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Two-photon bremsstrahlung in low-frequency approximation

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Methods developed previously for estimating the amplitude for the nonrelativistic potential scattering of a charged particle accompanied by the emission of a single low-frequency photon are extended and applied to the problem of two-photon emission. In the case where only one photon is soft, previously developed techniques, which involve asymptotic evaluation of integrals in configuration space, can be taken over with little modification. The simplest way to obtain a low-frequency approximation when both photons are soft is through analytic continuation of this latter approximation, valid when one photon is soft and the other hard, in the frequency of the hard photon. A direct examination of the relevant integral shows that this method is in fact justified. The low-frequency approximation for two-photon bremsstrahlung requires, as input, the physical (on-shell) amplitudes for radiationless scattering and for single-photon emission, and introduces an error which vanishes linearly with frequency. The result obtained here represents the nonrelativistic analog of a theorem of Brown and Goble [Phys. Rev. **173**, 1505 (1968)] who based their derivation on the principles of Lorentz invariance, analyticity, and gauge invariance.

I. INTRODUCTION

The emission of two photons by a charged particle as it is being scattered from a target is a fundamental process which can now be subjected to careful experimental study.[1] Recently, Gavrila *et al.*[2] have performed a highly accurate calculation of the nonrelativistic two-photon bremsstrahlung matrix element (combining analytic and numerical techniques) for the case where the scattering potential is purely Coulombic. An extension of this procedure to allow for a wider class of potentials will be more difficult since analytic solutions of the radiationless scattering problem will generally be unavailable. However, useful approximations can be obtained in those cases where one or both photons have energies small compared to the scattering energy. If one photon is "soft" one expects, on the basis of Low's analysis of single-photon bremsstrahlung,[3] that the two-photon matrix element can be expressed rather simply in terms of the physical (on-shell) one-photon amplitude, while if both photons are soft one would look for an approximation involving the on-shell scattering amplitude in the absence of radiation. In the latter case this expectation is borne out by the work of Brown and Goble[4] who, in an extension of Low's work, made use of analyticity and gauge-invariance properties in the context of a relativistic model. Here we study the problem in a nonrelativistic formulation based directly on the configuration-space representation of the matrix element. As expected, the general structure revealed by the Brown-Goble calculation is recovered, as are the low-frequency limiting forms obtained in the calculation of Ref. 2 for the purely Coulombic potential.

It is reasonable to expect that there will be a continuing interest in carrying out detailed and reliable calculations of nonrelativistic multiphoton free-free transition amplitudes, taking into account the internal structure of the atomic target and extending over a broad range of frequencies. The present calculation should provide some guidance in these more difficult undertakings, and this has been our primary motivation for recasting the Brown-Goble analysis in nonrelativistic form. It might also be mentioned that the amplitude for spontaneous two-photon bremsstrahlung appears as input to an approximate evaluation of the amplitude for scattering in an intense *external* field,[5] and this provides additional incentive for a study of the two-photon matrix element.

II. BREMSSTRAHLUNG MATRIX ELEMENTS

For the sake of orientation and to introduce notation we begin with a brief outline of the theory of free-free transitions in a form convenient for our purposes. The Hamiltonian of the system is represented as

$$H = H_0 + H_F + H' , \tag{2.1a}$$

where

$$H_0 = (\mathbf{p}_{\rm op})^2/2\mu + V \tag{2.1b}$$

is the sum of the kinetic and potential energies of the particle, H_F is the energy operator of the radiation field in occupation number representation, and H' is the particle-field interaction

$$H' = -(e/\mu c)\mathbf{p}_{\rm op}\cdot\mathbf{A} + (e^2/2\mu c^2)A^2 . \tag{2.1c}$$

