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Soap-bubble growth

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A simple theory describing the dynamics of two-dimensional arrays of soap bubbles is proposed, and compared with a recent experiment. The average area of bubbles scales linearly at late times. Agreement with experiment is satisfactory, although not conclusive.

A recent experiment[1] provides motivation to investigate the dynamics of two-dimensional soap-bubble arrays.[2] The walls of the bubbles are layers of soap that always meet at vertices in threes, forming angles of 120°, as shown in Fig. 1. Glazier, Gross, and Stavans[1] create bubbles between two closely spaced glass plates, and watch them grow for periods up to several weeks. They measure the average area of bubbles as a function of time, and find that after initial transients and independent of initial conditions, the average area of bubbles increases as $t^{0.6}$. This result is particularly striking, since a very simple dimensional argument indicates that the exponent should be 1.[2] From thermodynamics follows von Neumann's law,[3] that the area of every bubble increases or decreases as

$$\dot{A} = \kappa(n-6) \ , \tag{1}$$

where n is the number of sides of a bubble, and κ is a time-independent constant. Three-, four-, and five-sided bubbles shrink; those with more than seven sides grow. Two-sided bubbles are never seen. Since every member of the ensemble grows linearly, it is difficult to see how the ensemble as a whole could violate this rule at long times. I will propose here a simple theory that attempts to mimic the dynamics of these bubbles fairly realistically. The theory scales linearly at late times. However, its agreement with experiment is sufficiently good that it is unclear whether new and complicated correlations need to be included to make the theory even qualitatively correct, or whether the remaining discrepancies are simply due to finite-size effects in the experiment.

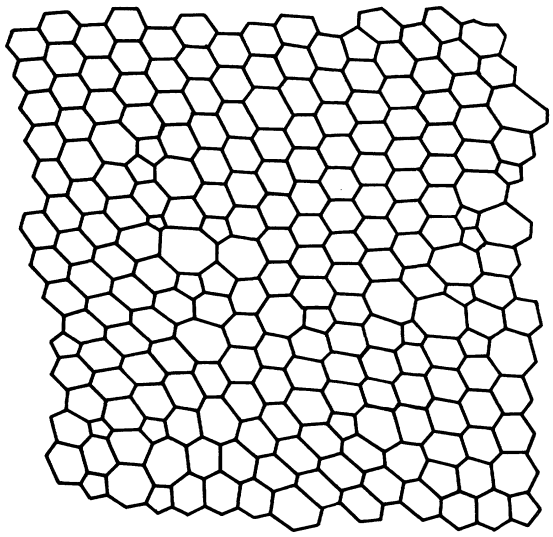

FIG. 1. This sketch shows actual cells from the experiment of Ref. 1. The experiment begins with more than 7000 cells.

One understands only half the dynamics of bubbles from von Neumann's law. In addition, one must know how a given bubble can alter its number of sides. This may happen in two ways. First, adjacent bubbles may exchange sides, as illustrated in Fig. 2(a). This process is observed experimentally to be so rare that the theory will neglect it. Second, a bubble may find itself next to a three-, four-, or five-sided bubble that shrinks down to zero area, and disappears. When a three-sided bubble disappears, its three neighbors lose a side. When a four-sided bubble disappears, two of its neighbors lose a side, and when a five-sided bubble disappears, two of its neighbors lose a side, and one of them gains a side. The last case is illustrated in Fig. 2(b). All of this dynamics is consistent with the geometrical constraint that the average number of sides of the bubbles is six, which follows from Euler's theorem, and the fact that every vertex branches into exactly three sides.[2]

Beenakker[4] has recently proposed a very simple way to model this system. Let us suppose that we are told the numbers of bubbles in our system of a given area, but not told which bubbles have how many sides. Then there is a unique way to assign numbers of sides to the bubbles which is consistent with the constraint that the average number of sides be six, and also minimizes the total surface area of the cells. Beenakker assumes that soap bubbles always adopt this energetically favorable config-

