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Systematic corrections to the rotating-wave approximation and quantum chaos

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Quantum chaos has been demonstrated in several studies of the interaction between N two-level quantum systems and a single-mode radiation field interacting in a resonant cavity. It has been shown that the mechanism for this chaos is dynamically identical to a periodically perturbed, near-separatrix motion of a pendulum. Identification of this mechanism followed from examination of the rotating-wave approximation and corrections to this approximation. Our earlier study established the nature of the corrections on the basis of numerical simulation experiments and heuristic arguments. In this paper we invoke an averaging procedure which enables us to systematically establish the form of these corrections. We first apply this procedure to the Rabi model in order to illustrate its utility and exhibit its structure, and then we apply it to the more complicated problem of quantum chaos.

I. INTRODUCTION

The question of whether the dynamical chaos observed in classical systems has an analogue in quantum mechanics has been debated in the recent literature.[1] A model was presented by Belobrov *et al.*[2] (BZT model) which strongly indicated that N two-level quantum systems interacting with a single-mode radiation field in a resonant cavity could be chaotic. The cavity is tuned to be resonant with the energy-level spacing of the two-level systems, and it is the feedback effect of the two-level systems on the radiation field which is responsible for the chaos. Belobrov *et al.* studied the putative chaos from the perspective of Liapunov exponents, whereas Milonni *et al.*[3] studied essentially the same model from the viewpoint of power spectra. We subsequently[4] showed that together with these two approaches it was possible to isolate the dynamical mechanism responsible for the chaos. We discovered the dynamical equivalent of a periodically perturbed Eberly-Chirikov pendulum hidden in the dynamics of the BZT model. This proved that the mechanism for quantum chaos in this model is the same mechanism which is generic for a large class of classical systems, as was originally demonstrated by Chirikov for classical, near-resonant systems.[5]

In classical physics, chaos may be readily studied by looking at phase-space trajectories. Rapid, exponential separation of initially adjacent trajectories is indicative of a positive Liapunov exponent which implies chaos.[6] There is no obvious analogue to phase space for a quantum system. Nevertheless, we took the point of view that the expectation values of a complete set of observables may be used to define an effective phase space for a quantum system. As time progresses, the expectation values change and a trajectory is produced. Power spectra for components of these trajectories and even Liapunov exponents can be obtained, just as for classical dynamical trajectories. When this approach is applied to the BZT model, five expectation values are involved. They evolve in time in accord with a system of five coupled first-order, ordinary differential equation which possess two conservation laws. This implies only three independent quantities. Three variables is the minimum number necessary for chaos,[7] although it is not always sufficient by any means.

When the rotating-wave approximation (RWA) is applied to these BZT equations, one more conserved quantity is created. This reduces the dynamics to just two independent variables, which is insufficient for chaos. Moreover, it renders the problem exactly solvable. We previously showed[4] that a change of variables in the RWA converts the dynamics into that of a spherical pendulum: the Eberly pendulum.[8] With a special choice of initial conditions, this RWA Eberly pendulum becomes a planar pendulum which operates close to its separatrix: a Chirikov pendulum.[5] We then showed that the full BZT model amounts to the RWA Eberly-Chirikov pendulum plus corrections. For relatively weak coupling between the two-level systems and the radiation field, these corrections take the form of a periodic perturbation. With the special initial conditions mentioned above, this produces a periodically perturbed, near-separatrix Chirikov pendulum dynamics, which is generic for chaos. It is also nonintegrable, but when we numerically simulate the periodically perturbed Chirikov pendulum equations, we reproduce the behavior seen in the full BZT model, in particular the time evolution of one important component of the trajectory and its power spectrum.

Our earlier work relied heavily on numerical simulation experiments and heuristic arguments in order to obtain the corrections to the RWA. In this paper we utilize an averaging method which has proved to be highly useful in classical dynamical systems.[9,10] With it we are able to systematically obtain the corrections to the RWA, thereby corroborating our earlier numerically empirical results. This averaging procedure does not seem to have been previously used in conjunction with the RWA approach. In order to illustrate its structure and utility, we first apply it to the Rabi model, which is not chaotic and is much simpler than the BZT model. The averaging technique easily and naturally produces the Bloch-Siegert shift[11] for the Rabi model.

The rest of the paper is organized as follows: Section II presents the averaging technique in a general setting, Sec. III applies it to the Rabi model, and Sec. IV presents the results for the BZT model.

II. THE AVERAGING THEOREM (REF. 10)

The reader is referred to the references for full details regarding the precise conditions for the validity of this averaging theorem and regarding the asymptotic accuracy of the result. Consider a coupled first-order system of ordinary differential equations

$$\dot{x}=\epsilon f(x,t,\epsilon)\ , \tag{1}$$

in which the dot denotes a time derivative (d/dt), $0\leq\epsilon<1$, $x\epsilon U\subseteq R^n$, and f: $R^n\times R\times R^+\to R^n$ is C^r, $r\geq 2$, bonded on bounded sets, and of period $T>0$ in t. The associated autonomous averaged system is defined by

$$\dot{y}=\epsilon\frac{1}{T}\int_0^T f(y,t,0)dt\equiv\epsilon\bar{f}(y)\ . \tag{2}$$

For our purposes, the averaging theorem may be expressed as the following.

Theorem: There exists a C^r change of coordinates $x=y+\epsilon\omega(y,t,\epsilon)$ such that (1) becomes

$$\dot{y}=\epsilon\bar{f}(y)+\epsilon^2 f_1(y,t,\epsilon)+O(\epsilon^3)\ , \tag{3}$$

where f_1 is of period T in t; and if $x(t)$ and $y(t)$ are solutions to (1) and (2) initiated at x_0 and y_0, respectively, at $t=0$, and $|x_0-y_0|=O(\epsilon)$, then $|x(t)-y(t)|=O(\epsilon)$ on a time scale $\sim 1/\epsilon$.

