RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 36, NUMBER 1 JULY 1, 1987

Horseshoe-shaped maps in chaotic dynamics of atom-field interaction

Asoke Nath and Deb Shankar Ray
Department of Physical Chemistry, Indian Association for the Cultivation of Science, Jadavpur, Calcutta 700032, India
(Received 2 September 1986)

We show that in addition to the length of the Bloch vector there exists an energylike integral of motion for the Maxwell-Bloch equations. This allows us to recast the semiclassical problem of atom-field interaction into a classical Hamiltonian problem of two coupled oscillators which contain both periodic and homoclinic orbits. Melnikov's method is then used to demonstrate the existence of Smale horseshoe-shaped maps in its dynamics which preclude the existence of any further analytic integral of motion. The treatment yields a systematic procedure for understanding the mechanism of instability in this fundamental model of quantum optics.

A system of N identical two-level atoms interacting with a single-mode classical electromagnetic field is one of the fundamental models of quantum optics. The mathematical description of this model is provided by the coupled Maxwell-Bloch equations[1] which admit of a wide variety of solutions pertaining to different physical situations.[2] While these solutions are regular, there exist characteristically irregular or chaotic solutions[3-9] which have become important recently in the context of chaotic dynamics of the atom-field interaction. It is now well known that stochasticity in this semiclassical dynamical system appears when the rotating-wave approximation (RWA) is not invoked[3,4] and the strength of the atom-field interaction is increased beyond a critical value.[3,6] The non-RWA version (as well as RWA version) of the Maxwell-Bloch equations is known to possess only one integral of motion, namely, the length of the Bloch vector. We show here that in addition to this integral there exists another classical-Hamiltonian-like constant of motion. This allows us to recast this semiclassical problem of the atom-field interaction into a purely classical Hamiltonian problem of two coupled oscillators. One of the oscillators is harmonic with periodic orbits while the other one is anharmonic with homoclinic orbits. We show that the small-coupling perturbation breaks the integrability of the uncoupled system by introducing a complicated kind of invariant set (obtained on the Poincaré map defined on each constant-energy hypersurface in some energy interval), a Smale horseshoe-shaped map, into the dynamics. Very recently Fox and Edison[3] have considered the Belobrov-Zaslavskii-Tartakovsky version of the Maxwell-Bloch equations on the basis of seminumerical arguments based on Chirikov's pendulum for the study of instability in the atom-field interaction. The present treatment, however, is completely analytical.

The method for finding the horseshoe-shaped maps is based on Melnikov's technique[10] extended to the system of two[11] and higher degrees of freedom[12] by Holmes and Marsden.[11,12] In Melnikov's method one is concerned with the perturbation of the homoclinic manifold in the Hamiltonian system which consists of an integrable part and a small perturbation. It is well known that if Melnikov's function admits of simple zeros than the stable and unstable manifolds, which for the unperturbed system of oscillators coincide as a smooth homoclinic manifold, intersect transversely for small perturbation. The Smale-Birkhoff[11] homoclinic theorem asserts the existence of invariant sets—horseshoe-shaped maps on the Poincaré map. Second, the existence of horseshoe-shaped maps precludes the existence of any further analytic integral of motion and explains qualitatively the mechanism of development of instability in this fundamental model of atom-field interaction.

The dynamics of interaction between N identical two-level atoms with a single-mode classical electromagnetic field is described by the following set of Maxwell-Bloch equations:[1]

$$\dot{x} = -\omega_0 y \ , \tag{1a}$$

$$\dot{y} = x + \frac{2d}{\hbar} E z \ , \tag{1b}$$

$$\dot{z} = -\frac{2d}{\hbar} E y \ , \tag{1c}$$

and

$$\ddot{E} + \omega^2 E = -4\pi \ddot{P} \ , \tag{2}$$

where x, y, and z are the usual Bloch-vector components. E is the single-mode classical field with frequency ω. The right-hand side of Eq. (2) is the macroscopic polarization source term for the N-atomic system which is connected to the microscopic polarization through the simple relation $P = Ndx$. ω_0 and d represent the transition frequency and the transition dipole for the atom, respectively. The overdot denotes differentiation with respect to time t.