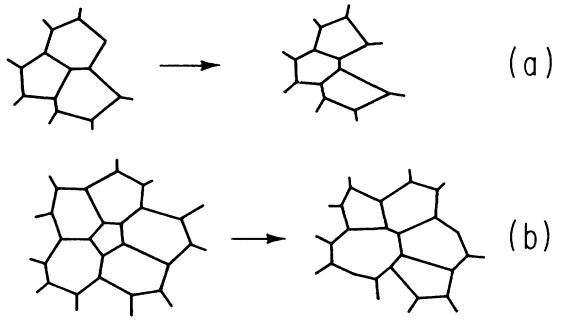

FIG. 2. In (a) is a sketch of how nearby bubbles exchange sides, and in (b) is a sketch of what happens when a five-sided bubble shrinks and disappears.

uration, and is able to obtain simple equations for their dynamics. One objection to this approach is that the bubbles are highly constrained, and it is not clear that they should easily succeed in obeying energetic demands. One may observe experimental configurations in which the assumption is badly violated. A second objection is that the solutions of Beenakker's equations disagree with experiment in a particularly troubling way. At long times he finds that a disordered array of bubbles condenses into a nearly perfect hexagonal array, which then becomes disordered again in a peculiar never-ending cycle. No hint of this behavior is seen experimentally, although the new experiments have, according to Beenakker's graphs, proceeded long enough for the oscillation to occur.

Let us proceed to write some simple equations for soap-bubble development, keeping as much realistic detail as seems feasible. We begin with the distribution function $g(A,n,t)$, which gives the number of bubbles that have an area A and number of sides n at time t. Von Neumann's law is expressed by

$$\frac{\partial}{\partial t}g(A,n,t) = -\frac{\partial}{\partial A}\kappa(n-6)g(A,n,t) \ . \tag{2}$$

Now we must add terms which account for the way that bubbles change their numbers of sides. When a four-sided bubble disappears, two of its neighbors lose sides. Which ones? The ones which join onto the four-sided bubble with sides of least length. As we are not keeping track of the lengths of sides for all bubbles, we will instead choose to eliminate sides from the two neighbors which have the smallest areas. So when a four-sided bubble shrinks to zero size, we will pick four bubbles at random from the available ensemble, find the two of those with the smallest area, and remove a side from each of them. When a five-sided bubble disappears, the bubble to gain a side will be the neighbor with the largest area. Obviously, many spatial correlations are being neglected with this approach.[5] The equation now looks like this:[6]

$$\frac{\partial}{\partial t}g(A,n) = -\frac{\partial}{\partial A}\kappa(n-6)g(A,n)+u(A)\frac{n-1}{S}g(A,n-1)-[u(A)+d(A)]\frac{n}{S}g(A,n)+d(A)\frac{n+1}{S}g(A,n+1) \ . \tag{3}$$

Here S gives the total number of sides of all bubbles in the system. The reason for the weighting factor n/S is that if one chooses a bubble's neighbor at random, an n-sided bubble has n/S chances of being chosen. The rate at which n-sided bubbles of a given area lose a side is $d(A)$, and the rate at which they gain is $u(A)$. The expression for $d(A)$ is a bit lengthy and will not be recorded in full here. The five-sided bubbles contribute

$$d(A) = \kappa g(0,5)5!\left[\frac{p^4(A)}{4!}+[1-p(A)]\frac{p^3(A)}{3!}\right]+\cdots \ ,$$

where $p(A)$ gives the probability that when we choose an edge in the system at random, the bubble it belongs to has an area greater than A. Three- and four-sided disappearing bubbles make a similar additive contribution to $d(A)$. The expression for $u(A)$ is

$$u(A) = \kappa g(0,5)5!\frac{[1-p(A)]^4}{4!} \ .$$

One may perform a simple scaling analysis and find that the solutions of this equation must have average area increasing as t^1 at late times. Numerical solution bears the analysis out. Results appear in Figs. 3 and 4. The agreement with experiment is not perfect, but plausible. The slopes of the theoretical curves in Fig. 3 are one to within 1% by the end of the run. Glazier, Gross, and Stavans[1] find a best fit of the experimental data to slopes of 0.6, but only about 100 bubbles are left in the system at the end of the experimental run, and edge effects, which eventually will bring the experiment to a halt and the growth exponent to zero, may be important. In Fig. 4(a) is shown the average area of bubbles sharing a given number of sides as a function of numbers of sides. Experimental bubbles with more than six sides are larger than theory would predict. Figure 4(b) shows the fraction of bubbles