This is a constructive theorem in that f_1 is explicitly determined in the following way. First, write

$$f(x,t,\epsilon)=\bar{f}(x)+\tilde{f}(x,t,\epsilon)\ . \tag{4}$$

Then solve for ω using the equation

$$\frac{\partial}{\partial t}\omega=\tilde{f}(y,t,\epsilon)\ . \tag{5}$$

With this solution, f_1 is given by

$$f_1(y,t,\epsilon)=[D_y f(y,t,\epsilon)]\omega(y,t,\epsilon)-[D_y\omega(y,t,\epsilon)]\bar{f}(y)\ . \tag{6}$$

Written out with explicit indexes, we have $x_i=y_i+\epsilon\omega_i$ and

$$\dot{y}_i=\epsilon\bar{f}_i+\epsilon^2\left[\frac{\partial f_i}{\partial y_j}\omega_j-\frac{\partial\omega_i}{\partial y_j}\bar{f}_j\right]\ . \tag{7}$$

It is instructive to give a simple example[10] to illustrate this method and the subtle aspects of its implementation. Consider

$$\dot{x}=\epsilon x\sin^2 t\ . \tag{8}$$

Clearly, $\bar{f}(x)=\frac{1}{2}x$ because $1/2\pi\int_0^{2\pi}\sin^2 t\,dt=\frac{1}{2}$. Note that in computing $\bar{f}$, x is treated as a constant, even though we know that it really depends on t. This reflects the fact that for small ϵ, x is slowly varying, i.e., *secular*, whereas $\sin^2 t$ is a fast oscillation. Thus, we have

$$f=\bar{f}+\tilde{f}=\tfrac{1}{2}x-\tfrac{1}{2}x\cos(2t) \tag{9}$$

because $\sin^2 t=\frac{1}{2}[1-\cos(2t)]$. Therefore, (5) becomes (note the change from x to y)

$$\frac{\partial}{\partial t}\omega=-\frac{y}{2}\cos(2t)\ . \tag{10}$$

This yields

$$\omega=-\frac{y}{4}\sin(2t) \tag{11}$$

because we treat y as secular in integrating (10). Equation (3) becomes

$$\dot{y}=\epsilon\frac{y}{2}+\epsilon^2\left[\frac{1}{2}[1-\cos(2t)]\left[-\frac{y}{4}\sin(2t)\right]-\left[-\frac{1}{4}\sin(2t)\right]\frac{y}{2}\right]+O(\epsilon^3)\ . \tag{12}$$

One may readily verify

$$x(t)-y(t)=\exp(\tfrac{1}{2}\epsilon t)[\,|x_0-y_0|-\tfrac{1}{4}\epsilon x_0\sin(2t)+O(\epsilon^2)] \tag{13}$$

by solving (8) and (12). This confirms the prediction of the theorem, even for this diverging example.

III. THE RABI MODEL

The Rabi model not only proves useful for exhibiting a relatively simple application of the averaging theorem, but it also exhibits what occurs when a single two-level system is driven by a single-mode radiation field *without* any feedback because there is no resonant cavity. For this problem the underlying Hamiltonian is

$$H=\tfrac{1}{2}\hbar\omega_0\sigma_z+\hbar\lambda\sigma_x\cos(\omega t) \tag{14}$$

in which $\hbar$ is Planck's constant divided by 2π, $\hbar\omega_0$ is the energy of the level spacing of the two-level system, σ_z is a Pauli matrix, λ is the coupling strength for the electromagnetic field, σ_x is another Pauli matrix, and $\cos(\omega t)$ represents a single-mode field with frequency ω. The resonant situation corresponds with the choice $\omega=\omega_0$ which we use from here on. The Heisenberg equations of motion are

$$\dot{\sigma}_x=-\omega\sigma_y\ , \tag{15}$$

$$\dot{\sigma}_y=\omega\sigma_x-2\lambda\sigma_z\cos(\omega t)\ , \tag{16}$$

$$\dot{\sigma}_z=2\lambda\sigma_y\cos(\omega t)\ . \tag{17}$$

Let us define x, y, and z to be the expectation values of the Pauli operators with respect to the initial state of the system:

$$x=E_x(\sigma_x),\quad y=E_x(\sigma_y),\quad z=E_x(\sigma_z)\ . \tag{18}$$

The linearity of Eqs. (15)–(17) implies that these expectation values satisfy exactly the system of three coupled ordinary, first-order differential equations

$$\dot{x} = -\omega y \ , \tag{19}$$

$$\dot{y} = \omega x - 2\lambda z \cos(\omega t) \ , \tag{20}$$

$$\dot{z} = 2\lambda y \cos(\omega t) \ . \tag{21}$$

We now transform to a rotating frame

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ . \tag{22}$$

This produces the equivalent system of equations

$$\dot{a} = -\lambda z \sin(2\omega t) \ , \tag{23}$$

$$\dot{b} = -\lambda z [\cos(2\omega t) + 1] \ , \tag{24}$$

$$\dot{z} = \lambda a \sin(2\omega t) + \lambda b [\cos(2\omega t) + 1] \ , \tag{25}$$

which is the system we will analyze with the averaging theorem. We want to study how the two-level system evolves in time by observing how it changes from the upper state to the lower state and back again. This is given by $z = E_x(\sigma_z)$.

The averaging theorem may be applied as follows. First rescale time to the dimensionless time

$$t' = \omega t \ . \tag{26}$$

Let $\epsilon \equiv \lambda/\omega$ and rewrite Eqs. (23)–(25) in terms of t', but then drop the prime, yielding

$$\dot{a} = -\epsilon z \sin(2t) \ , \tag{27}$$

$$\dot{b} = -\epsilon z [\cos(2t) + 1] \ , \tag{28}$$

$$\dot{z} = \epsilon \{ a \sin(2t) + b [\cos(2t) + 1] \} \ . \tag{29}$$

Make the identifications for Eqs. (1) and (3):

$$x \rightarrow \begin{bmatrix} a \\ b \\ z \end{bmatrix} \ , \tag{30}$$

$$y \rightarrow \begin{bmatrix} \bar{a} \\ \bar{b} \\ \bar{z} \end{bmatrix} \ , \tag{31}$$

$$f \rightarrow \begin{bmatrix} -z \sin(2t) \\ -z[\cos(2t)+1] \\ a \sin(2t) + b[\cos(2t)+1] \end{bmatrix} \ . \tag{32}$$

Clearly,

$$\bar{f}(y) \rightarrow \begin{bmatrix} 0 \\ -\bar{z} \\ \bar{b} \end{bmatrix} \tag{33}$$

and

$$\tilde{f}(y) \rightarrow \begin{bmatrix} -\bar{z} \sin(2t) \\ -\bar{z} \cos(2t) \\ \bar{a} \sin(2t) + \bar{b} \cos(2t) \end{bmatrix} \tag{34}$$

and

$$\omega(y) \rightarrow \begin{bmatrix} \frac{1}{2}\bar{z}[\cos(2t)-1] \\ -\frac{1}{2}\bar{z}\sin(2t) \\ -\frac{1}{2}\bar{a}[\cos(2t)-1] + \frac{1}{2}\bar{b}\sin(2t) \end{bmatrix} \ . \tag{35}$$