The vector potential, in dipole approximation, can be expanded as

$$\mathbf{A} = \sum_i (2\pi c^2/\omega_i L^3)^{1/2} \boldsymbol{\lambda}_i (a_i + a_i^\dagger) , \tag{2.2}$$

in units with $\hbar = 1$. In this expression L^3 represents the quantization volume, a_i the annihilation operator for the ith mode, ω_i the frequency, and $\boldsymbol{\lambda}_i$ the (linear) polarization vector. In the absence of the interaction H' the state vector takes the form

$$|\Phi_{\mathbf{p}N}^{(\pm)}\rangle = |u_{\mathbf{p}}^{(\pm)}\rangle |N\rangle , \tag{2.3}$$

where the photon state $|N\rangle$ is defined by the collection of occupation numbers $N_1, N_2, \ldots,$ for the various modes. The field energy is $E_N = \sum_i \omega_i N_i$. The particle states $|u_{\mathbf{p}}^{(\pm)}\rangle$ are fixed by specifying the asymptotic momentum $\mathbf{p}$ along with the boundary conditions—either incoming wave $(-)$ or outgoing wave $(+)$ at infinity—as described in more detail below. We then have

$$(H_0 + H_F)|\Phi_{\mathbf{p}N}^{(\pm)}\rangle = E_{\mathbf{p}N}|\Phi_{\mathbf{p}N}^{(\pm)}\rangle , \tag{2.4}$$

with $E_{\mathbf{p}N} = E_N + E_{\mathbf{p}}$ and $E_{\mathbf{p}} = p^2/2\mu$.

Consider a transition from state $|\Phi_{\mathbf{p}N}\rangle \equiv |\Phi_\alpha\rangle$ to state $|\Phi_{\mathbf{p}'N'}\rangle \equiv |\Phi_{\alpha'}\rangle$. Adopting standard distorted-wave techniques[6] to the radiation problem at hand, we express the S-matrix element as

$$S_{\alpha'\alpha} = S_{\alpha'\alpha}^{(1)} + S_{\alpha'\alpha}^{(2)} , \tag{2.5}$$

with

$$S_{\alpha'\alpha}^{(1)} = \langle \Phi_{\alpha'}^{(-)} | \Phi_\alpha^{(+)} \rangle = -(2\pi i)\delta(E_{\mathbf{p}'} - E_{\mathbf{p}}) t(\mathbf{p}',\mathbf{p}) \delta_{N'N} . \tag{2.6}$$

The t-matrix element is related to the amplitude f of the outgoing scattered wave in the absence of radiation by

$$t(\mathbf{p}',\mathbf{p}) = -(4\pi/2\mu)(2\pi)^{-3} f(\mathbf{p}',\mathbf{p}) . \tag{2.7}$$

Radiative effects are contained in the matrix element $S_{\alpha'\alpha}^{(2)}$ defined by

$$S_{\alpha'\alpha}^{(2)} = -(2\pi i)\delta(E_{\alpha'} - E_\alpha) U_{\alpha'\alpha} , \tag{2.8}$$

with

$$U_{\alpha'\alpha} = \langle \Phi_{\alpha'}^{(-)} | [H' + H' G^{(+)}(E_\alpha) H'] | \Phi_\alpha^{(+)} \rangle \tag{2.9}$$

and

$$G^{(+)}(W) = \lim_{\epsilon \to 0+} (W + i\epsilon - H)^{-1} . \tag{2.10}$$

In the following the one- and two-photon emission amplitudes in lowest nontrivial order of perturbation theory will be of particular concern. The matrix element for single-photon bremsstrahlung in first order is

$$\langle \Phi_{\alpha'}^{(-)} | H' | \Phi_\alpha^{(+)} \rangle = -(e/\mu c)(2\pi c^2/\omega L^3)^{1/2} \boldsymbol{\lambda} \cdot \langle u_{\mathbf{p}'}^{(-)} | \mathbf{p}_{\rm op} | u_{\mathbf{p}}^{(+)} \rangle , \tag{2.11}$$

and the two-photon emission amplitude in second order is

$$\langle \Phi_{\alpha'}^{(-)} | H' G_0^{(+)}(E_\alpha) H' | \Phi_\alpha^{(+)} \rangle = (e/\mu c)^2 (2\pi c^2/\omega_1 L^3)^{1/2} (2\pi c^2/\omega_2 L^3)^{1/2} M(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2) . \tag{2.12}$$

The matrix element M, of primary interest here, is defined as

$$M(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2) = \langle u_{\mathbf{p}'}^{(-)} | \boldsymbol{\lambda}_2\cdot\mathbf{p}_{\rm op} G_0^{(+)}(E_{\mathbf{p}} - \omega_1) \boldsymbol{\lambda}_1\cdot\mathbf{p}_{\rm op} | u_{\mathbf{p}}^{(+)} \rangle + \langle u_{\mathbf{p}'}^{(-)} | \boldsymbol{\lambda}_1\cdot\mathbf{p}_{\rm op} G_0^{(+)}(E_{\mathbf{p}} - \omega_2) \boldsymbol{\lambda}_2\cdot\mathbf{p}_{\rm op} | u_{\mathbf{p}}^{(+)} \rangle , \tag{2.13}$$

with

$$G_0^{(+)}(W) = \lim_{\epsilon \to 0+} (W + i\epsilon - H_0)^{-1} . \tag{2.14}$$

