Path integrals in multiply connected spaces and the fractional angular momentum quantization

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The homotopy theory of path integrals in two-dimensional multiply connected spaces is used to reexamine the wave function and angular momentum of a charged particle on a circle in various configurations. A topological phase factor arising from the nontrivial representation of the fundamental group leads to the fractional angular momentum quantization.

I. INTRODUCTION

Homotopy theory is a powerful tool for the study of path integrals in multiply connected spaces.[1–3] The use of homotopy theory gives rise to a multivalued function defined over the group manifold which is characteristic to the system in question.[1] A simple model yielding the concept of spin is the spherical top, whose group manifold M is of SO(3). It has been shown that the manifold M is doubly connected. The associated classical paths fall into two homotopy classes corresponding to the integral and half-integral spins[1] which are consistent with the result from the Lie algebra of SO(3).

For the rotation group SO(2) in two dimensions parametrized by the angle of rotation ϕ, the group manifold M consists of the points on a circle, i.e., a one-dimensional sphere S^1 which is, however, infinitely connected.[4] The fractional angular momentum (spin) naturally appears if the nontrivial representation of the fundamental group of M, which is responsible for a multivalued wave function, is considered.

The possibility of fractional angular momentum in two-space has been proposed in a different context.[5] However, the idea of the fractional angular momentum for a system consisting of an electric charge and an impenetrable, current-carrying solenoid gives rise to much controversy. Jackiw and Redlich[6] have pointed out that there is a difference between the kinetic angular momentum and the canonical angular momentum whenever velocity-dependent forces apply.[7] The generator of rotation is the canonical angular momentum prescribed by Noether's theorem as a conserved quantity. The canonical angular momentum has conventional discrete eigenvalues if the requirement of single valuedness is imposed on the wave function. Therefore, flux-dependent eigenvalues of the kinetic angular momentum do not lead to the fractional angular momentum quantization.[6] It has also been pointed out that the difference between the kinetic and the canonical angular momenta cannot be removed by a so-called singular gauge transformation, because such a transformation changes the magnetic field.[8]

We fully agree with Jackiw and Redlich that the generator of rotation should be the canonical angular momentum. Now let us consider quantization of the canonical angular momentum itself. The SO(2) group does not give a unique definition for the angular momentum; one may add any arbitrary constant to the angular momentum eigenvalues. Equivalently, an arbitrary angular phase can be added to the wave function.[6] Neither completeness of angular momentum eigenfunctions nor hermiticity of operators is able to eliminate an arbitrary common phase of all the eigenfunctions which would shift all eigenvalues of the angular momentum by an equal amount.[6] We show in the present paper that the angular phase factor of wave functions is equivalent to the nontrivial representation of the fundamental group in the homotopy theory of path integrals. The phase factor is indeed well defined according to the symmetry of a system under consideration and is physically meaningful.

In Sec. II, the homotopy theory of path integrals in two-dimensional multiply connected space is briefly reviewed. A method to determine the one-dimensional, unitary representation of the fundamental group is given. In Secs. III–V, the wave functions and angular momentum eigenvalues of a charged particle on a circle in various configurations are obtained to illustrate our theory.

II. PATH INTEGRALS IN MULTIPLY CONNECTED SPACES

The space of our interest M is a plane R^2 minus a disc Δ,

$$M = R^2 - \Delta \ . \tag{2.1}$$

The propagator for a particle moving from $\mathbf{r}'(t')$ to $\mathbf{r}''(t'')$ is given by the path integral expression[9]

$$K(\mathbf{r}'',t'';\mathbf{r}',t') = \int D(\mathbf{r}(t))\exp\left[i\int_{t'}^{t''} d\tau L(\mathbf{r}(\tau);\dot{\mathbf{r}}(\tau);\ \tau)\right] , \tag{2.2}$$

where L is the classical Lagrangian and $\hbar = c = 1$ throughout. For a free motion

$$L = \frac{\mu}{2}\dot{\mathbf{r}}^2 , \tag{2.3}$$

μ being the mass of the particle.

