\sqrt{2\pi}} e^{-\eta^2/2} \tanh\{\beta[m + \sqrt{\alpha(1-q)}\eta + \sqrt{\alpha q} z]\} \right]^2 . \tag{31}$$

One might also want to consider higher-order moments $\langle \langle \mu(t)\rangle_\eta^n \rangle_z$ or, in general, the probability density

$$P(M) = \langle \delta(M - \langle \mu(t)\rangle_\eta)\rangle_z \; . \tag{32}$$

Close to T_c, $P(M)$ is a Gaussian with mean m and variance $\alpha q = \alpha m^2/(1-\alpha)$ for $\alpha \neq 1$. At zero temperature, Eq. (31) has two solutions: $q = 1$ and $q = q_0(\alpha) < 1$. The first solution, $q = 1$, is unstable for all α. The second solution $q_0(\alpha)$ is easily calculated for small α,

$$\lim_{\alpha\to 0} q_0(\alpha) = 1 - \frac{8}{\pi^2} e^{-1/\alpha} \; , \tag{33}$$

and for $\alpha \to \alpha_c$, $q_0(\alpha) \simeq m^2$. Since $q_0(\alpha) < 1$, the spin autocorrelation is time dependent and hence a time-dependent noise exists even at $T = 0$. For a particular realization of synaptic bonds we expect that many attractors are time dependent with time scales which depend sensitively on the particular realization. Apparently averaging destroys all periodicity. The averaged dynamics is stochastic and the averaged correlation function decays in time to a value $q_0 < 1$.

Another interesting question is the possibility of a spin-glass state, more precisely a stationary state with zero overlap $m = 0$ and a nonzero time-persistent correlation $q \neq 0$. For such a transition to occur, one must have

$$\frac{\partial F}{\partial q}(q=0,\, m=0) = 1 = \left[\int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi}} e^{-T_{\mathrm{SG}}^2 y^2/2\alpha} \times \cosh^{-2} y \right]^2 . \tag{34}$$

This can never happen, because $\partial F(0,0)/\partial q$ is monotonically increasing with $1/T$ and as $T \to 0$ we find $\partial F/\partial q = 2/\pi < 1$. All higher-order moments are not determined self-consistently and hence are slaved to m and q.

The time decay of the averaged autocorrelation is characterized by an effective relaxation rate

$$\Gamma_{\rm eff}^{-1}=\int_0^{\infty}dt\,\tilde{C}(t)\;, \tag{35}$$

which can be obtained from Eq. (27),

$$\Gamma_{\rm eff}^{-1}=\Gamma^{-1}+\int_0^{\infty}d\tau\,g_c(\tau)(1-e^{-\Gamma\tau})\;, \tag{36}$$

with

$$g_c(t_1-t_2)=\langle\tanh[\beta h(t_1)]\tanh[\beta h(t_2)]\rangle_{\eta,z}-\langle\langle\tanh[\beta h(t_1)]\rangle_\eta\langle\tanh[\beta h(t_2)]\rangle_\eta\rangle_z\;. \tag{37}$$

We look for a possible divergence of the relaxation time $\Gamma_{\rm eff}^{-1}$, which implies an algebraic decay of $\tilde{C}(t)$. In that case, the main contribution to $g_c(t)$ is due to contractions of the Gaussian fields $h(t_1)$ and $h(t_2)$ which involve the same time (either t_1 or t_2) except for one cumulant, i.e., $g_c(t)=K\tilde{C}(t)$ with

$$K=\alpha\beta^2\int_{-\infty}^{+\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\left[\int_{-\infty}^{+\infty}\frac{d\eta}{\sqrt{2\pi}}e^{-\eta^2/2}\cosh^{-2}\{\beta[m+\sqrt{\alpha q}\,z+\sqrt{\alpha(1-q)}\,\eta]\}\right]^2\;. \tag{38}$$

Substituting this result into Eq. (36) yields

$$(1-K)\Gamma_{\rm eff}^{-1}\simeq{\rm const}\;. \tag{39}$$

We first consider temperatures $T\geq T_c$, such that $m=q=0$. Comparison with Eq. (34) shows that the condition for criticality of $\Gamma_{\rm eff}^{-1}$ is the same as the condition for a spin-glass transition to occur. Hence $\Gamma_{\rm eff}^{-1}$ cannot diverge: it remains finite at $T_c(\alpha)$ and also in the limit $T\to0$ for $\alpha>\alpha_c$. In the retrieval phase, $T<T_c(\alpha)$, the constant K can be easily calculated for $T=0$ and $q\neq1$,

$$K=\frac{2}{\pi}\frac{e^{-m^2/\alpha(1+q)}}{(1-q^2)^{1/2}}\;. \tag{40}$$

In the limit $\alpha\to0$, $K=\frac{1}{2}$, whereas for $\alpha\to\alpha_c$, $K\to2/\pi$ and it remains $2/\pi$ for $\alpha_c\leq\alpha<1$. Hence we expect $\Gamma_{\rm eff}$ to be finite everywhere in the whole phase diagram.