For convenience, we put Eqs. (1) and (2) in dimensionless form through the following change of variables:

$$\tau = \omega_0 t, \quad \mathcal{E} = \frac{2d}{\omega_0} E, \quad \mu = \frac{\omega}{\omega_0} \ ,$$

and

$$\beta = \frac{8\pi N d^2}{\hbar \omega_0} \ .$$

Since we will be concerned with the resonant atom-field interaction we use the resonance condition $\mu = 1$.

 © 1987 The American Physical Society

The system of Eqs. (1) and (2), which thus contain only a single parameter β, therefore, reduces to

$$\dot{x} = -y \ , \tag{2a}$$

$$\dot{y} = x + \mathcal{E} z \ , \tag{2b}$$

$$\dot{z} = -\mathcal{E} y \ , \tag{2c}$$

$$\ddot{\mathcal{E}} + \mathcal{E} = -\beta \ddot{x} \ . \tag{3}$$

Here the overdot represents differentiation with respect to the scaled time τ.

It is well known that this non-RWA version of the Maxwell-Bloch equations (2) and (3) possesses only first integral of motion

$$x^2 + y^2 + z^2 = 1 \ . \tag{4}$$

We note here that in addition to this integral there exists a second integral of motion which is the following energylike quantity:

$$\begin{aligned} H &= \dot{\mathcal{E}}^2 + \mathcal{E}^2 + 2\beta z - 2\beta \dot{\mathcal{E}} y + \beta^2 y^2 \\ &= (\dot{\mathcal{E}} - \beta y)^2 + \mathcal{E}^2 + 2\beta z \ . \end{aligned} \tag{5}$$

The presence of the two integrals of motion reduces the dimension of the autonomous system [(2) and (3)] from five to three. We shall show that the system does not possess any further integral (thus proving nonintegrability). Moreover, this reduction in dimension greatly facilitates the study of this dynamical system through the Poincaré section.

Let us now look into the second integral (5) more closely. We first eliminate z from this integral by making use of the relation (4), then H becomes

$$H = \dot{\mathcal{E}}^2 + \mathcal{E}^2 + \beta^2 y^2 + 2\beta\sqrt{1 - (x^2 + y^2)} - 2\beta \dot{\mathcal{E}} y \ .$$

Since $\dot{x} = -y$ we have

$$H = \dot{\mathcal{E}}^2 + \mathcal{E}^2 + \beta^2 \dot{x}^2 + 2\beta\sqrt{1 - (x^2 + \dot{x}^2)} + 2\beta \dot{\mathcal{E}} \dot{x} \ .$$

Rescaling x and $\dot{x}$ as $X = \beta x$ and $P_X = \dot{x}$ which are canonically conjugate pairs of dynamical variables, and denoting $\mathcal{E}$ and $\dot{\mathcal{E}}$ by $\mathcal{E}$ and $P_{\mathcal{E}}$, another set of conjugate field variables, we can write down the constant of motion H as the sum of the Hamiltonian of two subsystems $F(X,P_X)$ and $G(\mathcal{E},P_{\mathcal{E}})$ and the perturbation $H'(X,P_X,\mathcal{E},P_{\mathcal{E}})$ as follows (we follow the notation of Holmes and Marsden[11]):

$$H = G(\mathcal{E},P_{\mathcal{E}}) + F(X,P_X) + \epsilon H'(X,P_X,\mathcal{E},P_{\mathcal{E}}) \ , \tag{6}$$

where

$$G(\mathcal{E},P_{\mathcal{E}}) = \mathcal{E}^2 + P_{\mathcal{E}}^2 \ ,$$

$$F(X,P_X) = P_X^2 + 2\sqrt{\beta^2 - (X^2 + P_X^2)} \ ,$$

and

$$H'(X,P_X,\mathcal{E},P_{\mathcal{E}}) = 2P_X P_{\mathcal{E}} \ .$$

We note that while F contains only the atomic variables, G is a function of purely field variables. ϵ is a smallness parameter introduced in the perturbative momentum-coupling term needed for usual perturbative analysis.

