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Unitary point of view on the puzzling problem of nonlinear systems driven by colored noise

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It is shown that the projection approach offers a straightforward way to elucidate the connections among the current theories on the effects of colored noise. An error indicator with a simple analytical expression is built up which allows us to evaluate the error associated with each approximation.

It is well known[1] that the Fokker-Planck equation for the time evolution of the probability distribution $\sigma(\mathbf{a};t)$ can be derived from the more detailed description

$$\frac{\partial}{\partial t}\rho(\mathbf{a},\mathbf{b};t)=\mathcal{L}\rho(\mathbf{a},\mathbf{b};t) \ , \tag{1}$$

via contraction over the set of irrelevant variables $\mathbf{b}$ $[\sigma(\mathbf{a};t)\equiv\int d\mathbf{b}\rho(\mathbf{a},\mathbf{b};t)]$. This program is carried out through a projection method[1] a basic step of which is the repartition of the dynamical operator $\mathcal{L}$ into a perturbation term, $\mathcal{L}_1$, an unperturbed part concerning the variables of interest $\mathbf{a}$, $\mathcal{L}_a$, and another unperturbed part concerning the irrelevant variables $\mathcal{L}_b$. It has been pointed out[1,2] that this choice must be made keeping clearly in mind the physical nature of the problem under investigation. Faetti, Fronzoni, and Grigolini[2] illustrated this basic aspect in the case of systems of Hamiltonian nature and showed how to divide the dynamical operator $\mathcal{L}$ so as to fit the constraint of canonical distribution. This allowed them to recover the nonlinear fluctuation-dissipation relations discovered by Lindenberg and Seshadri[3] (see also Ref. 4 and the more recent and remarkable work by Ramshaw and Lindenberg[5]). The very same problem has been discussed also in the case where Eq. (1) does not satisfy the requirement of detailed balance.[1] In Ref. 1, however, this discussion has been limited to the case of a linear system driven by colored additive noise.

In this Rapid Communication, we extend that discussion to the more general case

$$\begin{aligned} \dot{x}&=\varphi(x)+\psi(x)\xi(t) \ , \\ \dot{\xi}&=-\frac{1}{\tau}\xi+f(t) \ , \end{aligned} \tag{2}$$

$f(t)$ being white Gaussian noise defined by

$$\begin{aligned} \langle f(t)\rangle&=0 \ , \\ \langle f(0)f(t)\rangle&=2Q\delta(t) \ . \end{aligned} \tag{3}$$

In this case the operator $\mathcal{L}$ reads

$$\mathcal{L}=-\frac{\partial}{\partial x}\varphi(x)-\xi\frac{\partial}{\partial x}\psi(x)+\frac{1}{\tau}\left[\frac{\partial}{\partial\xi}\xi+\langle\xi^2\rangle\frac{\partial^2}{\partial\xi^2}\right] \ . \tag{4}$$

We identify the set of variables of interest $\mathbf{a}$ with x and the irrelevant ones $\mathbf{b}$ with ξ. It seems natural (in the former way of dividing $\mathcal{L}$) to assume $\mathcal{L}_a$, $\mathcal{L}_b$, and $\mathcal{L}_1$ to be, respectively, defined by

$$\begin{aligned} \mathcal{L}_a&=-\frac{\partial}{\partial x}\varphi(x) \ , \\ \mathcal{L}_b&=\frac{1}{\tau}\left[\frac{\partial}{\partial\xi}\xi+\langle\xi^2\rangle\frac{\partial^2}{\partial\xi^2}\right] \ , \\ \mathcal{L}_1&=-\xi\frac{\partial}{\partial x}\psi(x) \ . \end{aligned} \tag{5}$$