The time-dependent response function can also be calculated exactly. We differentiate Eq. (22)

$$\partial\mu(t_1)/\partial h(t_2)=\Theta(t_1-t_2)\Gamma e^{-\Gamma(t_1-t_2)}\times\cosh^{-2}[\beta h(t_2)] \tag{41}$$

and average over the internal noise to obtain

$$G(t)=\langle\partial\mu(t_1)/\partial h(t_2)\rangle_h=\Theta(t_1-t_2)\Gamma e^{-\Gamma(t_1-t_2)}G(\omega=0)\;. \tag{42}$$

Here,

$$G(\omega=0)=\int_{-\infty}^{+\infty}\frac{dy}{\sqrt{2\pi}}e^{-y^2/2}\beta\cosh^{-2}(\beta\sqrt{\alpha}\,y) \tag{43}$$

is the static susceptibility. The imaginary part of $G(\omega)$ is a simple Lorentzian at all temperatures, as one would expect, since the static disorder only generates internal noise, but does not modify the response.

IV. TIME EVOLUTION OF TWO INITIAL CONFIGURATIONS

It is remarkable that the equation for q coincides with the equation for the overlap function

$$\hat{q}=\lim_{t\to\infty}\left[\frac{1}{N}\sum_i\langle s_i(t)\overline{s}_i(t)\rangle_{s,\eta,z}\right]$$

determined in Ref. 5 for a discrete time dynamics (see Eq. 27 of Ref. 5). $\{s_i(t)\}$ and $\{\overline{s}_i(t)\}$ denote the time evolution of two configurations which start out with macroscopic overlaps with *one* of the stored patterns, i.e.,

$$m_0^\nu=\frac{1}{N}\sum_{i=1}^N\xi_i^\nu\langle s_i(t_0)\rangle=m_0\delta_{\nu1}$$

and

$$\overline{m}_0^\nu=\frac{1}{N}\sum_{i=1}^N\xi_i^\nu\langle\overline{s}_i(t_0)\rangle=\overline{m}_0\delta_{\nu1}\;.$$

To study this quantity in our continuous time model we duplicate the spin variables $s_{i\gamma}$ $(\gamma=1,2)$. The equations of motion for s_{i1} and s_{i2} are given by (3), but the external noise sources f_{i1} and f_{i2} are *uncorrelated.* Repeating all the steps discussed in Sec. 2 we end up with a problem of two Ising spins in time-dependent local fields $h_\gamma=m_\gamma\xi_1+\sqrt{\alpha}y_\gamma(t)$ which are correlated by

$$\langle y_\alpha(t)y_\beta(t')\rangle_y=C_{\alpha\beta}(t-t')\;.$$

The overlap of the two configurations with pattern $\{\xi_i^1\}$ is the same, i.e., $m_\gamma=m$ [Eq. (23)], provided we take the limit $t_0\to-\infty$, such that the initial overlap is lost. The diagonal elements of $C_{\alpha\beta}$ are the usual spin autocorrelation function [Eq. (27)], whereas the off-diagonal elements are magnetization correlations by definition,

$$C_{12}(t-t')=\langle\mu_1(t)\mu_2(t')\rangle_{\eta,z}\;. \tag{44}$$

Note that $C_{\alpha\alpha}(t,t)=1$ but $C_{12}(t,t)\neq1$, in general. From the two spin Glauber dynamics we find

$$2C_{12}(0)=\langle\mu_1(t)\tanh[\beta h_2(t)]\rangle_{\eta,z}+\langle\mu_2(t)\tanh[\beta h_1(t)]\rangle_{\eta,z}\;. \tag{45}$$

To get a self-consistent solution we also need the delayed autocorrelation $C_{12}(t',t+t')$ which we get from the generalization of Eq. (26) to a two-spin problem,

$$\Gamma^{-1}\partial_tC_{\alpha\beta}(t',t+t')=-C_{\alpha\beta}(t',t+t')+\mu_\alpha(t')\tanh[\beta h_\beta(t+t')]\;. \tag{46}$$

Averaging over the local fields requires the calculation of

$$g_{\alpha\beta}(t-t')\equiv\langle \tanh\{\beta[m+\sqrt{\alpha}y_\alpha(t)]\}\tanh\{\beta[m+\sqrt{\alpha}y_\beta(t')]\}\rangle_{\eta,z} \ . \tag{47}$$

Note that this average only depends on $C_{\alpha\alpha}(t,t)$ (which equals one for Ising spins) and $C_{12}(t-t')$, so that (46) becomes a self-consistent equation for $C_{12}(t-t')$ after averaging. It is then straightforward to show that Eqs. (45) and (46) have a time-independent solution $C_{12}(t)=\hat{q}$, which is the same as the time-persistent part of the autocorrelation. Hence in the subspace of points belonging to one attractor the initial condition is lost in a finite time and there is only one characteristic distance $d=\frac{1}{2}(1-q)$ in this subspace.