Higher-order contributions to the S matrix can be generated from the expansion $G^{(+)} = G_0^{(+)} + G_0^{(+)} H' G_0^{(+)} + \cdots$, but we shall not be concerned with such terms here.

To conclude this preliminary discussion we record, for later reference, the asymptotic forms in configuration space of the wave function and free Green's function. An explicit summation of the partial-wave expansion of $u_{\mathbf{p}}^{(+)}(\mathbf{r})$ for $r \to \infty$ provides us with the asymptotic form

$$u_{\mathbf{p}}^{(+)}(\mathbf{r}) \sim u_{\mathbf{p};\rm in}^{(+)}(\mathbf{r}) + u_{\mathbf{p};\rm out}^{(+)}(\mathbf{r}), \quad r \to \infty , \tag{2.15}$$

with

$$u_{\mathbf{p};\rm in}^{(+)}(\mathbf{r}) = (2\pi)^{-3/2}(i/2pr)(4\pi) \times e^{-ipr}(1 + L^2/2ipr)\delta(\Omega_{-\hat{\mathbf{r}}} - \Omega_{\hat{\mathbf{p}}}) . \tag{2.16}$$

Here $\mathbf{L} = -i\mathbf{r}\times\boldsymbol{\nabla}$ is the angular momentum operator. The directional δ function arises, in the course of the partial-wave summation procedure, from an application of the closure relation

$$\sum_{l,\nu}{}' (-1)^l Y_l^\nu(\hat{\mathbf{r}})[Y_l^\nu(\hat{\mathbf{p}})]^* = \delta(\Omega_{-\hat{\mathbf{r}}} - \Omega_{\hat{\mathbf{p}}}) . \tag{2.17}$$

The appearance of a correction term of order $1/r$ in Eq (2.16) results from our having retained the first *two* terms in the asymptotic expansion of the radial wave function in each partial wave.[7] In a similar way one finds (with corrections of order $1/r^2$ ignored)

$$u_{\mathbf{p};\rm out}^{(+)}(\mathbf{r}) = (2\pi)^{-3/2} e^{ipr}/r(1 - L^2/2ipr) f(p\hat{\mathbf{r}},\mathbf{p}) . \tag{2.18}$$

[These results must be modified slightly, in a manner discussed previously,[7] to account for the presence, in the potential $V(r)$, of a long-range Coulomb tail.] Since $[u_{\mathbf{p}}^{(-)}(\mathbf{r})]^* = u_{-\mathbf{p}}^{(+)}(\mathbf{r})$ the asymptotic behavior of this function can be obtained directly from the forms given above. Finally, with $W = q^2/2\mu$, the Green's function can be represented as

$$\langle \mathbf{r}' | G_0^{(+)}(W) | \mathbf{r} \rangle \equiv G_0^{(+)}(\mathbf{r}',\mathbf{r};W) \sim -(2\mu/4\pi)(2\pi)^{3/2} \times e^{iqr}/r(1 - L^2/2iqr) u_{-q\hat{\mathbf{r}}}^{(+)}(\mathbf{r}') , \tag{2.19}$$

valid for $r \to \infty$, $r' \ll r$.

III. ONE SOFT PHOTON

As a first step in the development of an approximation for the two-photon amplitude given in Eq. (2.13) it will be convenient to replace the momentum operators which appear there by coordinate operators; this is done with the aid of the commutation relation

$$\mathbf{p}_{\rm op}=-\mu[\mathbf{r}_{\rm op},H_0] \;. \tag{3.1}$$

Then, with $M=M_1+M_2$ and

$$M_2(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2)=M_1(\mathbf{p}',\mathbf{p};\omega_2\boldsymbol{\lambda}_2,\omega_1\boldsymbol{\lambda}_1) \;, \tag{3.2}$$

one obtains the configuration-space representation

$$M_1(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2)=-\mu\omega_1\omega_2\int d^3r\int d^3r' u_{-\mathbf{p}'}^{(+)}(\mathbf{r}')\boldsymbol{\lambda}_2\cdot\mathbf{r}'G_0^{(+)}(\mathbf{r}',\mathbf{r};E_{\rm p}-\omega_1)\boldsymbol{\lambda}_1\cdot\mathbf{r}u_{\mathbf{p}}^{(+)}(\mathbf{r}) \;. \tag{3.3}$$