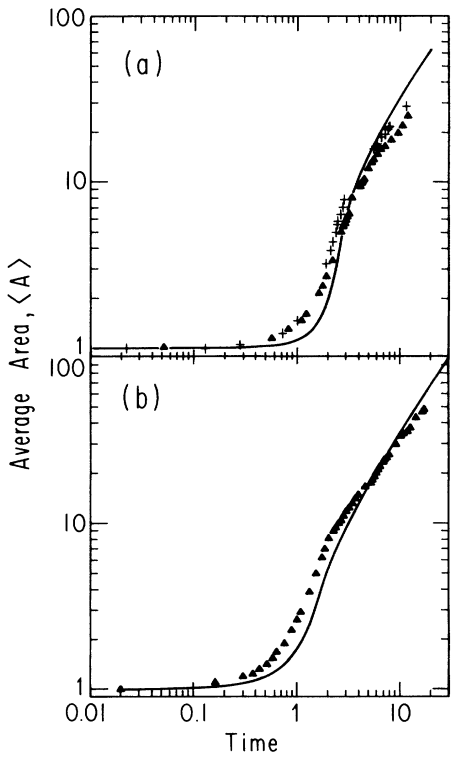

FIG. 3. The average area of bubbles is shown as a function of time. Theory is shown in the solid lines, and data from Ref. 1 in the remaining symbols. The time axes are scaled by measurements of the constant κ appearing in von Neumann's law. In (a) the theory begins with an initial distribution containing 90% six-sided bubbles, 10% an even mixture of fives and sevens. Two experimental runs with a similar initial condition are displayed as well. In (b) are shown theoretical and experimental runs beginning with 80% six-sided bubbles. The theory is fairly insensitive to details of the initial area distributions; these were chosen so as to be close to the conditions of the experiments.