Are there other solutions, $C_{12}(t)\neq q$? To investigate such a possibility we decompose $C_{\alpha\beta}(t)=q+\widetilde{C}_{\alpha\beta}(t)$ and the noise accordingly, $y_\alpha(t)=z+\eta_\alpha(t)$, with

$$\langle \eta_\alpha(t_1)\eta_\beta(t_2)\rangle=\widetilde{C}_{\alpha\beta}(t_1-t_2) \ .$$

The diagonal components $\widetilde{C}_{\alpha\alpha}(t)$ decays in time and is the solution of Eqs. (27)–(31). The off-diagonal component $\widetilde{C}_{12}(t)$ has yet to be determined from Eq. (46), which is a *homogeneous* equation for $\widetilde{C}_{12}(t)$. Hence one might expect a phase transition to a low-temperature phase with a spontaneous nonzero value of $\widetilde{C}_{12}(t)=C_{12}(t)-q$. We now show that a continuous transition cannot occur in the model under consideration. We assume $\widetilde{C}_{12}(t)$ to be small and expand $g_{12}(t)$ up to linear order

$$g_{12}(t)=q+K\widetilde{C}_{12}(t) \ , \tag{48}$$

where the constant K is given by Eq. (38). The linear equation is solved by an exponential ansatz $\widetilde{C}_{12}(t)=A_\delta e^{-\Gamma\delta t}$ with $(1+\delta)=K$. Hence a nonzero A_δ first appears for $\delta=0$, when $K=1$. This is identical with the criterion for a divergence of the relaxation time of the autocorrelation [Eq. (39)]. Hence such a transition cannot occur and $\widetilde{C}_{12}=0$ for all T and $\alpha<1$.

V. CONCLUSIONS AND OUTLOOK

We have shown that many dynamical properties of an asymmetrically diluted neural network can be obtained analytically in the limit of strong dilution [$c=O(1/N)$], and many stored patterns ($p=\alpha Nc$) for a continuous time dynamics. The problem can be reduced to a self-consistent Glauber dynamics of a single Ising spin in a time-dependent field. The static disorder generates internal time-dependent noise, which does not influence the response function. For low temperatures we still find stable retrieval states. The effective relaxation rate of the autocorrelation remains finite for all values of T and α and a transition into a spin-glass phase does not occur. However, even at $T=0$ the long-time limit of the spin autocorrelation function is not equal to one in the retrieval phase, due to the presence of the internal noise. This indicates a complicated structure of attractors. In fact, we expect many of the attractors to become time dependent with time scales which depend sensitively on the particular realization. Additional information on the structure of the attractors is obtained by studying the time-evolution of two different configurations starting in the vicinity of one stored pattern. We find that the overlap of the two configurations equals the time-persistent part of the autocorrelation. As a consequence there is only one characteristic distance between configurations on the attractor.

All three criteria for the appearance of (1) a time-persistent part of the autocorrelation, (2) a divergence of the relaxation time of the autocorrelation, and (3) a distance between two configurations which is different from the long-time limit of the autocorrelation, coincide whenever they apply. In the paramagnetic phase ($m=0$) all three criteria can be used to determine the spin-glass transition, whereas in the retrieval phase ($m\neq 0$) only the latter two are applicable.

One interesting open problem is the size and shape of the basins of attraction. So far no analytical means exist to attack this problem, which requires the solution of an initial value problem. Another open question is, how much of our results depend on the particle choice of dynamics (Glauber)? Certainly the basins of attraction will depend on the particular dynamics, but in addition, even the attractors might do so. Finally it would be interesting to generalize the calculations to a model with either a small amount of correlated bonds and/or weaker dilution. In both cases one can still reduce the many-spin problem to the problem of the dynamics of a single spin. However, the self-consistent equations are more complicated and the absence of a fluctuation-dissipation theorem makes it difficult to extract the static properties.

ACKNOWLEDGMENTS

We wish to thank B. Derrida and M. Mézard for interesting discussions.

APPENDIX

Results were obtained in the main text on the basis of a soft spin dynamics for a neural network. A recent functional-integral formulation of Glauber dynamics[8] offers the possibility of performing all calculations for Ising spins directly. We want to show how to rederive our results by this method.