Let us first begin with unperturbed system $G \otimes F$ (see Fig. 1). Now the G system is simply a harmonic oscillator so that one can use a canonical change of coordinates to action angle variables (I,θ) through the relation $\mathcal{E} = \sqrt{2I}\sin\theta$ and $P_{\mathcal{E}} = \sqrt{2I}\cos\theta$, such that θ is 2π periodic, $I > 0$. Then G becomes simply $2I$. The coordinates of the F system are retained unaltered. Therefore $G(I) = 2I$,

$$F(X,P_X) = P_X^2 + 2\sqrt{\beta^2 - (X^2 + P_X^2)} \ ,$$

$$H' = 2\sqrt{2} P_X \sqrt{I} \cos\theta \ .$$

The equations of motion for the unperturbed system $G \otimes F$ are

$$\dot{X} = \frac{\partial F}{\partial P_X} \text{ and } \dot{P}_X = -\frac{\partial F}{\partial P_X} \ , \tag{7}$$

$$\dot{\theta} = 2 \text{ and } \dot{I} = 0 \ . \tag{8}$$

The fixed point for the unperturbed G system is (0,0) which is evidently an elliptic fixed point. The fixed points for the unperturbed F system are (0,0) and $(0, \pm\sqrt{\beta^2 - 1})$. A linear-stability analysis around these fixed points reveals that of these three fixed points (0,0) is hyperbolic and the other two are elliptic for $\beta > 1$.

In order to study the classical perturbative dynamics using Melnikov's method the unperturbed system must contain a homoclinic orbit. In the present problem the F system satisfies this requirement. Since this orbit passes through the hyperbolic fixed point (0,0) the energy of the F-system on this orbit is 2β. The homoclinic orbit can then be obtained from the following equation:

$$P_X^2 + 2\sqrt{\beta^2 - (X^2 + P_X^2)} = 2\beta \ ,$$

which can be rewritten as

$$P_X^4 - 4(\beta - 1)P_X^2 + 4X^2 = 0 \ . \tag{9}$$

Note the graphical plot of the homoclinic orbit (X vs P_X) which is reminiscent of that of the Duffing's oscillator [in fact, they become identical if P_X is replaced by X and vice

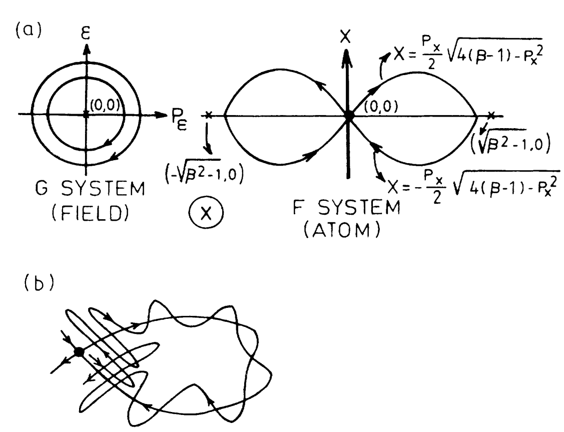

FIG. 1. (a) Unperturbed periodic and homoclinic orbits; ($\beta > 1$) for G and F systems. Elliptic and hyperbolic fixed points are denoted by x and ●, respectively. (b) Perturbed homoclinic orbit; transverse intersection of stable and unstable orbits ($\beta > 1$).

versa in Eq. (9)].

In many weakly nonlinear problems the unperturbed system does not possess any homoclinic orbit. However, to exploit Melnikov's technique, sometimes suitable averaging and truncation are used to cook up an effective Hamiltonian system which contains a homoclinic orbit. Such procedures, however, are not always correct.[11] In the present problem our unperturbed system $G\otimes F$ naturally possesses a homoclinic orbit. This natural occurrence has made our analysis quite straightforward.

The Hamiltonian system associated with the F system possesses the homoclinic orbit which can be obtained as the solution of Eq. (9),

$$\left(\frac{dX}{dt}\right)^4-4(\beta-1)\left(\frac{dX}{dt}\right)^2+4X^2=0 .$$

Unfortunately, this equation does not possess any explicit solution in terms of t. Instead, we have an implicit solution as follows:

$$\xi=\tanh[2\xi\pm\sqrt{4/C}(t-t_0)] , \qquad (10)$$

where

$$\xi=1/\sqrt{2}\left(\sqrt{\tfrac{1}{2}-2X/C}-\sqrt{\tfrac{1}{2}+2X/C}\right) ,$$

and

$$C=4(\beta-1) .$$

We shall see, however, that for our purpose we need not have to have the explicit solutions X and P_X.