Note that this transformation from the velocity form of the matrix element to the length form is not valid separately for M_1 and M_2; rather, in forming the sum there is a cancellation of terms whose presence can be traced to the fact that G_0 satisfies an *inhomogeneous* differential equation.

In the remainder of this section we consider the case where $\omega_1 \ll E_{\rm p}$, with $\omega_2 \gtrsim E_{\rm p}$. The limit in which both ω_1 and ω_2 are small compared with the scattering energy is studied in Sec. IV. For clarity in the following discussion we focus our attention on the amplitude M_1, with M then obtained by symmetrization with respect to the photon indexes.

We first observe that the integration over r in the expression (3.3) for M_1 diverges in the limit $\omega_1 \to 0$. Generalizing an earlier treatment of the single-photon bremsstrahlung amplitude[7] we define the approximation for the integral (3.3) by the condition that terms which remain finite in the limit $\omega_1 \to 0$ are ignored. In the case where the potential is short ranged the terms retained are of order ω_1^{-2} and ω_1^{-1}; taking into account the factor ω_1 multiplying the integral in Eq. (3.3), it follows that the leading approximation to M_1 is of order ω_1^{-1} with a correction of order ω_1^0. (The addition of a Coulomb tail to the potential results in the appearance of logarithmic terms which modify the nature of the correction term, in a manner indicated below.)

For ω_2 large enough, the integration domain in Eq. (3.3) corresponding to $r' \to \infty$, $r' > r$ does not involve a slowly varying exponential and accordingly does not lead to a near singularity. Following the approximation rule introduced above (retain only terms which are singular for $\omega_1 \to 0$) we confine the integration, as a first step, to the region $R<r<\infty$, $0<r'<r$; R is a constant, large enough so that the asymptotic forms given in Sec. II for the wave function and Green's function are applicable. With $u_{\mathbf{p}}^{(+)}(\mathbf{r})$ decomposed as in Eq. (2.15) it is readily seen that the integral involving $u_{\mathbf{p};\rm out}^{(+)}(\mathbf{r})$ may be neglected since a rapidly varying exponential factor appears in the integrand; that is, $u_{\mathbf{p}}^{(+)}(\mathbf{r})$ may be replaced by its incoming-wave component given in Eq. (2.16). Similar reasoning may be used to justify extension of the integration domain down to the origin—the contribution from the domain $r<R$ is finite in the limit $\omega_1 \to 0$. As a final simplification the restriction $r'<r$ may be dropped since the additional contribution coming from the domain $r'>r$ is nonsingular for $\omega_1 \to 0$, $\omega_2 \neq 0$. More explicitly, we write $\int_0^r dr' = \int_0^\infty dr' - \int_r^\infty dr'$ and neglect $\int_r^\infty dr'$. One may justify this directly (even when long-range Coulomb effects are included), since for r and r' both large, the integrand can be expressed in terms of known asymptotic forms and the integrations carried out explicitly. The nonsingular nature of the term to be neglected is then evident.

At this point the remainder of the calculation is essentially identical to the one described earlier [leading to Eq. (3.24) of Ref. 7] for the single-photon amplitude, with the radiationless scattering amplitude of the previous calculation here replaced by the amplitude for scattering with the emission of a single photon. This amplitude, corresponding to arbitrary photon frequency ω and polarization $\boldsymbol{\lambda}$, is given (in dipole approximation) by the expression

$$\overline{M}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})=(-i\mu\omega)\int d^3r[u_{\mathbf{p}'}^{(-)}(\mathbf{r})]^*\boldsymbol{\lambda}\cdot\mathbf{r}u_{\mathbf{p}}^{(+)}(\mathbf{r}) \;, \tag{3.4}$$

with $p'^2/2\mu+\omega=p^2/2\mu$ expressing energy conservation. The result of the calculation is a low-frequency approximation ($\omega_1 \ll E_{\rm p}$) for $M=M_1+M_2$ of the form