Following Ref. 8, we consider a single-spin flip dynamics for the probability $P\{\sigma,t\}=P(\sigma_1,\sigma_2,\ldots,\sigma_N,t)$ of a network of N Ising spins $\sigma_i=\pm1$,

$$\partial_t P\{\sigma,t\}=-\sum_{i=1}^{N}(2-\zeta_i)\frac{\Gamma}{2}[1-\sigma_i\tanh(\beta h_i)]P\{\sigma,t\} \ . \tag{A1}$$

ζ_i denotes the spin flip operator ($\zeta_i\sigma_i=-\sigma_i\zeta_i$), and the local field h_i is given by

$$h_i(t)=b_i(t)+\sum_{j\,(\neq i)} c_{ij}J_{ij}\sigma_j \ , \tag{A2}$$

where $b_i(t)$ denotes an additional external field. In Ref. 8 a functional integral for dynamic spin correlations and response functions is derived, which we use in the form

$$Z[l,b]=\int D(\tilde{h},h)D(\tilde{s},s)D(\tilde{\varphi}\varphi)e^{-L} \ , \tag{A3}$$

with

$$L=-\int dt \sum_{\substack{j,i=1\\(i\neq j)}}^{N} c_{ij}J_{ij}\tilde{h}_i(t)s_j(t)+L_0 \ . \tag{A4}$$

L_0 is local in the site index i and has the form

$$L_0=\int dt \sum_{i=1}^{N} \{\tilde{s}_i(s_i-\varphi_i)+\tilde{h}_i(h_i-b_i)-l_i s_i +\tilde{\varphi}_i[\Gamma^{-1}\partial_t\varphi_i+\varphi_i-\tilde{s}_i(1-\varphi_i^2) -\tanh(\beta h_i)]\} \ . \tag{A5}$$

The first term in L can be treated in exactly the same way as the corresponding term of the soft spin model. After the reduction to a single-spin problem is performed, we integrate over $\tilde{h}$, h, $\tilde{s}$, and s. Analogously to Eq. (15), we obtain, for the case of Glauber dynamics,

$$L_s+L_{0s}=-\int dt \left\{ l(t)\varphi(t)+\tilde{\varphi}[\Gamma^{-1}\partial_t\varphi+\varphi-\Gamma^{-1}l(1-\varphi^2)-\tanh\left[\beta\left[b+\sqrt{\alpha}y(t)+\sum_{\nu=1}^{s} m^{\nu}\xi^{\nu}\right]\right]\right\} \ . \tag{A6}$$

The Gaussian distribution of the random field $y(t)$ is determined by

$$\langle y(t)y(t')\rangle = C(t-t') \ . \tag{A7}$$

Here $C(t-t')$ denotes the correlation function $\langle\sigma(t)\sigma(t')\rangle$ of a single Ising spin with a self-consistent Glauber dynamics defined by Eqs. (A6) and (A7).

Spin correlations and response functions may be obtained from $Z[l,b]$ by differentiating with respect to l or b; e.g.,

$$\begin{aligned}\mu(t)&=\langle\sigma(t)\rangle=\{\delta Z[l,b]/\delta l(t)\}_{l=b=0} \ ,\\ C(t-t')&=\langle\sigma(t)\sigma(t')\rangle=\{\delta^2 Z[l,b]/\delta l(t)\delta l(t')\}_{l=b=0} \ ,\\ G(t-t')&=\{\delta\langle\sigma(t)\rangle/\delta b(t')\}_{b=0}=\{\delta^2 Z[l,b]/\delta l(t)\delta b(t')\}_{l=b=0} \ .\end{aligned} \tag{A8}$$

The self-consistent Glauber dynamics [(A6) and (A7)] is identical to the one we used in the main text as a limiting form of the soft spin dynamics. It is now straightforward to show that (A8) leads to (23), (27), and (43). For the purpose of illustration let us consider $C(t-t')$. From (A8) we get

$$C(t-t')=\langle\{\varphi(t)+\Gamma^{-1}\tilde{\varphi}(t)[1-\varphi^2(t)]\}\{\varphi(t')+\Gamma^{-1}\tilde{\varphi}(t')[1-\varphi^2(t')]\}\rangle_{l=0} \ . \tag{A9}$$

For $l=0$ the equation of motion for φ becomes linear and (A9) is easily evaluated, using

$$\langle\tilde{\varphi}(t)\varphi(t')\rangle=\Theta(t'-t)\Gamma\exp[\Gamma(t-t')]$$

and $\Theta(t=0)=0$.

$$C(t-t')=\langle\varphi(t)\varphi(t')\rangle +\Theta(t'-t)e^{\Gamma(t-t')}\langle 1-\varphi^2(t)\rangle +\Theta(t-t')e^{\Gamma(t'-t)}\langle 1-\varphi^2(t')\rangle \ . \tag{A10}$$

Inserting

$$\varphi(t)=\Gamma\int_{-\infty}^{t} dt'\, e^{-\Gamma(t-t')}\tanh[\beta h(t')] \tag{A11}$$

in (A10) we recover Eq. (27).

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