Putting all of this in (7) yields

$$\frac{d\bar{a}}{dt} = \epsilon^2 \{ \tfrac{1}{2}\bar{a}[\tfrac{1}{2}\sin(4t) - \sin(2t)] + \tfrac{1}{2}\bar{b}[1 - \cos(2t) - \sin^2(2t)] \} + O(\epsilon^3) \ , \tag{36}$$

$$\frac{d\bar{b}}{dt} = -\epsilon\bar{z} + \epsilon^2[-\tfrac{1}{2}\bar{a}\sin^2(2t) - \tfrac{1}{4}\bar{b}\sin(4t)] + O(\epsilon^3) \ , \tag{37}$$

$$\frac{d\bar{z}}{dt} = \epsilon\bar{b} + \epsilon^2[-\tfrac{1}{2}\bar{z}\sin(2t)] \ . \tag{38}$$

These equations appear to be just as complicated as Eqs. (27)–(29). However, they have the advantage that they are well ordered in the parameter ϵ, which is not the case for Eqs. (27)–(29). This means that the secular behavior and the fast oscillation behavior can be easily separated. For example, to first order in ϵ we simply get the RWA equations

$$\frac{d\bar{b}}{dt} = -\epsilon\bar{z} \ , \tag{39}$$

$$\frac{d\bar{z}}{dt} = \epsilon\bar{b} \ , \tag{40}$$

which have the well-known RWA solution for $\bar{z}$:

$$\bar{z}(t) = \bar{z}(0)\cos(\epsilon t) + \frac{d\bar{z}}{dt}(0)\sin(\epsilon t) \tag{41}$$

with the unshifted (but scaled) frequency $\epsilon = \lambda/\omega$. Because $\epsilon \ll 1$, this is the slow, secular behavior, whereas the fast oscillations have scaled frequency 2.

We can also apply the averaging theorem to the system (36)–(38). This produces the twice averaged equations[12]

$$\frac{d\bar{\bar{a}}}{dt} = \epsilon^2 \tfrac{1}{4}\bar{\bar{b}} \ , \tag{42}$$

$$\frac{d\bar{\bar{b}}}{dt} = -\epsilon\bar{\bar{z}} + \epsilon^2(-\tfrac{1}{4}\bar{\bar{a}}) \ , \tag{43}$$

$$\frac{d\bar{\bar{z}}}{dt} = \epsilon\bar{\bar{b}} \ . \tag{44}$$

These equations provide a systematic correction to the RWA of order ϵ^2. Their solution is easily obtained by noting that

$$\frac{d\bar{\bar{b}}}{dt} = -\epsilon^2\bar{\bar{b}} - \tfrac{1}{16}\epsilon^4\bar{\bar{b}} \ , \tag{45}$$

which has the solution

$$\bar{\bar{b}}(t) = \bar{\bar{b}}(0)\cos(\beta t) + \frac{d\bar{\bar{b}}(0)}{dt}\sin(\beta t) \ , \tag{46}$$

in which β is defined by

$$\beta=\epsilon\left[1+\frac{\epsilon^2}{16}\right]^{1/2}, \tag{47}$$

which contains the well-known Bloch-Siegert shift![11] Solving for $\bar{\bar{a}}$ and $\bar{\bar{z}}$ yields

$$\bar{\bar{a}}(t)=\bar{\bar{a}}(0)+\tfrac{1}{4}\epsilon[\bar{\bar{z}}(t)-\bar{\bar{z}}(0)], \tag{48}$$

$$\bar{\bar{z}}(t)=\bar{\bar{z}}(0)+\frac{\epsilon}{\beta}\bar{\bar{b}}(0)\sin(\beta t)-\frac{\epsilon}{\beta}\frac{d\bar{\bar{b}}(0)}{dt}(0)[\cos(\beta t)-1], \tag{49}$$

in which we could replace $d\bar{\bar{b}}(0)/dt$ by $-\epsilon\bar{\bar{z}}(0)-(\epsilon^2/4)\bar{\bar{a}}(0)$.

It is illuminating to represent the dynamics with a second-order differential equation for $z(t)$. The relationship between z and $\bar{\bar{z}}$, according to the averaging theorem, is

$$z=\bar{z}+\epsilon\{-\tfrac{1}{2}\bar{a}[\cos(2t)-1]+\tfrac{1}{2}\bar{b}\sin(2t)\}+O(\epsilon^3). \tag{50}$$

Taking two time derivatives of this equation and substituting (36)–(38) on the right-hand side up to order ϵ^2, yields

$$\ddot{z}=-\epsilon^2[1+3\cos(2t)]\bar{z}+2\epsilon[\bar{a}\cos(2t)-\bar{b}\sin(2t)]+O(\epsilon^3). \tag{51}$$

The functions $\bar{a}$, $\bar{b}$, and $\bar{z}$ may be approximated by $\bar{\bar{a}}$, $\bar{\bar{b}}$, and $\bar{\bar{z}}$ to order ϵ^2. We may then write the $\ddot{z}$ equation, consistent to order ϵ^2, as

$$\ddot{z}=-\epsilon^2 z-2\epsilon\bar{\bar{b}}\sin(2t)-\tfrac{1}{2}\epsilon^2\cos(2t)(5z+1), \tag{52}$$

which represents a *Rabi oscillator with a periodic perturbation*. The perturbation contains $\bar{\bar{b}}$ which is determined from (46). In Fig. 1, the numerical solution to (52) is compared with the numerical solutions obtained from the full Rabi model given by (23)–(25). There is apparently exact agreement, given the resolution limit of the line plots.