$$\begin{aligned} M(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2) &\cong \frac{1}{\omega_1}\boldsymbol{\lambda}_1\cdot\mathbf{p}'\overline{M}(\mathbf{p}'+\mu\omega_1\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_1\cdot\mathbf{p}',\mathbf{p};\omega_2\boldsymbol{\lambda}_2) \\ &\quad -\frac{1}{\omega_1}\boldsymbol{\lambda}_1\cdot\mathbf{p}\overline{M}(\mathbf{p}',\mathbf{p}-\mu\omega_1\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_1\cdot\mathbf{p};\omega_2\boldsymbol{\lambda}_2) \;. \end{aligned} \tag{3.5}$$

As remarked earlier, this approximation provides not only the correct residue of the pole at $\omega_1=0$ but also the leading correction term; the error is of order ω_1 for $\omega_1 \to 0$ with ω_2 fixed and nonvanishing. A second noteworthy feature is that, with corrections of order ω_1^2 ignored, the relations

$$(1/2\mu)(\mathbf{p}-\mu\omega_1\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_1\cdot\mathbf{p})^2=p^2/2\mu-\omega_1 \;, \tag{3.6a}$$

$$(1/2\mu)(\mathbf{p}'+\mu\omega_1\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_1\cdot\mathbf{p}')^2=p'^2/2\mu+\omega_1 \;, \tag{3.6b}$$

are satisfied so that, as input in Eq. (3.5), only on-shell values of $\overline{M}$ need be known. The result (3.5) is in conformity with Low's theorem,[3,4] which is expected to hold for a general scattering process accompanied by the emission of a single soft photon. Another point of comparison is provided by the calculation of the two-photon amplitude with $V(r)$ purely Coulombic.[2] The limiting form, corresponding to a single soft photon, with only the leading term retained, was quoted in Ref. 2; that result is consistent with the approximation (3.5).

The modification of Eq. (3.5) due to the effect of a long-range Coulomb tail, $V(\mathbf{r})\sim g/r$ for $r\to\infty$, can be obtained straightforwardly by including the appropriate logarithmic distortions in the asymptotic forms of the wave functions and Green's function, in a manner very similar to that described previously.[7] The result is to introduce an additional factor $B(p',q_2)$ in the first term on the right-hand side (rhs) of Eq. (3.5) and a factor $B(p,q_1)$ in the second term, where $q_i^2/2\mu=E_p-\omega_i$, $i=1,2$. The function $B(p',p)$ is defined in Eq. (3.19) of Ref. 7. When expanded in terms of $\beta=(p-p')/p$ one finds

$$B(p',p)=1-i(g\mu/p)\beta\ln(|\beta|/2)+O(\beta)\ , \tag{3.7}$$

indicating the origin of the logarithmic corrections associated with the Coulombic behavior of the potential at great distances. Coulomb effects are more difficult to keep track of in the case, considered below, in which both photons are soft. These additional complications should be surmountable but we have not attempted to do so here. In the following discussion the potential is assumed to be of short range.

IV. TWO SOFT PHOTONS

As a first approach to the problem of estimating the two-photon amplitude when both photons are soft we adopt the approximation (3.5) and replace the single-photon amplitude $\overline{M}$ appearing in that expression by a low-frequency approximation of the form derived previously.[7] This procedure, which has the merit of simplicity, is based on the assumption that the expression (3.5) may be analytically continued from large to small values of ω_2. The validity of the analytic continuation can be justified through a more careful analysis of the integral (3.3), as discussed subsequently.

To begin, we introduce the decomposition

$$\overline{M}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})=\overline{M}_{\rm LFA}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})+\overline{R}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})\ , \tag{4.1}$$

with the low-frequency approximation defined as

$$\overline{M}_{\rm LFA}(\mathbf{p}',\mathbf{p};\omega\boldsymbol{\lambda})=(1/\omega)\boldsymbol{\lambda}\cdot\mathbf{p}'t(\mathbf{p}'+\mu\omega\boldsymbol{\lambda}/\boldsymbol{\lambda}\cdot\mathbf{p}',\mathbf{p}) -(1/\omega)\boldsymbol{\lambda}\cdot\mathbf{p}t(\mathbf{p}',\mathbf{p}-\mu\omega\boldsymbol{\lambda}/\boldsymbol{\lambda}\cdot\mathbf{p})\ , \tag{4.2}$$