The propagator on the covering space of M, denoted by M^*, is well defined. In our case M^* is similar to the Riemann sheets in a complex plane. In the original space M the propagator is a linear combination of covering space propagators,[10,11]

$$K(\mathbf{r}'',t'';\mathbf{r}',t')=\sum_{n=-\infty}^{\infty} a(f_n)K_n(\mathbf{r}'',t'';\mathbf{r}',t') , \tag{2.4}$$

where K_n, which belongs to the nth homotopy class $\Gamma_n(\mathbf{r}'',\mathbf{r}')$, is the sum over paths which loop around the disc $(n-1)$ times for $n>1$ (negative n for clockwise loops). A third class is formed by attaching the heads of paths in one class to the tails of those in the other. There is a group structure with this operation and the group is called the fundamental group[10] of M denoted by F. The coefficient $a(f_n)$ in Eq. (2.4) should be a one-dimensional unitary representation of the fundamental group F,

$$|a(f_n)|=1, \quad a(f_n)a(f_m)=a(f_nf_m) \tag{2.5}$$

where $f_n,f_m \in F$. In our case F is a discrete group and

$$a(f_n)=\exp(-in\delta) . \tag{2.6}$$

The arbitrary parameter (but with $0\le\delta<2\pi$) is independent of $\mathbf{r}$ and the winding numbers. There are two results directly obtained from Eqs. (2.4)–(2.6).

(1) The nontrivial representation ($\delta\neq0$) of the fundamental group F results in the multivalued propagator as well as the wave function.[11] The proof[11] is trivial. According to the definition of the winding number, in cylindrical coordinates we have

$$K_n(r'',\phi''\pm2\pi,t'';r',\phi',t')=K_{n\pm1}(r'',\phi'',t'';r',\phi',t') . \tag{2.7}$$

Using Eq. (2.7) it is easy to show that the propagator as well as the wave function satisfies the Bloch condition

$$K(r'',\phi''+2\pi,t'';r',\phi',t') =\exp(i\delta)K(r'',\phi'',t'';r',\phi',t') . \tag{2.8}$$

(2) The representation reduces to the trivial one ($\delta=0$) if for any line through the center of disc Δ there are symmetric relations as

$$K_n(\mathbf{r}_0,t'';\mathbf{r}',t')=K_{-n+1}(\mathbf{r}_0,t'';\mathbf{r}',t') . \tag{2.9}$$

$\mathbf{r}_0$ and $\mathbf{r}'$ lie on opposite sides of the disc along the line. In the cylindrical coordinates, one has that $\phi_0-\phi'=-\pi$, where ϕ_0 and ϕ' are the polar angles of $\mathbf{r}_0$ and $\mathbf{r}'$, respectively.

The proof is also very simple. The propagator can be written as

$$K(\mathbf{r}_0,t'';\mathbf{r}',t')=\sum_{n=-\infty}^{\infty}\exp[-i(-n+1)\delta]K_{-n+1} \tag{2.10}$$

by changing the index of summation. Replacing K_{-n+1} by K_n we obtain an alternative form

$$K=\sum_{n=-\infty}^{\infty} a'(f_n)K_n , \tag{2.11}$$

where

$$a'(f_n)=\exp[-i(-n+1)\delta] . \tag{2.12}$$

$a'(f_n)$ should also be a representation of the fundamental group. According to the properties of (2.5), we must have the relation

$$\exp[-i(-n+1)\delta-i(-m+1)\delta] =\exp\{-i[-(n+m)+1]\delta\} , \tag{2.13}$$

for any n and m. Therefore,

$$\delta=2N\pi , \tag{2.14}$$

where N is an integer or zero. Since $0\le\delta<2\pi$, the only one possibility is $\delta=0$. There is a simple physical argument[11] leading to the condition (2.9). If one considers the diffraction pattern resulting from scattering of a free particle by the impenetrable obstacle Δ, the pattern should be symmetric with respect to the line $\mathbf{r}_0-\mathbf{r}'$. Here we may suppose that the source of particles is located at $\mathbf{r}'$.