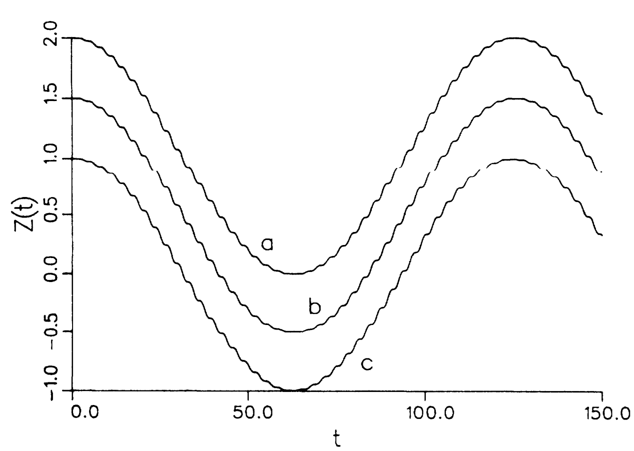

FIG. 1. Results of the averaging approximation for the Rabi model. (a) Numerical integration of the full Rabi model, Eqs. (27)–(29), for $a(0)=b(0)=0$, $z(0)=1$, and $\epsilon=0.05$. To allow comparison the trajectory solution has been given a vertical offset of $+1$. (b) Numerical integration of the averaging approximation, Eq. (52). The solution has been offset by $+0.5$. (c) Numerical integration of the "$\frac{3}{2}$" equation (53).

While empirically exploring numerical simulations of *ad hoc* or heuristically derived equations to serve in the place of (52), we discovered the equation

$$\ddot{z}=-\tfrac{3}{2}\epsilon^2 z\mp 2\epsilon(1-z^2)^{1/2}\sin(2t)-\tfrac{1}{2}\epsilon^2\cos(2t)(5z+1), \tag{53}$$

in which z also appears in the periodic perturbation term, unlike (52). The choice of a plus or minus sign was made numerically and depends on the initial conditions. Moreover, the $\epsilon^2 z$ term also has the factor $\frac{3}{2}$, not seen in (52). Most curiously, the line plots of (53) show a secular Rabi oscillation upon which there are superposed small periodic perturbations of frequency 2, just as for (52) and (23)–(25). The secular motion appears to have frequency ϵ, in spite of the presence of the $\frac{3}{2}$. This has puzzled us for some time. Now, we can show, using the results above which were obtained from the averaging theorem, that (53) is equivalent with (52), to order ϵ^2. Squaring (50) implies

$$z^2=\bar{z}^2+\epsilon\bar{z}\,\bar{b}\sin(2t)+O(\epsilon^2). \tag{54}$$

Equations (36)–(38) imply that $\bar{z}^2+\bar{b}^2$ is conserved up to order ϵ. Let the conserved value be C. Therefore, we obtain

$$C-z^2=\bar{b}^2-\epsilon\bar{z}\,\bar{b}\sin(2t)+O(\epsilon^2) \tag{55}$$

and

$$\pm(C-z^2)^{1/2}=\bar{b}\left[1-\tfrac{1}{2}\epsilon\frac{\bar{z}}{\bar{b}}\sin(2t)\right]+O(\epsilon^2). \tag{56}$$

Finally, this means

$$\bar{b}\cong\pm(C-z^2)^{1/2}+\tfrac{1}{2}\epsilon\bar{z}\sin(2t)+O(\epsilon^2). \tag{57}$$

Using this in (52) gives

$$\ddot{z}=-\epsilon^2 z\mp 2\epsilon(C-z^2)^{1/2}\sin(2t)-\epsilon^2\bar{z}\sin^2(2t)-\tfrac{1}{2}\epsilon^2\cos(2t)(5z+1)+O(\epsilon^3). \tag{58}$$

Using averaging on the ϵ^2 term, we may write, accurate to order ϵ^2,

$$\ddot{z}=-\tfrac{3}{2}\epsilon^2 z\mp 2\epsilon(C-z^2)^{1/2}\sin(2t)-\tfrac{1}{2}\epsilon^2\cos(2t)(5z+1)+O(\epsilon^3). \tag{59}$$

This reduces to (53) when $C=1$, as it was during our numerical experiments.

In Sec. IV we exhibit these same phenomena for the BZT model. As is the case here with the Rabi model, the averaging theorem provides a systematic procedure for obtaining periodic perturbation corrections to the RWA.

IV. THE BZT MODEL

The Hamiltonian for the BZT model is[13]

$$H=\tfrac{1}{2}\hbar\omega_0 S_z+\hbar\omega(a^\dagger a+\tfrac{1}{2})+\hbar\lambda S_x(a+a^\dagger), \tag{60}$$

in which $S_z=\sum_{j=1}^{N}\sigma_{zj}$, $S_x=\sum_{j=1}^{N}\sigma_{xj}$, and σ_{lj} is the lth Cartesian component for the jth two-level system, and in which $a^\dagger$ and a are photon creation and annihilation operators, respectively. We have in mind, explicitly, the quantum nature of the radiation. This is in contrast to the Rabi model in which the radiation field is treated semiclassically, i.e., by introduction of a c-number external electromagnetic field. In this model, we have a quantized single-mode field, and both the two-level systems and the field evolve in time. Heisenberg equations are readily derived and are

$$\dot{S}_x=-\omega S_y \ , \tag{61}$$

$$\dot{S}_y=\omega S_x-2\lambda(a+a^\dagger)S_z \ , \tag{62}$$

$$\dot{S}_z=2\lambda(a+a^\dagger)S_y \ , \tag{63}$$

$$\dot{a}+\dot{a}^\dagger=-\omega i(a-a^\dagger) \ , \tag{64}$$

$$i(\dot{a}-\dot{a}^\dagger)=\omega(a+a^\dagger)+2\lambda S_x \ . \tag{65}$$

Unlike Eqs. (15)–(17), these are nonlinear by virtue of two, bilinear nonlinearities in Eqs. (62) and (63). Moreover, there is an S_x feedback term in (65) which has no analogue at all in the Rabi model. It is this term which becomes the central focus of our further studies.

Once again, we introduce as real variables, the quantum expectation values of the relevant operators by defining

$$\begin{aligned} &Nx=E_x(S_x),\quad Ny=E_x(S_y),\quad Nz=E_x(S_z)\ ,\\ &Q=E_x(a^\dagger+a), P=E_x(ia-ia^\dagger)\ . \end{aligned} \tag{66}$$