Let us now turn toward the perturbed Hamiltonian H. The perturbation H' is smooth and 2π periodic in θ. The equations of motion corresponding to H are

$$\dot{X}=\frac{\partial F}{\partial P_X}+\epsilon\frac{\partial H'}{\partial P_X},\ \dot{P}_X=-\frac{\partial F}{\partial X}-\epsilon\frac{\partial H'}{\partial X} , \qquad (11)$$

$$\dot{\theta}=2+\epsilon\frac{\partial H'}{\partial I},\ \dot{I}=-\epsilon\frac{\partial H'}{\partial \theta} . \qquad (12)$$

For $\epsilon=1$, these are exactly the same Maxwell-Bloch equations [(2) and (3)] with which we have started. Those have been recast into a classical two-degree-of-freedom Hamiltonian problem. For $\epsilon=0$, we have the uncoupled field and atomic system which is integrable and one gets the invariant curves on the Poincaré map. The question is what happens when $\epsilon>0$, but sufficiently small. We shall see that the Smale horseshoe-shaped map appears in this case.

Following Holmes and Marsden[11,12] this two-degree-of-freedom autonomous system can be reduced to a one-degree-of-freedom nonautonomous system using Whittakar's method. In this process one actually eliminates the action I from the Eqs. (11) and (12) using the integral of motion H. Then from the system of equations containing H one further eliminates the variable t which is conjugate to H and the resultant equations of motion are written by expressing the coordinates and momenta as functions of the angle variable θ.

We need not explicitly follow here this procedure but can directly use the theorem of Holmes and Marsden[11] to calculate Melnikov's function which actually measures the leading nontrivial distance between the stable and unstable orbit in a direction transverse to the dynamic variable θ. In practice, the calculation involves the following integration of the Poisson bracket $\{F,H'\}$ around the homoclinic orbit of the unperturbed system

$$\bar{X}(t-t_0),\ \bar{P}_X(t-t_0) ,$$
$$M(t_0)=\int_{-\infty}^{+\infty}\{F,H'\}dt . \qquad (13)$$

Since

$$H'=2\sqrt{2}P_X\sqrt{I}\cos\theta$$

and

$$F=P_X^2+2\sqrt{\beta^2-(X^2+P_X^2)} ,$$

we have

$$\{F,H'\}=-\frac{4\sqrt{2I}\cos\theta\bar{X}}{\sqrt{\beta^2-(\bar{X}^2+\bar{P}_X^2)}} . \qquad (14)$$

Again the energy of the homoclinic orbit is 2β. So $2I=h-2\beta,(h>2\beta)$ where $H(\mathcal{E},P_{\mathcal{E}},X,P_X)=h$. Because of Eq. (8) we have $\theta=2t$, the frequency being 2. Therefore, the Melnikov function is given by

$$M(t_0)=-4\sqrt{(h-2\beta)}\int_{-\infty}^{+\infty}\frac{\cos(2t)\bar{X}(t-t_0)dt}{\sqrt{\beta^2-[\bar{X}(t-t_0)^2+\bar{P}_X(t-t_0)^2]}} . \qquad (15)$$

Changing the variables of integration from t to τ using $t-t_0=\tau$ we obtain

$$M(t_0)=-4\sqrt{(h-2\beta)}\int_{-\infty}^{+\infty}\frac{\cos[2(t_0+\tau)]\bar{X}(\tau)d\tau}{\sqrt{\beta^2-[\bar{X}(\tau)^2+\bar{P}_X(\tau)^2]}} , \qquad (16)$$

or

$$M(t_0)=4\sqrt{(h-2\beta)}[A\sin(2t_0)-B\cos(2t_0)] ,$$

where

$$A=\int_{-\infty}^{+\infty}\frac{\sin(2\tau)\bar{X}(\tau)d\tau}{\sqrt{\beta^2-[\bar{X}(\tau)^2+\bar{P}_X(\tau)^2]}} ,$$

and

$$B=\int_{-\infty}^{+\infty}\frac{\cos(2\tau)\bar{X}(\tau)d\tau}{\sqrt{\beta^2-[\bar{X}(\tau)^2+\bar{P}_X(\tau)^2]}} .$$