If a system does not have the symmetric relations (2.9), then δ does not have to be zero. The nontrivial representation may be considered. A physical example that shows the violation of condition (2.9) is the Aharonov-Bohm interference,[12–14] the pattern of which is no longer symmetric with respect to the line $\mathbf{r}_0-\mathbf{r}'$. The interference fringes shift from the line of symmetry due to the presence of an inaccessible magnetic flux.[13]

The nonzero phase parameter δ was first used by Schulman in the literature.[1] In the present paper we show that the phase parameter can be determined only by the symmetry of space, if the propagator K_n can be evaluated exactly in a moving coordinate frame which moves along with particle in classical sense.[15] For a quadratic Lagrangian the propagator can be evaluated directly from the classical action.[16,17] The calculation to determine δ using the method developed in this paper may be simple.

III. ANGULAR MOMENTUM AND WAVE FUNCTION OF A PARTICLE ON A CIRCLE

Let us consider an electron confined on a circle with radius R which is the simplest example suited to discuss quantum mechanics in two-dimensional, multiply connected spaces.

The Lagrangian for the electron is

$$L=\tfrac{1}{2}I\dot{\phi}^2 , \tag{3.1}$$

where $0\le\phi\le2\pi$ and $I=\mu R^2$. The propagator is

$$K(\phi'',t'';\phi',t')=\sum_{n=-\infty}^{\infty}\exp(-in\delta)K_n(\phi'',t'';\phi',t') . \tag{3.2}$$

$K_n(\phi'',t'';\phi',t')$ can be lifted to the propagator in covering space M^*, which is a line in our case. The relations between two end points (ϕ'',ϕ') of paths in M and end points $(\tilde{\phi}'',\tilde{\phi}')$ in M^* are

$$\tilde{\phi}''_n=\phi''+2n\pi, \quad \tilde{\phi}'=\phi'$$

and

$$K_n(\phi'',t'';\phi',t')=\tilde{K}(\tilde{\phi}''_n,t'';\tilde{\phi}',t') , \quad (3.3)$$

where $-\infty<\tilde{\phi}<\infty$ and $\tilde{K}$ is a one-dimensional free-particle propagator in M^*. For a quadratic Lagrangian, the propagator can be evaluated from the classical action S_c via the Van Vleck-Plauli formula,[16,17]

$$\tilde{K}(\tilde{\phi}''_n,t'';\tilde{\phi}',t') = \left[\frac{i}{2\pi}\frac{\partial^2 S_c}{\partial\tilde{\phi}'\partial\tilde{\phi}''_n}\right]^{1/2}\exp[iS_c(\tilde{\phi}''_n,t'';\tilde{\phi}',t')] . \quad (3.4)$$

Using the Lagrangian (3.1) one obtains the propagator from (3.4)

$$\tilde{K}(\tilde{\phi}''_n,t'';\tilde{\phi}',t')=\left[\frac{I}{2\pi iT}\right]^{1/2}\exp\left[\frac{iI}{2T}(\Delta\tilde{\phi}_n)^2\right] , \quad (3.5)$$

where $T=t''-t'$ and $\Delta\tilde{\phi}_n=\tilde{\phi}''_n-\tilde{\phi}'$. The propagator belonging to the nth homotopy class in M then is

$$K_n(\phi'',t'';\phi',t')=\left[\frac{I}{2\pi iT}\right]^{1/2}\exp\left[\frac{iI}{2T}(\phi+2n\pi)^2\right] , \quad (3.6)$$

with $\phi=\phi''-\phi'$ and $\phi'\geq\phi''\geq 0$.

Now we determine the phase parameter δ according to the symmetry of the propagator. For any line $(\mathbf{r}'-\mathbf{r}'')$ through the center of the circle we have

$$\phi=\phi''-\phi'=-\pi . \quad (3.7)$$

The substitution of $\phi=-\pi$ into Eq. (3.6) yields that $K_n=K_{-n+1}$; namely, the condition (2.9) is satisfied. Therefore, δ should be zero.