It has been shown[6] how to extend the projection operator approach so as to properly deal with the problem posed by a nonvanishing $\mathcal{L}_a$. This theoretical investigation has been more recently supplemented by a detailed comparison with the results of analog and digital simulation.[7,8] By using the projection method of Refs. 6–8 we obtain

$$\frac{\partial}{\partial t}\sigma(x;t)=\left[-\frac{\partial\varphi}{\partial x}+\left(\int_0^t W(s)ds\right)\right]\sigma(x;t) \ , \tag{6}$$

where (to order $\mathcal{L}_1^2$)

$$W(s)=\langle\xi^2\rangle\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)e^{-[1/\tau-\Pi(x)]s} \ , \tag{7}$$

the exponential $\exp(-\Pi(x)s)$ being formally defined by

$$e^{-\Pi(x)s}=\sum_{r=0}^{\infty}\frac{(-s)^r}{r!}\Pi^{(r)}(x) \ , \tag{8}$$

with

$$\Pi^{(r)}(x)=\pi^{(r)}(x)/\psi(x) \ ; \tag{9}$$

$$\begin{aligned} \pi^{(0)}(x)&=\psi(x) \ , \\ \pi^{(r)}(x)&=\pi^{(r-1)}(x)\varphi'(x)-\varphi(x)\pi^{(r-1)'}(x) \ . \end{aligned} \tag{10}$$

If we are interested in the steady-state property we can set the upper limit of time integration on the right-hand side (rhs) of Eq. (6) equal to infinity. By adapting the fourth-order calculation recently carried out by Faetti *et al.*[8] in the purely additive case to the more general multiplicative case of Eq. (1) we then obtain (a detailed demonstration will be given in a more extended paper)

$$\frac{\partial}{\partial t}\sigma(x;t)=\left[-\frac{\partial}{\partial x}\varphi(x)+\sum_{n=0}^{\infty}D\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\Pi^{(n)}(x)\tau^n+\tfrac{3}{2}D\tau^2\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\Pi^{(1)\prime}(x)\varphi(x)\right]\sigma(x;t)\ , \tag{11}$$

where $D\equiv\langle\xi^2\rangle\tau$. This equation is exact up to order τ^2. This is a remarkable result which needs to be properly commented on. The second term on the rhs of Eq. (11) is nothing but the Laplace transform at zero frequency of the operator of Eq. (7). This coincides with the predictions of the best Fokker-Planck approximation.[9,10] The third term on the rhs of Eq. (11) was obtained by extending to the multiplicative case the method of Ref. 8. From the contributions to order $\mathcal{L}_1^4$ a suitable nonstandard diffusion term is drawn which has the property of remaining competitive with the best Fokker-Planck approximation even in the seemingly safe limit $D\to 0$. This is rendered evident by replacing that nonstandard diffusion term with an "equivalent" one of standard form (according to the prescriptions of Ref. 11). The resulting term is precisely the third term on the rhs of Eq. (11). It is expected that the contributions to order $\mathcal{L}_1^{2n}(n>2)$ will provide analogous terms of the order $D\tau^n$.

The remarkable result of Eq. (11) means that it is useless to use the best Fokker-Planck equation at orders higher than the first in τ, without supplementing it with suitable corrections stemming from the perturbation terms of order $\mathcal{L}_1^n$ (with $n>2$) [third term on the rhs of Eq. (11)].

The assumption that $\Pi^{(1)\prime}(x)$ can be neglected has a twofold effect. First, from Eqs. (9) and (10) we obtain

$$\Pi^{(n)}(x)=[\Pi^{(1)}(x)]^n\ . \tag{12}$$

Furthermore, we are allowed to disregard the last term on the rhs of Eq. (11). This equation has thus to be replaced by

$$\frac{\partial}{\partial t}\sigma(x;t)=\left[-\frac{\partial}{\partial x}\varphi(x)+D\frac{\partial}{\partial x}\Psi(x)\frac{\partial}{\partial x}\times\frac{\psi(x)}{1-\tau\Pi^{(1)}(x)}\right]\sigma(x;t)\ , \tag{13}$$

which coincides precisely with the Fox theory.[12,13]

We thus reach the following remarkable result: Although the Fox theory can be seen as being, so to speak, an approximation to another approximation[7,14] (i.e., the best Fokker-Planck approximation[9,10]), it has the significant effect of making negligible the corrections to the Fokker-Planck structure stemming from the perturbation terms of the order $\mathcal{L}_1^n(n>2)$. As already noticed by Faetti, Fronzoni, Grigolini, and Mannella[7] it has also the remarkable effect of producing an exact equilibrium distribution at $\tau=\infty$. We shall come back to this interesting property in the final part of this paper.