with $E_{p'}=E_p-\omega$. [From this form, and the analogous result (3.5), it is clear how to write down an approximation for scattering with the emission of one soft photon and N hard photons, in terms of the N-photon amplitude.] To first order in ω the t-matrix element appearing in Eq. (4.2) is on shell by virtue of relations, of the type shown in Eqs. (3.6), satisfied by the shifted momenta. Since $\overline{M}_{\rm LFA}$ correctly accounts for terms of order ω^{-1} and ω^0 in the low-frequency limit, the remainder $\overline{R}$ is of order ω in that limit.

The result obtained by combining Eqs. (4.1) and (4.2) with the expression (3.5) for the two-photon amplitude takes on its simplest form when the t matrix is expressed in terms of the energy and momentum-transfer variables. That is, we write

$$t(\mathbf{p}',\mathbf{p})\equiv T(E_p,(\mathbf{p}'-\mathbf{p})^2)\ . \tag{4.3}$$

In fact, the t matrix appears in Eq. (4.2) with shifted momenta. Since it need only be represented correctly to first order, we may write, for example,

$$t(\mathbf{p}'+\mu\omega_2\boldsymbol{\lambda}_2/\boldsymbol{\lambda}_2\cdot\mathbf{p}',\mathbf{p}-\mu\omega_1\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_1\cdot\mathbf{p}) \cong T(E_p-\omega_1,(\mathbf{p}'-\mathbf{p})^2)+2\mu(\omega_2\boldsymbol{\lambda}_2/\boldsymbol{\lambda}_2\cdot\mathbf{p}'+\omega_1\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_1\cdot\mathbf{p})\cdot(\mathbf{p}'-\mathbf{p})(\partial/\partial\tau)T(E_p,\tau)\big|_{\tau=(\mathbf{p}'-\mathbf{p})^2}\ , \tag{4.4}$$

with similar expansions introduced for the other t-matrix elements which appear with different arguments. We then arrive, after a bit of algebra, to the approximation

$$M\cong M_{\rm LFA}+R\ , \tag{4.5}$$

with

$$M_{\rm LFA}(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2)=(\omega_1\omega_2)^{-1}[(\boldsymbol{\lambda}_1\cdot\mathbf{p}')(\boldsymbol{\lambda}_2\cdot\mathbf{p}')T(E_p,\tau)+(\boldsymbol{\lambda}_1\cdot\mathbf{p})(\boldsymbol{\lambda}_2\cdot\mathbf{p})T(E_{p'},\tau) -(\boldsymbol{\lambda}_1\cdot\mathbf{p}')(\boldsymbol{\lambda}_2\cdot\mathbf{p})T(E_p-\omega_2,\tau)-(\boldsymbol{\lambda}_1\cdot\mathbf{p})(\boldsymbol{\lambda}_2\cdot\mathbf{p}')T(E_p-\omega_1,\tau)] \tag{4.6}$$

and $\tau=(\mathbf{p}'-\mathbf{p})^2$. Remarkably, all of the correction terms which are explicitly of order $1/\omega_1$ or $1/\omega_2$ have canceled in the expression (4.6).[8] (Terms of this type would be introduced if the T matrix were expanded about the energy E_p, but this is not necessary, nor would it be appropriate in the neighborhood of a resonance which caused the T matrix to vary rapidly with energy. In this connection it should be recalled that Feshbach and Yennie[9] pointed out some time ago that low-frequency approximations could be expressed in a form which remains valid in the presence of scattering resonances, and further that this could be used to advantage in the analysis of resonance phenomena.)

A simple diagrammatic interpretation can be given to each of the terms on the rhs of Eq. (4.6). The first term corresponds to radiationless scattering at energy E_p followed by the emission of two photons. In the second term the two photons have been emitted before the scattering takes place; the value for the scattering energy, $E_{p'}=E_p-\omega_1-\omega_2$, is then the appropriate one. The third and fourth terms account for emission of one photon before and one photon after the scattering event; again, the T-matrix element appears with the appropriate energy variable.