The total propagator (3.2) becomes

$$K(\phi'',t'';\phi',t') = \left[\frac{I}{2\pi iT}\right]^{1/2}\exp\left[\frac{iI\phi^2}{2T}\right]\Theta_3\left[\frac{\pi i\phi}{T},\frac{2\pi I}{T}\right] , \quad (3.8)$$

where

$$\Theta_3(z,y)\equiv\sum_{n=-\infty}^{\infty}\exp(i\pi yn^2+i2nz) \quad (3.9)$$

is the Jacobi Θ function. On the other hand, the propagator can be written as a sum over energy eigenstates according to the conventional quantum mechanics,

$$K(\phi'',t'';\phi',t') = \sum_{m=-\infty}^{\infty}\psi_m(\phi'')\psi_m^*(\phi')\exp[-iE_m(t''-t')] . \quad (3.10)$$

By virtue of Jacobi's transformation,

$$\Theta_3(z,y)=(-iy)^{-1/2}\exp(z^2/i\pi y)\Theta_3(z/y,-1/y) ,$$

the propagator (3.8) can be reduced to the form (3.10) with

$$\psi_m=\frac{1}{\sqrt{2\pi}}\exp(im\phi),\quad E_m=\frac{m^2}{2I} . \quad (3.11)$$

In this example the angular momentum is quantized in standard manner and the energy eigenfunctions of angular momentum are single valued.

Schulman[1] has made a generalization of the results (3.8) and (3.11) including the nonzero phase parameter δ. To make a possible interpretation of the nonzero phase, let us suppose that the ring is rotating about the axis with angular velocity ω. In a coordinate frame, which rotates with the ring,[15] the angle and the angular velocity are given by

$$\beta=\phi-\omega t,\quad \dot{\beta}=\dot{\phi}-\omega . \quad (3.12)$$

The Lagrangian then is

$$L=\frac{I}{2}(\dot{\beta}+\omega)^2=\frac{I}{2}\dot{\beta}^2+I\omega\dot{\beta}+\frac{I\omega^2}{2} . \quad (3.13)$$

The propagator belonging to the nth homotopy class in the rotating coordinate frame is obtained from (3.6) by the transformation $\phi=\beta+\omega T$,

$$K_n=\left[\frac{I}{2\pi iT}\right]^{1/2}\exp\left[\frac{iI}{2T}(\beta+2n\pi)^2+i\gamma(\beta+2n\pi)+i\frac{\gamma^2}{2I}T\right] , \quad (3.14)$$

where $\phi=\phi''-\phi'$, $\beta=\beta''-\beta'$, $T=t''-t'$, and $\gamma=I\omega$.

For $\beta=-\pi$ one finds that

$$K_{-n+1}=K_n\exp[-i2\gamma(2n-1)\pi] . \quad (3.15)$$

The propagator is no longer symmetric. Following the same procedure as that of (2.10)–(2.13), one obtains the nontrivial phase

$$\delta=2\pi\gamma . \quad (3.16)$$

Thus the total propagator can be expressed as

$$K=\left[\frac{I}{i2\pi T}\right]^{1/2}\exp\left[i\frac{I\phi^2}{2T}\right]\Theta_3\left[\pi\left[\frac{I\phi}{T}-\gamma\right],\frac{2I\pi}{T}\right] . \quad (3.17)$$

In much the same fashion that (3.11) results from (3.8) via (3.10), the wave function and the energy spectrum for (3.13) follow from (3.17);

$$\psi_m=\frac{1}{\sqrt{2\pi}}\exp(im\phi+i\gamma\phi)$$

and (3.18)

$$E_m=\frac{1}{2I}(m+\gamma)^2 .$$

Our results are straightforward. The two systems, (3.1) and (3.13), are different only by a constant angular velocity ω. Classically, the equations of motion for the two systems are the same in form, but their angular momenta differ by γ. In quantum mechanics the two systems are described by the same Schrödinger equation with different boundary conditions. The shifts of the angular momenta and of the kinetic energies are the same as the classical results. In other words, the correspondence principle is satisfied.

Notice that the angle ϕ is defined in the region $0 \le \phi \le 2\pi$ and the multivalued phase parameter is common for all eigenfunctions. An orthonormal basis, even though multiplied by a common multivalued phase factor, satisfies the orthogonality relation and the completeness condition. From the physical point of view, nothing is wrong with the multivalued wave functions.