Even the decoupling theory,[15] in the form recently generalized by Fox and Roy,[16] can be recovered within the context of the projection method. Let us make the assumption that

$$\Pi^{(1)}(x)\simeq\Pi^{(1)}(x_s)=\varphi'(x_s)-\frac{\psi'(x_s)}{\psi(x_s)}\varphi(x_s)\ , \tag{14}$$

where $x_s\equiv\langle x(t)\rangle$. Equation (11) then reduces to Eq. (13) with $\Pi^{(1)}(x)$ replaced by that of Eq. (14) (i.e., precisely the decoupling theory of Ref. 16). In addition to making the corrections to the standard Fokker-Planck form vanish, the decoupling theory makes the diffusion coefficient of Eq. (13) always positive. This appealing aspect, however, is associated with the approximation of Eq. (14) and, thereby, with an error, which can be evaluated within the context of the projection method.[6] In the additive case $\varphi=ax-\beta x^3$ ($a>0$, $\beta>0$) it is straightforward to show that the relative error on the diffusion coefficient at x is given by $\tau\beta(x^2-x_s^2)D/(a-3\beta x_s^2)$, where x_s has presumably to be intended as being the mean value of x within a well [within the standard decoupling theory[15] relative error would read $\tau\beta(x^2-\langle x^2\rangle)D/(a-3\beta\langle x^2\rangle)$]. This shows that upon increasing $\beta\tau$ we must have recourse to noises of decreasing intensity for the decoupling theory to work. This error is precisely the main reason why the decoupling theory fails in correctly reproducing anharmonic effects like the transition from the one-to-two-mode distribution, which takes place upon increase of τ.[7] Note that both the Fox theory and the best Fokker-Planck approximation, on the contrary, are proven by Eq. (11) to be exact at the order τ.

As a quite remarkable aspect of Eq. (11), we would like to point out that it suggests that if $\Pi^{(1)\prime}(x)$ exactly vanishes the resulting Fokker-Planck equation turns out to be exact. This important property can be shown as follows. Let us make the change of variables

$$y=\varphi(x)/[\Pi^{(1)}(x)\psi(x)]\ . \tag{15}$$

When $\Pi^{(1)\prime}(x)=0$, Eq. (2) is then proven to read

$$\dot y=\Pi^{(1)}y+\xi\ ,\qquad \dot\xi=-\frac{1}{\tau}\xi+f(t)\ . \tag{16}$$

This is a linear stochastic differential equation, the explicit form of the "exact" contracted Fokker-Planck equation to be associated with it is well known[1] (see also, the final part of this paper). By coming back to the original variable x the exact Fokker-Planck equation reduces to Eq. (13). Note that in this special case the best Fokker-Planck equation,[9,10] the decoupling theory,[15,16] and the Fox theory[13] coincide.

The class of systems satisfying the condition $\Pi^{(1)\prime}(x)=0$ is very extended. A simple case of special interest is $\varphi=0$. In this case it is easily proven that Eq. (6) leads to precisely the same result as the Kubo theory of cumulants[17] (another example of exact treatment). When $\varphi\neq 0$ we find that for $\Pi^{(1)\prime}(x)$ to vanish $\psi(x)$ must be related to $\varphi(x)$ through

$$\psi(x)=C\varphi(x)/\exp\left[A\int[1/\varphi(x')]dx'\right]\ ,$$

where A and C are arbitrary constants. An interesting case fulfilling this constraint is $\varphi=\alpha x-\beta x^3, \psi=x^3$ dealt with by Suzuki.[18]

In Ref. 1 a different repartition of $\mathcal{L}$ (leading to $\mathcal{L}_a=0$) has been suggested which has been proven to lead to a faster attainment of the correct equilibrium distribution of the variable x (note that this advantage is now counterbalanced by the agile way of dealing with $\mathcal{L}_a \neq 0$ developed in Ref. 6). This repartition is naturally introduced by adopting the variables x and $v \equiv \varphi(x)+\xi\psi(x)$ rather than x and ξ. As discussed in Ref. 19 this choice is dictated by the rules for the determination of the best basis set (the so-called Mori basis set[19]). We then obtain