The nonlinearity creates the dilemma caused by the inequality of products of averages and averages of products. Exact factorization of averaged products is valid only in the limit $N\to\infty$, for which it can be shown that there is a relative error of $O(1/N)$. This point is addressed in greater detail in the Appendix. Introducing the scaling $\lambda_N=\sqrt{N}\lambda$, $A=Q/\sqrt{N}$, and $B=P/\sqrt{N}$, we obtained the closed factorized system of equations

$$\dot{x}=-\omega y \ , \tag{67}$$

$$\dot{y}=\omega x-2\lambda_N zA \ , \tag{68}$$

$$\dot{z}=2\lambda_N yA \ , \tag{69}$$

$$\dot{A}=-\omega B \ , \tag{70}$$

$$\dot{B}=\omega A+2\lambda_N x \ . \tag{71}$$

This is equivalent to a particular *ab initio* semiclassical theory in which Eqs. (67)–(69) remain the same and equations (70)–(71) are replaced by the Maxwell equation[4]

$$\ddot{A}=-\omega^2 A-2\lambda_N\omega x \ . \tag{72}$$

Therefore, the BZT model may be understood as either a factorization approximation in a fully quantum problem, or as an exact treatment of a corresponding semiclassical problem. The latter interpretation is closer to the spirit of the Rabi model, and will be followed here. Thus, while we do not make a definitive statement about a fully quantal treatment because of the factorization approximation, we do make a definitive statement for the semiclassical case: Chaos is manifested for special conditions. The specific interpretation of the N two-level-systems–field interaction given in either (60) or (72) is a subtle question,[3,4] and not all types of elementary couplings translate into (60) or (72). For us, this is an $\mathbf{A}\cdot\mathbf{p}$ type of coupling with a vector potential which satisfies a nodal boundary condition at the reflector ends of the cavity.[4,13]

In addition to (22), which we again use here, we also use the rotation

$$\begin{bmatrix} r\\ s\end{bmatrix}=\begin{bmatrix}\cos(\omega t) & \sin(\omega t)\\ -\sin(\omega t) & \cos(\omega t)\end{bmatrix}\begin{bmatrix}A\\ B\end{bmatrix}\ . \tag{73}$$

The dynamics is now expressed by

$$\dot{a}=-\lambda_N z\{r\sin(2\omega t)+s[\cos(2\omega t)-1]\}\ , \tag{74}$$

$$\dot{b}=-\lambda_N z\{-s\sin(2\omega t)+r[\cos(2\omega t)+1]\}\ , \tag{75}$$

$$\dot{z}=\lambda_N[(ra-sb)\sin(2\omega t)+(rb+sa)\cos(2\omega t)+(rb-sa)]\ , \tag{76}$$

$$\dot{r}=\lambda_N\{a\sin(2\omega t)+b[\cos(2\omega t)-1]\}\ , \tag{77}$$

$$\dot{s}=\lambda_N\{-b\sin(2\omega t)+a[\cos(2\omega t)+1]\}\ , \tag{78}$$

which is the system we will analyze with the averaging theorem. We will again rescale time with $t'=\omega t$ and let $\epsilon\equiv\lambda_N/\omega$. Rewriting the equations and dropping the prime on t' yields

$$\dot{a}=-\epsilon z\{r\sin(2\omega t)+s[\cos(2\omega t)-1]\}\ , \tag{79}$$

$$\dot{b}=-\epsilon z\{r[\cos(2t)+1]-s\sin(2t)\}\ , \tag{80}$$

$$\dot{z}=\epsilon[(ra-sb)\sin(2t)+(rb+sa)\cos(2t)+(rb-sa)]\ , \tag{81}$$

$$\dot{r}=\epsilon\{a\sin(2t)+b[\cos(2t)-1]\}\ , \tag{82}$$

$$\dot{s}=\epsilon\{a[\cos(2t)+1]-b\sin(2t)\}\ , \tag{83}$$

which are the analogue to Eqs. (27)–(29). As in the Rabi model, these equations are now well ordered in ϵ. The averaging theorem will convert them into a system in terms of a systematic expansion in ϵ.

We make the identification for Eqs. (1) and (3):

$$x\to\begin{bmatrix}a\\ b\\ z\\ r\\ s\end{bmatrix}\ , \tag{84}$$

$$y\to\begin{bmatrix}\bar{a}\\ \bar{b}\\ \bar{z}\\ \bar{r}\\ \bar{s}\end{bmatrix}\ , \tag{85}$$

$$f \to \begin{pmatrix} -zr\sin(2t) - zs[\cos(2t)-1] \\ -zr[\cos(2t)+1] + zs\sin(2t) \\ (ra - sb)\sin(2t) + (rb + sa)\cos(2t) + rb - sa \\ a\sin(2t) + b[\cos(2t)-1] \\ a[\cos(2t)+1] - b\sin(2t) \end{pmatrix}, \quad (86)$$

$$\bar{f}(y) \to \begin{pmatrix} \bar{z}\,\bar{s} \\ -\bar{z}\,\bar{r} \\ \bar{r}\,\bar{b} - \bar{s}\,\bar{a} \\ -\bar{b} \\ \bar{a} \end{pmatrix}, \quad (87)$$

$$\tilde{f}(y) \to \begin{pmatrix} -\bar{z}\,\bar{r}\sin(2t) - \bar{z}\,\bar{s}\cos(2t) \\ -\bar{z}\,\bar{r}\cos(2t) + \bar{z}\,\bar{s}\sin(2t) \\ (\bar{r}\,\bar{a} - \bar{s}\,\bar{b})\sin(2t) + (\bar{r}\,\bar{b} + \bar{s}\,\bar{a})\cos(2t) \\ \bar{a}\sin(2t) + \bar{b}\cos(2t) \\ \bar{a}\cos(2t) - \bar{b}\sin(2t) \end{pmatrix}, \quad (88)$$

$$\omega(y) \to \tfrac{1}{2} \begin{pmatrix} \bar{z}\,\bar{r}[\cos(2t)-1] - \bar{z}\,\bar{s}\sin(2t) \\ -\bar{z}\,\bar{r}\sin(2t) - \bar{z}\,\bar{s}[\cos(2t)-1] \\ -(\bar{r}\,\bar{a} - \bar{s}\,\bar{b})[\cos(2t)-1] + (\bar{r}\,\bar{b} + \bar{s}\,\bar{a})\sin(2t) \\ -\bar{a}[\cos(2t)-1] + \bar{b}\sin(2t) \\ \bar{a}\sin(2t) + \bar{b}[\cos(2t)-1] \end{pmatrix}. \quad (89)$$