IV. ANGULAR MOMENTUM SPECTRUM OF AN ELECTRON ON A RING IN THE PRESENCE OF LONG-RANGE MAGNETIC FLUX

Let us suppose there is a flux line, $\Phi = \alpha\Phi_0$, coinciding with axis of the ring. Here $\Phi_0 = 1/e$ is the fundamental fluxon and $0 \le \alpha < 1$. The Lagrangian is

$$L = \frac{I}{2}\dot{\phi}^2 + \alpha\dot{\phi} \ . \tag{4.1}$$

Since the magnetic flux is time independent, the vector potential established by the inaccessible magnetic flux cannot change the angular velocity of the electron classically.

The propagator can be evaluated by changing the variable,

$$\Theta = \phi + \frac{\alpha}{I}t \tag{4.2}$$

and

$$L = \frac{I}{2}\dot{\theta}^2 - \frac{\alpha^2}{2I} \ . \tag{4.3}$$

Lifting to the covering space M^*, the propagator belonging to the paths of the nth homotopy class is

$$\tilde{K}(\tilde{\theta}''_n, t''; \tilde{\theta}', t') = \left[\frac{I}{2\pi i T}\right]^{1/2} \exp\left[\frac{iI}{2T}(\Delta\tilde{\Theta}_n)^2 - \frac{i\alpha^2 T}{2I}\right] , \tag{4.4}$$

where $\Delta\tilde{\Theta}_n = \tilde{\Theta}''_n - \tilde{\Theta}'$. In the original variable ϕ,

$$\begin{aligned} K_n(\phi'', t''; \phi', t') &= \left[\frac{I}{2\pi i T}\right]^{1/2} \exp\left[\frac{iI}{2T}\left[\phi + \frac{\alpha}{I}T + 2n\pi\right]^2 - \frac{i\alpha^2 T}{2T}\right] \\ &= K_n^0(\phi'', t''; \phi', t')\exp[i\alpha(\phi + 2n\pi)] , \end{aligned} \tag{4.5}$$

where

$$K_n^0(\phi'', t''; \phi', t') = \left[\frac{I}{2\pi i T}\right]^{1/2} \exp\left[\frac{iI}{2T}(\phi + 2n\pi)^2\right] \tag{4.6}$$

is the propagator in the absence of the magnetic flux and $\phi = \phi'' - \phi'$.

Because of the inaccessible magnetic flux, the symmetric condition (2.9) is broken. For $\phi = -\pi$ one obtains[18] from Eq. (4.5)

$$K_{-n+1} = K_n \exp[-i2\alpha(2n-1)\pi] \ . \tag{4.7}$$

Repeating the procedure (2.10)–(2.13) again, one finds

$$\delta = 2\pi\alpha \ . \tag{4.8}$$

The total propagator,

$$K = K^0 \exp(i\alpha\phi) , \tag{4.9}$$

is then multivalued. Here K^0 is the propagator (3.8). The corresponding eigenfunctions of angular momentum and eigenvalues of energy are

$$\psi_m = \frac{1}{\sqrt{2\pi}}\exp[i(m+\alpha)\phi] , \tag{4.10}$$

$$E_m = \frac{m^2}{2I} \ . \tag{4.11}$$

The eigenvalues of canonical angular momentum are shifted by a fractional number α. However, the kinetic angular momentum is not changed by the inaccessible time-independent magnetic flux, because there is no torque applied to the electron. In the literature the vector potential of the inaccessible magnetic flux is formally removed from the Hamiltonian by the singular gauge transformation[19] in order to reject the Aharonov-Bohm (AB) effect.[20] However, the wave functions become multivalued (4.10) in the new gauge. The multivalued wave functions (4.10) which incur a phase change under a rotation around the magnetic flux naturally cause the AB interference. It has been pointed out that an AB interference experiment, which can be used to test the existence of fractional angular momentum, is easily interpreted with the topological phase factor defined on a circle, that is, the group manifold of SO(2).[21] The AB effect is also studied with path integration by deducing winding number contributions, where the flux-dependent phase is included in each winding propagator.[22]