$$\begin{aligned} &\mathcal{L}_a=0 , \\ &\mathcal{L}_1=-\frac{1}{\tau}\frac{\partial}{\partial v}\varphi(x)-v\frac{\partial}{\partial x}-\frac{\partial}{\partial v}v^2\frac{\psi'(x)}{\psi(x)} , \\ &\mathcal{L}_b=\frac{\partial}{\partial v}v\gamma(x)+\frac{D}{\tau^2}\frac{\partial^2}{\partial v^2}\psi^2(x) , \end{aligned} \tag{17}$$

where $\gamma(x)\equiv 1/\tau-\Pi^{(1)}(x)$. According to Refs. 1 and 6 we must use the projection operator P defined by

$$\begin{aligned} P\rho(x,v;t) &= \rho_{\mathrm{eq}}(v/x)\sigma(x;t) \\ &\equiv \rho_{\mathrm{eq}}(v/x)\int dv\,\rho(x,v;t) , \end{aligned} \tag{18}$$

where $\rho_{\mathrm{eq}}(v/x)$ denotes the x-dependent "thermal bath" equilibrium distribution. $\rho_{\mathrm{eq}}(v/x)$ is characterized by the significant properties

$$\int dv\,\rho_{\mathrm{eq}}(v/x)=1,\ \mathcal{L}_b\rho_{\mathrm{eq}}(v/x)=0 , \tag{18$'$}$$

which are lost when $\gamma(x)<0$, thereby preventing the projection method from working in this critical region. Since the v-equilibrium distribution is not independent of x, the projection operator P also turns out to be dependent on x. This involves only minor technical difficulties in comparison with the ordinary case of projection operators independent of the variable of interest[1] (note that in the linear and additive case $\mathcal{L}_b$, and therefore P, turns out to be independent of x, thereby recovering the conditions behind this ordinary case).

Prior to the application of a rigorous method of adiabatic elimination of fast variables it is not possible to predict whether the Itô or the Stratonovich form of diffusion equation has to be used.[20] In many cases[2,21] neither form is reliable. In this case a calculation at the order $\mathcal{L}_1^2$ with the projection operator P of Eq. (17) leads us to

$$\frac{\partial}{\partial t}\sigma(x;t)=\left[\frac{D}{\tau^2}\frac{\partial}{\partial x}\frac{\psi(x)}{\gamma(x)}\frac{\partial}{\partial x}\frac{\psi(x)}{\gamma(x)}-\frac{1}{\tau}\frac{\partial}{\partial x}\frac{\varphi(x)}{\gamma(x)}\right]\sigma(x;t) , \tag{19}$$

which formally coincides with the Stratonovich diffusion equation associated with the naive Langevin equation of Ref. 22. The authors of Ref. 22 note that the diffusion coefficient of this equation is always positive ($\gamma^2>0$ even when $\gamma<0$). Actually it appears clear from the analysis of the present paper [see our remarks on the conditions of Eq. (18′)] that Eq. (19) is completely invalidated in the region $\gamma(x)<0$. The equilibrium distribution of Eq. (19) is easily proven to coincide with the equilibrium distribution of the Fox theory (as already pointed out by Jung and Hänggi.[22] The authors of Ref. 22 rely indeed on this property to recover the same conclusion as that by Faetti *et al.*:[7] *The Fox theory provides an exact equilibrium distribution in the limiting case* $\tau\to\infty$.

This incontrovertible conclusion has been reached by Faetti *et al.*[7] by using the same procedure as that adopted (within a somewhat different context) by Sancho, San Miguel, Katz, and Gunton.[9] In the additive case $\psi'=0$ the same conclusion can be reached by noticing that the "intensity" of $\mathcal{L}_1$ [Eq. (17)] becomes vanishingly small for $1/\tau\to 0$, thereby making the calculation at the order $\mathcal{L}_1^2$ virtually exact. Equation (19) (the result of this second-order calculation), as above remarked, has the same equilibrium distribution as the Fox theory, rendering it exact for $1/\tau\to 0$.