Using all of this in (7) leads to a sizable amount of algebra, the output of which is

$$\frac{d\bar{a}}{dt} = \epsilon\bar{z}\,\bar{s} + \tfrac{1}{2}\epsilon^2\{\bar{r}\,\bar{s}\,\bar{b}[3\sin(2t) - 2\sin(2t)\cos(2t)] + \bar{r}\,\bar{s}\,\bar{a}[2\cos^2(2t) - 1 - \cos(2t)]$$
$$+ \bar{r}^{\,2}\bar{a}\sin(2t)[\cos(2t)-1] + \bar{r}^{\,2}\bar{b}[\cos^2(2t) - \cos(2t)]$$
$$+ \bar{s}^{\,2}\bar{b}[2\cos(2t) - 1 - \cos^2(2t)] - \bar{s}^{\,2}\bar{a}\sin(2t)\cos(2t) + \bar{z}\,\bar{b}[3\cos(2t) - 3] + \bar{z}\,\bar{a}\sin(2t)\}, \quad (90)$$

$$\frac{d\bar{b}}{dt} = -\epsilon\bar{z}\,\bar{r} + \tfrac{1}{2}\epsilon^2\{\bar{r}\,\bar{s}\,\bar{b}[2\sin(2t) - 1 + \cos(2t)] - \bar{r}\,\bar{s}\,\bar{a}[2\sin(2t)\cos(2t) + \sin(2t)] - \bar{r}^{\,2}\bar{a}\sin^2(2t) - \bar{r}^{\,2}\bar{b}\sin(2t)\cos(2t)$$
$$+ \bar{s}^{\,2}\bar{b}[\sin(2t)\cos(2t) - \sin(2t)] + \bar{s}^{\,2}\bar{a}[1 + \sin^2(2t) - \cos(2t)] + \bar{z}\,\bar{a}[\cos(2t)-1] - \bar{z}\,\bar{b}[3\sin(2t)]\}, \quad (91)$$

$$\frac{d\bar{z}}{dt} = \epsilon(\bar{r}\,\bar{b} - \bar{s}\,\bar{a}) + \tfrac{1}{2}\epsilon^2\{\bar{a}\,\bar{b}[4 - 4\cos(2t)] + \bar{b}^{\,2}[3\sin(2t)] - \bar{a}^{\,2}\sin(2t) - \bar{z}\,\bar{s}^{\,2}\sin(2t) - \bar{z}\,\bar{r}^{\,2}\sin(2t)\}, \quad (92)$$

$$\frac{d\bar{r}}{dt} = -\epsilon\bar{b} + \tfrac{1}{2}\epsilon^2\{\bar{r}\,\bar{z}\sin(2t)[\cos(2t)-1] + \bar{z}\,\bar{s}[3\cos(2t) - 3]\}, \quad (93)$$

$$\frac{d\bar{s}}{dt} = \epsilon\bar{b} + \tfrac{1}{2}\epsilon^2\{\bar{r}\,\bar{z}[\cos(2t)-1] - \bar{z}\,\bar{s}[3\sin(2t)]\}. \quad (94)$$

Although these equations are much more complicated than Eqs. (79)–(83), they are well ordered in ϵ.

To first order in ϵ, the equations are identical with the RWA for the BZT model, which is also identical with the Jaynes-Cumming model[3,4]

$$\frac{d\bar{a}}{dt} = \epsilon\bar{z}\,\bar{s}, \quad (95)$$

$$\frac{d\bar{b}}{dt} = -\epsilon\bar{z}\,\bar{r}, \quad (96)$$

$$\frac{d\bar{z}}{dt} = \epsilon(\bar{r}\,\bar{b} - \bar{s}\,\bar{a}), \quad (97)$$

$$\frac{d\bar{r}}{dt} = -\epsilon\bar{b}, \quad (98)$$

$$\frac{d\bar{s}}{dt} = \epsilon\bar{a}. \quad (99)$$

These equations support three conservation laws

$$\bar{a}^{\,2} + \bar{b}^{\,2} + \bar{z}^{\,2} = 1, \quad (100)$$

$$\bar{r}^{\,2} + \bar{s}^{\,2} + 2\bar{z} = \xi, \quad (101)$$

$$\bar{r}\,\bar{a} + \bar{s}\,\bar{b} = P. \quad (102)$$

The first is conservation of total probability for the two-level system, the second is conservation of total energy for the combined system of the two-level system, radiation field, and interaction, and the third is generated by the RWA as was mentioned in the Introduction. It is instructive to render this dynamics as a second-order differential equation for $\bar{z}(t)$:

$$\frac{d^2\bar{z}(t)}{dt^2} = -\epsilon^2(1 + \xi\bar{z} - 3\bar{z}^{\,2}), \quad (103)$$

which is readily obtained from Eqs. (97), (100), and (101).

The Eberly transformation

$$\bar{\bar{z}}=\cos\rho \tag{104}$$

leads to the equivalent dynamics

$$\ddot{\rho}=\epsilon^2\sin\rho+\epsilon^2\frac{P^2\cos\rho}{\sin^3\rho}\ , \tag{105}$$

which is precisely a spherical pendulum in a vertically uniform gravitational field with azimuthal angular momentum P. The $\epsilon^2\sin\rho$ term arises from the σ_x feedback term in (65).

Of special interest to us has been the particular initial conditions $\bar{\bar{a}}(0)=\bar{\bar{b}}(0)=0$, $\bar{\bar{z}}(0)=1$, $\bar{\bar{r}}(0)\neq 0$, and $\bar{\bar{s}}(0)=0$. This implies $P=0$, so that (105) reduces to a planar pendulum

$$\ddot{\rho}=\epsilon^2\sin\rho\ , \tag{106}$$

with initial conditions $\rho(0)=0$ and $\dot{\rho}(0)=\pm 2\epsilon\bar{\bar{r}}(0)$, which is small, i.e., of order ϵ. This implies a near separatrix motion of the planar pendulum ($\rho=0$ is the vertical position with the pendulum bob up). This problem may be completely integrated in terms of Jacobian elliptic functions.