The remainder in the approximation (4.5) is

$$R(\mathbf{p}',\mathbf{p};\omega_1\boldsymbol{\lambda}_1,\omega_2\boldsymbol{\lambda}_2)=(1/\omega_1)(\boldsymbol{\lambda}_1\cdot\mathbf{p}')\bar{R}(\mathbf{p}'+\mu\omega_1\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_1\cdot\mathbf{p}',\mathbf{p};\omega_2\boldsymbol{\lambda}_2)-(1/\omega_1)(\boldsymbol{\lambda}_1\cdot\mathbf{p})\bar{R}(\mathbf{p}',\mathbf{p}-\mu\omega_1\boldsymbol{\lambda}_1/\boldsymbol{\lambda}_1\cdot\mathbf{p};\omega_2\boldsymbol{\lambda}_2)+(\omega_1\boldsymbol{\lambda}_1\leftrightarrow\omega_2\boldsymbol{\lambda}_2)\ . \tag{4.7}$$

The first two terms on the rhs arise from the approximation procedure described above and the terms with photon indexes interchanged have been added to impose Bose symmetry. These latter terms vanish in the limit $\omega_1\to 0$, with ω_2 fixed and finite. Therefore, they are not contained in the approximation (3.5), nor are they excluded by that approximation. It will be seen below that these terms appear naturally when the calculation is performed in an explicitly symmetric manner. The diagrammatic interpretation of the terms appearing in Eq. (4.7) is clear. Each represents a process in which one of the photons is emitted in either the initial or final state while the other is emitted during the scattering process. This latter emission is described not by the full single-photon bremsstrahlung amplitude but rather the remainder, defined in Eq. (4.1), as required to avoid double counting of terms already included. The momenta appearing as arguments in the remainder term are shifted in such a way as to preserve the physically appropriate energy conservation conditions. It should be emphasized that each of the subsystem scattering operators that appear in the low-frequency approximation—the T matrices in Eq. (4.6) and the remainder terms in Eq. (4.7)—is a physical, on-shell amplitude and can be related directly to measurements of cross sections for processes of a simpler kind than the two-photon process of primary interest. Presumably, results of a similar nature can be established for transitions in which more than two soft photons are radiated.

The approximation contained in Eqs. (4.5)–(4.7) displays the singularity structure expected from simple perturbation-theory considerations. That is, the amplitude M_{LFA} contains the double-pole terms, which arise from sums of products of single poles according to the relation

$$\omega_1^{-1}(\omega_1+\omega_2)^{-1}+\omega_2^{-1}(\omega_1+\omega_2)^{-1}=\omega_1^{-1}\omega_2^{-1}\ ,$$

and the remainder R represents the single-pole terms corresponding to only one photon emitted in an initial or final state. According to the asymptotic evaluation method which was described earlier and used in the derivation of Eq. (4.5), the error in the approximation must vanish in the limit $\omega_1\to 0$, ω_2 fixed and finite, and also in the limit $\omega_2\to 0$, ω_1 fixed and finite. This, in itself, does not exclude the possibility that terms have been neglected which are nonvanishing in the limit $\omega_1\to 0$, $\omega_2\to 0$, with ω_1/ω_2 fixed and finite. The simplest function of this type which also satisfies the requirement of symmetry is $\omega_1\omega_2/(\omega_1+\omega_2)^2$. However, a second-order pole in the variable $(\omega_1+\omega_2)$ is inconsistent with perturbation theory and may be ruled out. Terms of a more complicated structure, obtained by introducing additional powers of $(\omega_1+\omega_2)^{-1}$, or by allowing the appearance of branch-point singularities, may be ruled out on similar grounds. In the absence of such terms the approximation (4.5) is seen to be exact in the low-frequency limit, the leading terms in the error vanishing like ω_1 or ω_2 in that limit.[10]

The derivation given above of the low-frequency approximation (4.5) was based on analytic continuation of the approximation (3.5) from large to small values of ω_2. In addition, a term was added to the remainder (4.7) to conform to the Bose symmetry requirement. The same result may be obtained through a more direct procedure. This involves an asymptotic evaluation of the original integral for the two-photon amplitude, in which the two photons are treated on an equal footing, so that Bose symmetry is manifest at the outset and no analytic continuation is required. For completeness we now briefly outline this alternative derivation.