Within the context of the projection method it is possible to supplement Eq. (19) with the explicit expression of the first nonvanishing corrections to it. These are too involved to be discussed in this short note. It is illuminating, however, to compare the results of the two different repartitions used in this paper in the linear case $\varphi(x)=-\alpha x$. The former and the latter way of dividing $\mathcal{L}$ in this simple case lead us to

$$\frac{\partial}{\partial t}\sigma(x;t)=\alpha\left[\frac{\partial}{\partial x}x+\frac{\langle\xi^2\rangle\tau}{\alpha(1+\alpha\tau)}\frac{\partial^2}{\partial x^2}\right]\sigma(x;t) \tag{20}$$

and

$$\frac{\partial}{\partial t}\sigma(x;t)=\alpha(1+\alpha\tau)^{-1}\left[1+\frac{\alpha\tau}{(1+\alpha\tau)^2}+2\frac{(\alpha\tau)^2}{(1+\alpha\tau)^4}+\cdots\right]\left[\frac{\partial}{\partial x}x+\frac{\langle\xi^2\rangle\tau}{\alpha(1+\alpha\tau)}\frac{\partial^2}{\partial x^2}\right]\sigma(x;t) , \tag{21}$$

respectively.

First of all, let us note that Eqs. (20) and (21) are characterized by the same equilibrium distribution [we denote it by $\sigma_{\mathrm{eq}}(x)$]. Furthermore, the friction term appearing in Eq. (21) is proven to be nothing but the expansion of the eigenvalue ϵ,

$$\epsilon=\left\{\left[\alpha+\frac{1}{\tau}\right]-\left[\left[\alpha+\frac{1}{\tau}\right]^2-\frac{4\alpha}{\tau}\right]^{1/2}\right\}\Big/2 \tag{21$'$}$$

in terms of the perturbation parameter $4\alpha\tau/(1+\alpha\tau)^2$. It is thus straightforward to show that at $\alpha<1/\tau$ and $\alpha>1/\tau$ this eigenvalue turns out to coincide with α and $1/\tau$, respectively.

Then the two results of Eqs. (20) and (21) must be compared with the exact correlation function

$$\langle x(0)x(t)\rangle=C_1e^{-\alpha t}+C_2e^{-t/\tau} , \tag{22}$$

while bearing in mind that they imply statistical averaging over

$$\rho_{\mathrm{eq}}^{(A)}(x,\xi)=\sigma_{\mathrm{eq}}(x)\rho_{\mathrm{eq}}(\xi)$$

and

$$\rho_{eq}^{(B)}(x,v)=\sigma_{eq}(x)\rho_{eq}(v) \ ,$$

respectively [$\rho_{eq}(\xi)$ and $\rho_{eq}(v)$ denote the Gaussian equilibrium distributions of ξ and v, respectively]. Only $\rho_{eq}^{(B)}(x,v)$ is a genuine equilibrium distribution of the whole operator $\mathcal{L}$. In other words, the former repartition leads to a preparation equivalent to setting $C_2=0$, whereas the latter one implies $C_1\alpha+C_2/\tau=0$.

In the spirit of adiabatic elimination of fast variables[1] the drift term should be proportional to α or $1/\tau$ according to whether $\alpha<1/\tau$ or $1/\tau<\alpha$. This important theoretical constraint is only partially fulfilled by the former repartition (namely in the case $\alpha<1/\tau$), whereas it is completely satisfied by the second one. However, the former repartition when $\alpha<1/\tau$ leads to an exact result already at the order $\mathcal{L}_1^2$, while the second one reaches the exact values α and $1/\tau$ after a resummation at infinite order in $\mathcal{L}_1$. Note that the second-order calculation resulting in Eq. (19) corresponds to replacing the whole term between square brackets in Eq. (21) with 1.

These results (concerning the simple linear case) suggest that the latter repartition has to be preferred to the former one in the high-memory limit (where it produces more accurate dynamical properties), whereas the former one (though leading to an exact equilibrium distribution even in the limit $\tau\rightarrow\infty$) turns out to be more advantageous in the short-τ region where it provides a faster convergence to the correct drift term. The importance of a description equivalent to the former repartition has also been stressed recently by Der.[23]

We can conclude by saying that the projection method of Refs. 6–8 has the capability of establishing clear relations among the current theories on the nonlinear systems driven by colored noise, including also the most recent ones.[13,14,16,22]

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