In the Rabi model, the twice averaged equations,[12] (42)–(44), still implied a second-order equation, (45), which possessed a Bloch-Siegert shifted frequency. Here, the averaged equations are

$$\frac{d\bar{\bar{a}}}{dt}=\epsilon\bar{\bar{s}}\,\bar{\bar{z}}+\tfrac{1}{2}\epsilon^2(\tfrac{1}{2}\bar{\bar{r}}^{\,2}\bar{\bar{b}}-\tfrac{3}{2}\bar{\bar{s}}^{\,2}\bar{\bar{b}}-3\bar{\bar{z}}\,\bar{\bar{b}})\ , \tag{107}$$

$$\frac{d\bar{\bar{b}}}{dt}=-\epsilon\bar{\bar{r}}\,\bar{\bar{z}}+\tfrac{1}{2}\epsilon^2(-\tfrac{1}{2}\bar{\bar{r}}^{\,2}\bar{\bar{a}}+\tfrac{3}{2}\bar{\bar{s}}^{\,2}\bar{\bar{a}}-\bar{\bar{z}}\,\bar{\bar{a}})\ , \tag{108}$$

$$\frac{d\bar{\bar{z}}}{dt}=\epsilon(\bar{\bar{r}}\,\bar{\bar{b}}-\bar{\bar{s}}\,\bar{\bar{a}})+\tfrac{1}{2}\epsilon^2(4\bar{\bar{a}}\,\bar{\bar{b}})\ , \tag{109}$$

$$\frac{d\bar{\bar{r}}}{dt}=-\epsilon\bar{\bar{b}}+\tfrac{1}{2}\epsilon^2(-3\bar{\bar{s}}\,\bar{\bar{z}})\ , \tag{110}$$

$$\frac{d\bar{\bar{s}}}{dt}=\epsilon\bar{\bar{a}}+\tfrac{1}{2}\epsilon^2(-\bar{\bar{r}}\,\bar{\bar{z}})\ . \tag{111}$$

They do not reduce to a simple second-order equation for $\bar{\bar{z}}$ because two of the conservation laws, (101) and (102), are no longer identities. However, it is still possible to obtain periodic perturbation corrections to order ϵ^2 for Eq. (103) through use of the averaging theorem.

According to the averaging theorem,

$$z=\bar{z}+\tfrac{1}{2}\epsilon\{(\bar{r}\,\bar{b}+\bar{s}\,\bar{a})\sin(2t)-(\bar{r}\,\bar{a}-\bar{s}\,\bar{b})[\cos(2t)-1]\}+O(\epsilon^2)\ . \tag{112}$$

Taking two time derivatives of z and substituting Eqs. (90)–(94) on the right-hand side yields

$$\begin{aligned}\ddot{z}=\epsilon^2\{&-\bar{a}^{\,2}-\bar{b}^{\,2}-(\bar{r}^{\,2}+\bar{s}^{\,2})\bar{z}\\&+[\bar{b}^{\,2}+\bar{a}^{\,2}+\bar{z}(\bar{s}^{\,2}-3\bar{r}^{\,2})]\cos(2t)+4\bar{z}\,\bar{s}\,\bar{r}\sin(2t)\}\\&+\epsilon[2(\bar{r}\,\bar{a}-\bar{s}\,\bar{b})\cos(2t)-2(\bar{r}\,\bar{b}+\bar{s}\,\bar{a})\sin(2t)]\\&+O(\epsilon^3)\ .\end{aligned} \tag{113}$$

As in the Rabi model, we will keep the periodic perturbation to order ϵ only and in the secular term we may substitute the conservation laws (100)–(102) which hold true to order ϵ. This produces

$$\begin{aligned}\ddot{z}=-\epsilon^2(1+\xi\bar{z}-3\bar{z}^{\,2})+\epsilon[2(\bar{r}\,\bar{a}-\bar{s}\,\bar{b})\cos(2t)&\\-2(\bar{r}\,\bar{b}+\bar{s}\,\bar{a})\sin(2t)]\ .&\end{aligned} \tag{114}$$

Consistent to order ϵ^2 for the secularity and order ϵ for the perturbation gives

$$\begin{aligned}\ddot{z}=-\epsilon^2(1+\xi z-3z^2)+2\epsilon[\bar{\bar{r}}\,\bar{\bar{a}}-\bar{\bar{s}}\,\bar{\bar{b}})\cos(2t)&\\-(\bar{\bar{r}}\,\bar{\bar{b}}+\bar{\bar{s}}\,\bar{\bar{a}})\sin(2t)]\ ,&\end{aligned} \tag{115}$$

which is a periodically perturbed pendulum dynamics, and in which the perturbation contains $\bar{\bar{a}}$, $\bar{\bar{b}}$, $\bar{\bar{r}}$, and $\bar{\bar{s}}$ which are determined by Eqs. (95)–(99) for this order in ϵ. Numerical simulations of Eq. (115) are compared in Fig. 2 with numerical simulations of the full BZT model given by Eqs. (79)–(83). The agreement is excellent, although not identical anymore within the resolution of the line plots.

While empirically exploring numerical simulations of *ad hoc* or heuristically derived equations to serve in place of (115), we discovered the equation

$$\ddot{\rho}=\tfrac{3}{2}\epsilon^2\sin\rho\mp 2\epsilon\sqrt{\xi-2\cos\rho}\,\sin(2t)\ , \tag{116}$$

in which z also appears in the periodic perturbation term, unlike (115). Moreover, the $\epsilon^2\sin\rho$ term also has the factor $\frac{3}{2}$, not seen in (115) or in (106), the Eberly transform of the secular part of (115) for special initial conditions. The line plots of numerical simulations of (116) show a secular behavior upon which there are

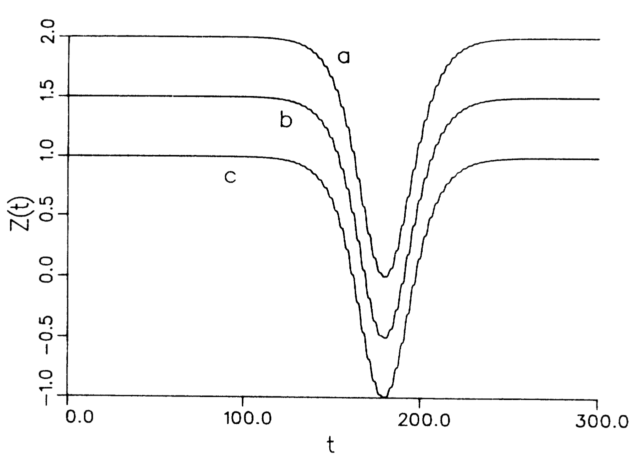

FIG. 2. Results of the averaging approximation for the BZT model. (a) Numerical integration of the full BZT model, Eqs. (74)–(78), for $a(0)=b(0)=s(0)=0$, $z(0)=1$, $r(0)=10^{-3}$, and $\epsilon=0.05$. As in Fig. 1 the trajectory has been given a vertical offset of $+1$. (b) Solution to the averaging approximation, Eq. (115); offset $=+0.5$. (c) Solution to the $\frac{3}{2}$ equation (116) with the initial condition $\rho(0)=0$, $\dot{\rho}(0)=2\epsilon r(0)$, and with $z(t)=\cos\rho(t)$.

small, rapid oscillations of period 2, just as for (115) or (79–(83). The secular motion appears to have a frequency determined by ϵ, in spite of the presence of the $\frac{3}{2}$. The explanation of this curious behavior follows from the averaging theorem just as it did for the Rabi model, as was explained in Sec. III.