Consider once again the matrix element $M=M_1+M_2$, with components M_1 and M_2 defined in Eqs. (3.2) and (3.3). The wave functions and Green's function appearing in each of these integrals are replaced by their asymptotic forms, as given in Sec. II. These integrals are now evaluated in a manner very similar to that described earlier in connection with the derivation of Eq. (3.5), but with the assumption $\omega_2\gg\omega_1$ now dropped. This makes the calculation rather more involved since one must now deal separately with the regions of configuration space corresponding to $r>r'$ and $r<r'$, with the appropriate asymptotic form of the Green's function adopted for each region.

A typical integral encountered is of the form $\int_0^\infty dr\int_0^r dr' I(r',r)$ and this is rewritten as $\int_0^\infty dr\int_0^\infty dr' I(r',r)-\int_0^\infty dr\int_r^\infty dr' I(r',r)$. The first integral can be evaluated, to the required accuracy, fairly easily; the calculation is similar to that performed in the derivation of Eq. (3.5). Evaluation of the second integral is more tedious, but still straightforward. One obtains in this way the approximation

$$M\cong N^{(1)}-N^{(2)}\ , \tag{4.8a}$$

with

$$N^{(1)}=2M_{\text{LFA}}+R\ , \tag{4.8b}$$

$$N^{(2)}=M_{\text{LFA}}\ . \tag{4.8c}$$

Here M_{LFA} and R are given by Eqs. (4.6) and (4.7), respectively, so that Eqs. (4.8) *do* reproduce the earlier results.

As outlined above the calculational procedure is oversimplified since the form (2.19) for the Green's function should be modified in the domain in which $|\mathbf{r}-\mathbf{r}'|$ is bounded with r and r' each very large. It is possible to estimate the error introduced by our having ignored this complication. The dominant correction is obtained by replacing $G_0(\mathbf{r}',\mathbf{r};q^2/2\mu)$, the Green's function in the presence of the scattering potential, by the "free" Green's function

$$-\frac{2\mu}{4\pi}\frac{\exp(iq\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\,.$$

(It may be shown, as justification for this replacement, that the component of the Green's function thus omitted leads to integrands with rapidly varying exponential factors from which no near singularities can be generated.) Next we introduce the variables $\boldsymbol{\rho}=\mathbf{r}-\mathbf{r}'$, $\mathbf{R}=(\mathbf{r}+\mathbf{r}')/2$, and write $\int d^3r \int d^3r' = \int d^3\rho \int d^3R$. The ρ integration, when restricted to a sphere of finite radius, is nonsingular and the dominant contribution to the R integration comes from terms of the form

$$\lim_{\epsilon\to 0+}\int_0^\infty dR\, R^2 \exp\{[-\epsilon \pm i(p'-p)]R\} = \pm 2i/(p-p')^3$$

in which both r and r' have been replaced by R as a first approximation. In this way the leading correction term is found to have a frequency dependence of the form $\omega_1\omega_2/(\omega_1+\omega_2)^3$. Higher-order corrections to this first estimate arise from terms containing additional factors of R^{-1} in the integrand and these introduce additional powers of $(\omega_1+\omega_2)$. Now we have pointed out that such terms, which vanish in the limit $\omega_1\to 0$, ω_2 fixed, or $\omega_2\to 0$, ω_1 fixed, have been neglected elsewhere in the calculation [i.e., in the derivation of Eqs. (4.8)] so that it is consistent to ignore them here as well, in evaluating the contribution from the domain ρ finite, $R\to\infty$. Presumably, a more accurate evaluation of the integral would show that terms of the form $\omega_1\omega_2/(\omega_1+\omega_2)^3$ and $\omega_1\omega_2/(\omega_1+\omega_2)^2$ actually cancel in the final result, since otherwise the physical amplitude would contain second- and third-order poles in the variable $\omega_1+\omega_2$ and this cannot be the case in second-order perturbation theory.

In summary, the low-frequency approximation contained in Eqs. (4.5)–(4.7) is confirmed by this alternative derivation. It has been shown that the error in the approximation vanishes in the separate limits $\omega_1\to 0$ or $\omega_2\to 0$. If we add the assumption that higher-order poles in $\omega_1+\omega_2$ are absent—which is reasonable on the basis of the expected singularity structure of the amplitude, as discussed above—the error must in fact vanish when ω_1 and ω_2 vanish, even when ω_1/ω_2 remains fixed and finite.

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[10]If the potential were Coulombic at great distances, arguments of the above type, based on analyticity assumptions, would require closer examination.