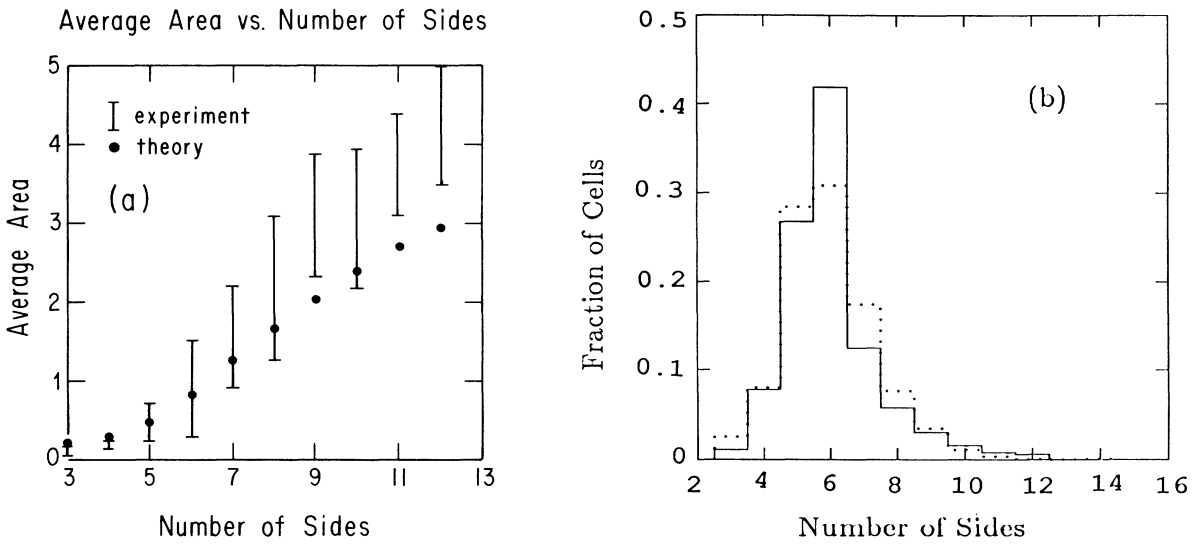

FIG. 4. In (a) is shown the average area of bubbles as a function of their numbers of sides. The experimental bars are from the data of Ref. 1. They are centered on the measured average area, and the width of the bar indicates the root-mean-square width of the measured area distribution. Experimental points lie along a line of greater slope than theory. In (b) is shown the theoretical number of cells (solid line) as a function of numbers of sides, at $t=1.9$ for the run of Fig. 3(a). This result is compared with the experiment of Aboav, Ref. 7 (dotted line). The coefficient defined by Aboav, $\mu_2=\sum_n \int dA(n-6)^2 g(A,n)$, equals 1.98 for both theory and experiment; Aboav reports this value rather than the time, which makes it difficult to be quite sure that theory and experiment correspond to exactly the same thing and may explain part of our discrepancy. Aboav has not reached the scaling limit. He sees μ_2 increase steadily to a value of 3 in his experiments, and guesses that it may continue increasing. In the present numerical study, μ_2 increases sharply to 5 at $t=3.5$, and then decreases approximately to 4 by $t=10$, where it remains.

with various numbers of sides in comparison with data of Aboav.[7] Further analysis of experimental data would allow more detailed comparison of theory and experiments, but the analysis is very tedious, and has not yet been carried out.

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[1]James A. Glazier, Steven P. Gross, and Joel Stavans, this issue, Phys. Rev. A **36**, 306 (1987).

[2]D. Weaire and N. Rivier, Contemp. Phys. **25**, 59 (1984).

[3]J. von Neumann, *Metal Interfaces* (American Society of Metals, Cleveland, OH, 1952), pp. 108–110.

[4]C. W. J. Beenakker, Phys. Rev. Lett. **57**, 2454 (1986).

[5]There is an obvious alternate choice, in which the bubbles to lose and gain sides are chosen entirely at random, with no regard to their areas. This version has the advantage of simplicity. It has the following disadvantage: When a system of bubbles begins as a nearly perfect lattice of six-sided cells, it passes through a stage in which small islands of small six-sided bubbles are surrounded by larger bubbles of varying numbers of sides. In experiments, these islands disappear rapidly because the small six-sided bubbles almost always are the ones to lose sides when some neighbor disappears. The effect is sufficiently striking, that the authors of Ref. 1 make it the starting point of a phenomenological theory. If bubbles lose sides without regard to area, then a population of small six-sided bubbles persists indefinitely. The curves in Fig. 3 are not much changed. However, the version of the theory presented in the main text is almost as simple, and more realistic. Further progress would involve detailed computations of bubble dynamics and correlations, and that is hard.

[6]There are some technical complications due to the fact that I do not consider two-sided bubbles. For $n=3$, I eliminate the factor of $d(A)$ multiplying $g(A,n)$ so as not to create any. Therefore, all terms multiplied by $d(A)$ are divided not by S but by S_3, the number of sides not contained in three-sided bubbles. In the numerical work, my array has no room for bubbles with more than 12 sides. I perform similar adjustments to keep from creating 13-sided bubbles. The equations are solved using an implicit method such as that described in W. H. Press *et al.*, *Numerical Recipes* (Cambridge Univ. Press, England, 1986), p. 626. The equations are first scaled by the average area, so they do not grow off the edge of the lattice on long runs, and the time step chosen small enough so that the average number of sides of the bubbles remains six to any desired accuracy.

[7]D. A. Aboav, Metallography **13**, 43 (1980).