The special initial conditions invoked above to get (106) imply that the solutions to (95)–(99) are $\bar{a}(t)=\bar{s}(t)=0$, $\bar{r}(t)=\pm[\xi-2\bar{z}(t)]^{1/2}$, and $\bar{b}(t)=\pm[1-\bar{z}^{\,2}(t)]^{1/2}$. Therefore, (112) becomes

$$z=\bar{z}+\tfrac{1}{2}\epsilon[\bar{r}\,\bar{b}\sin(2t)]+O(\epsilon^2)\ . \tag{117}$$

Squaring this gives

$$z^2=\bar{z}^{\,2}+\epsilon\bar{z}[\bar{r}\,\bar{b}\sin(2t)]+O(\epsilon^2)\ . \tag{118}$$

Therefore,

$$\begin{aligned}1-z^2&=1-\bar{z}^{\,2}-\epsilon\bar{z}\,\bar{r}\,\bar{b}\sin(2t)+O(\epsilon^2)\\&=\bar{b}^{\,2}-\epsilon\bar{z}\,\bar{r}\,\bar{b}\sin(2t)+O(\epsilon^2)\ ,\end{aligned} \tag{119}$$

in the second line of which we have used the solution for $\bar{b}$ above. Thus,

$$\pm(1-z^2)^{1/2}=\bar{b}-\tfrac{1}{2}\epsilon\bar{z}\,\bar{r}\sin(2t)+O(\epsilon^2) \tag{120}$$

or equivalently

$$\bar{b}=\pm(1-z^2)^{1/2}+\tfrac{1}{2}\epsilon\bar{z}\,\bar{r}\sin(2t)+O(\epsilon^2)\ . \tag{121}$$

Similarly,

$$\begin{aligned}\xi-2z&=\xi-2\bar{z}-\epsilon\bar{r}\,\bar{b}\sin(2t)+O(\epsilon^2)\\&=\bar{r}^{\,2}-\epsilon\bar{r}\,\bar{b}\sin(2t)+O(\epsilon^2)\end{aligned} \tag{122}$$

in the second line of which we have used the solution for $\bar{r}$ above. Thus,

$$\bar{r}=\pm\sqrt{\xi-2z}+\tfrac{1}{2}\epsilon\bar{b}\sin(2t)+O(\epsilon^2)\ . \tag{123}$$

Using (121) and (123) in (115) gives

$$\begin{aligned}\ddot{z}&=-\epsilon^2(1+\xi z-3z^2)-2\epsilon\bar{r}\,\bar{b}\,sin\,(2t)\\&=-\epsilon^2(1+\xi z-3z^2)\pm2\epsilon[(1-z^2)(\xi-2z)]^{1/2}\sin(2t)\\&\quad-\epsilon^2\sin^2(2t)[(1-z^2)^{1/2}\bar{b}+(\xi-2z)^{1/2}\bar{z}\,\bar{r}]+O(\epsilon^3)\\&=-\epsilon^2[1+\sin^2(2t)](1+\xi z-3z^2)\\&\quad\pm2\epsilon[(1-z^2)(\xi-2z)]^{1/2}\sin(2t)+O(\epsilon^3)\ .\end{aligned}$$

The last equality utilizes (121) and (123) to the desired order. Using averaging on the ϵ^2 term yields

$$\begin{aligned}\ddot{z}&=-\tfrac{3}{2}\epsilon^2(1+\xi z-3z^2)\\&\quad\pm2\epsilon[(1-z^2)(\xi-2z)]^{1/2}\sin(2t)+O(\epsilon^3)\ .\end{aligned}$$

The Eberly transformation, $z=\cos\rho$, converts this into Eq. (116), with initial conditions $\rho(0)=0$ and $\dot{\rho}(0)=\pm2\epsilon\bar{r}(0)$. Clearly, we have a periodically perturbed, near-separatrix motion of a pendulum, which is generic for chaos.[5]

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APPENDIX

For a single two-level system ($N=1$) in a single-mode electromagnetic field, the interaction may be modified in the form given by Jaynes and Cummings.[14] This makes the quantum model exactly solvable, and the variable z exhibits the phenomena of collapse and revival.[15,16] It is known that for this model, factorization of expectations of products yields equations which fail to reproduce collapse and revival. This raises serious doubts regarding our procedure for the BZT model[4] when it is applied to a single two-level system. Moreover, even if the Jaynes-Cummings interaction approximation is not invoked, Graham and Hohnerbach have shown by numerical analysis that collapse and revival still occur, although somewhat modified.[17] Recent experimental[18] work confirms these expectations for low-density Rydberg atomic beams. However, as N is increased, revivals are quenched, so that for only $N=9$ no revivals are seen at all in the numerical studies.[19] Moreover, the onset time for collapse increases in N as well. Thus, as N increases, the predictions of the factorized equations provide better and better agreement with the numerical studies.

In fact, the validity of the factorized equations for large N has been extensively studied.[20–23] A general framework, applicable to both the Jaynes-Cummings interaction and to the BZT model has been elucidated by Yaffe.[24] With it, it may be shown that factorization is exact for the BZT model in the limit $N\rightarrow\infty$. For finite N, the relative error is $O(1/N)$. While our earlier work[4] was applied to the case $N=1$, for which factorization is clearly invalid, our presentation in this paper is for N two-level systems in the large-N limit. Remarkably, Eqs. (67)–(71) are the same as were used for $N=1$, with the exception that the coupling strength is now λ_N, $\sqrt{N}$ times bigger than before. This increase in coupling brings the possibility of physical realizable manifestations of the equations closer.

For finite N, the Hamiltonian (60) has a discrete eigenspectrum and the time evolution of the Heisenberg operators is almost periodic.[17] For $N\rightarrow\infty$, this eigenspectrum may become truly continuous, which could justify the chaos observed in the factorized equations, (67)–(71). Therefore, the *bona fide* chaos we see in our semiclassical equations may represent chaos in a fully quantum system [governed by Hamiltonian (60)] in the limit $N\rightarrow\infty$. We believe this would provide the first example of genuine quantum chaos.

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