Since the explicit calculation of $\bar{X}(\tau)$ and $\bar{P}_X(\tau)$ from the implicit equation (10) is impossible, one must have to adapt the iterative procedures or other approximate methods for calculation of A and B. For the present problem, however, these are not necessary. Rather Eq. (17) can be written more compactly as

$$M(t_0)=4\sqrt{(h-2\beta)}\cos[2t_0+\phi(\beta)] ,$$

where ϕ is expressible in terms of A and B through the relations $A = R\sin\phi$ and $B = R\cos\phi$.

Since the Melnikov function $M(t_0)$ has simple zeros which are independent of ϵ, one can immediately infer that for $\epsilon > 0$, sufficiently small the stable and unstable manifolds intersect transversely, giving rise to scattered homoclinic points. The theorem of Holmes and Marsden then immediately asserts that the dynamical system [(11) and (12)] (equivalently the Maxwell-Bloch equations) has a complicated invariant set in the form of a horseshoe in its dynamics on the energy surface $H = h$. Second, the existence of the horseshoe-shaped map rules out the possibility of existence of any further analytic integral of motion (i.e., the model is nonintegrable).

In this Rapid Communication, we have recast the semiclassical problem of the atom-field interaction described by a set of coupled Maxwell-Bloch equations into a classical Hamiltonian problem of two interacting oscillators. Melnikov's method is then used to investigate the motion near the vicinity of homoclinic manifold under a small perturbation. We have shown that the transverse intersection of the stable and unstable manifolds generates the Smale horseshoe-shaped map. The treatment thus yields a systematic procedure for understanding the mechanism of instability in this fundamental model of quantum optics. We hope that this method of mapping the semiclassical problem into a classical Hamiltonian problem can be extended to systems with more than two degrees of freedom (i.e., fields with additional modes) to explore the possibility of observing Arnold diffusion, etc., in the problem of radiation-matter interaction.

We thank Dr. S. P. Bhattacharyya and Dr. Debendramadhaba Bhattacharyya for useful discussions. One of the authors (D.S.R.) thanks the Indian National Science Academy (INSA) for partial financial support under Grant No. BS/RP-417/1767. A.N. thanks the National Science Foundation for partial financial support under Grant No. 83-19310.

[1]L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).

[2]For example, Rabi's solution, Torrey's solution, solutions in the adiabatic-following limit and in the rate-equation-approximation limit, the self-induced transparency solution, etc. All these have been discussed in Ref. 1.

[3]P. I. Belobrov, G. M. Zaslavsky, and G. Kh. Tartakovsky, Zh. Eksp. Teor. Fiz. **71**, 1799 (1976) [Sov. Phys. JETP **71**, 1941 (1976)]; R. F. Fox and J. Edison, Phys. Rev. A **34**, 482 (1986).

[4]P. W. Milonni, J. R. Ackerhalt, and H. W. Galbraith, Phys. Rev. Lett. **50**, 966 (1983).

[5]R. Graham and M. Hohnerback, Phys. Lett. **101A**, 61 (1984).

[6]A. Nath and D. S. Ray, Phys. Lett. **116A**, 104 (1986).

[7]A. Nath and D. S. Ray, Phys. Lett. **117A**, 341 (1986).

[8]G. M. Zaslavsky, Phys. Rep. **80**, 157 (1981).

[9]J. R. Ackerbalt, P. W. Milonni, and M. L. Shih, Phys. Rep. **128**, 205 (1985).

[10]V. K. Melnikov, Trans. Moscow Math. Soc. **12**, 1 (1963).

[11]P. J. Holmes and J. E. Marsden, Commun. Math. Phys. **82**, 523 (1982).

[12]P. J. Holmes and J. E. Marsden, J. Math. Phys. **23**, 669 (1982).