V. ANGULAR MOMENTUM EIGENVALUES OF THE ELECTRON IN THE PRESENCE OF LONG-RANGE, TIME-DEPENDENT MAGNETIC FLUX

The single-valuedness requirement on wave functions is argued in relation to a time-dependent magnetic flux.[6] The theory presented in this paper leads to the same conclusion as that for the time-dependent case. Let us introduce a time-dependent magnetic flux passing through the center of the ring,

$$\Phi = \alpha\Phi_0 f(t) \ . \tag{5.1}$$

The Lagrangian is then

$$L = \frac{I}{2}\dot{\phi}^2 + \alpha\dot{\phi}f(t) , \tag{5.2}$$

where $f(t)$ is a function of time only. From (5.2) follows the equation of motion,

$$\ddot{\phi} + \frac{\alpha}{I}\dot{f}(t) = 0 \ . \tag{5.3}$$

If we introduce an angular variable such that[15]

$$\dot{\beta} = \dot{\phi} + \frac{\alpha}{I}f(t) , \tag{5.4}$$

the Lagrangian (5.2) becomes

$$L=\frac{I}{2}[\dot{\beta}(t)]^2-\frac{\alpha^2}{2I}f^2(t)\ . \tag{5.5}$$

The calculation of K_n is trivial,

$$K_n(\beta'',t'';\beta',t')=\left[\frac{I}{2\pi iT}\right]^{1/2}\exp\left[\frac{iI}{2T}(\beta+2n\pi)^2-i\frac{\alpha^2}{2I}G(t'',t')\right], \tag{5.6}$$

where

$$G(t'',t')=\int_{t'}^{t''}f^2(\tau)d\tau\ ,$$

$\beta=\beta''-\beta'$, and $T=t''-t'$. From (5.6) it is easy to verify that the symmetric relations (2.9) are satisfied for $\beta=-\pi$. Therefore, δ should be zero and the propagator is single valued.

In particular, suppose the magnetic flux is suddenly switched on at $t=0$. For the sake of simplicity we set

$$f(t)=H(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\ .\end{cases} \tag{5.7}$$

The total propagator for $t''>t'>0$ is

$$K(\phi'',t'';\phi',t')=\left[\frac{I}{i2\pi T}\right]^{1/2}\exp\left[i\frac{I}{2T}\left[\phi+\frac{\alpha T}{I}\right]^2-i\frac{\alpha^2}{2I}T\right]\times\Theta_3\left[\frac{\pi I}{T}\left[\phi+\frac{\alpha T}{I}\right],\frac{2I\pi}{T}\right]\ . \tag{5.8}$$

The corresponding wave function and kinetic energy are

$$\psi_m=\frac{1}{\sqrt{2\pi}}\exp(im\phi),\quad E_m=\frac{1}{2I}(m-\alpha)^2\ . \tag{5.9}$$

The canonical angular momentum in this case is an integer. However, the kinetic angular momentum shifts to $m-\alpha$ due to the torque applied by the induced electric field.

The dynamics of an electron with a time-dependent magnetic flux which induces a force field is different from what we discussed in Sec. IV. Nothing is strange with the flux-dependent kinetic angular momentum which is consistent with classical result. Since the generator of rotation should be the operator of canonical angular momentum, the fractional eigenvalues of kinetic angular momentum do not lead to the fractional angular momentum quantization.[6] The flux-dependent energy spectrum (5.9) is specified as the bound-state AB effect.[23]

VI. CONCLUSION

The SO(2) group does not give a unique definition for the angular momentum quantization. Moreover, in a two-dimensional multiply connected space there is no strong reason to impose the single-valuedness requirement on the wave function even though the potential is single valued; the eigenvalue of angular momentum is ambiguous by an additive constant. The ambiguity of angular momentum quantization can be removed using the path-integral formulation of quantum mechanics in multiply connected spaces. A topological phase arising from the homotopy theory of path integrals in multiply connected spaces is uniquely defined and physically meaningful. It is the topological phase defined on the group manifold of SO(2) that gives rise to the fractional angular momentum quantization.

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