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Quantum theory of a two-photon laser

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A quantum theory of a two-photon laser in a three-level atomic system of the cascade type is presented starting from an original microscopic Hamiltonian. The master equation of the laser operation is derived by using the method of Scully and Lamb [Phys. Rev. A **159**, 208 (1967)] for a single-mode laser containing a two-level medium. The photon-number distribution of the laser is studied in detail. The curve of the distribution may have a two-peak structure. One of the peaks located at the origin comes from the single-photon process and reflects the thermal property of the laser, while another comes from the two-photon process and reflects the coherent property of the laser. We cannot find an exact criterion for the threshold of the two-photon laser. The two-photon laser can approximately be considered as a coherent state for certain values of A/C, which can even be much smaller than the threshold for the nonresonant single-mode laser in a two-level system, but it is much larger than the threshold for the resonant one.

I. INTRODUCTION

The two-photon laser has been studied for two decades,[1–7] and has received increasing attention in recent years.[8–15] The first two-photon laser experiment was reported,[15] which was followed by Gao's in 1984.[16] Theoretical work has been done both semiclassically[6–8] and quantum mechanically.[3,4,10–14] Most of the quantum studies use an interaction Hamiltonian of the type $a_1 a_2 \sigma^\dagger + \text{H.c.}$ where a_i is the field annihilation operator and $\sigma^\dagger$ is the two-level creation operator. Wang and Haken[17] recently reached the same interaction Hamiltonian, the effective Hamiltonian, from the exact microscopic Hamiltonian. The effective one is equivalent to the exact one in the sense that the second-perturbation transition probabilities derived from the latter are equal to the first-perturbation transition probabilities derived from the former. But they failed to point out whether they are equivalent to each other for the photon statistics. Singh and Zubairy[5] have developed a quantum theory for the two-mode laser in a Λ-type atomic system starting from the exact microscopic Hamiltonian, but they treated only the situation of single-photon resonance, so that the interesting effects of detunings were not given in their paper.

In this paper we study the two-photon laser starting from the exact microscopic Hamiltonian using the method developed by Scully and Lamb[18,19] for a single-mode laser and we obtain the analytical results though they are very complicated. In Sec. II we give the exact Hamiltonian governing the two-photon laser in a cascade three-level atomic system. In Sec. III we derive the master equation for the light field. In Sec. IV we show the photon-number statistical distribution, and Sec. V is devoted to the case of two-photon resonance. In Sec. VI we state our conclusions.

II. HAMILTONIAN

The gain medium, placed in a cavity, is a three-level atomic system of the cascade type as shown in Fig. 1. The atoms have three levels, the upper level $|a\rangle$, the intermediate level $|b\rangle$, and the lower level $|c\rangle$. The transition between $|a\rangle$ and $|b\rangle$, and between $|b\rangle$ and $|c\rangle$ are mediated by a light field (cavity mode) with frequency Ω. The transition between $|a\rangle$ and $|c\rangle$ is forbidden. The Hamiltonian for the atom-field system is

$$H = H_0 + V , \tag{1}$$

$$H_0 = \sum_{\alpha=a,bc,} \hbar\omega_\alpha A_\alpha^\dagger A_\alpha + \hbar\Omega(a^\dagger a + \tfrac{1}{2}) , \tag{2}$$

$$V = \hbar g_1 a A_a^\dagger A_b + \hbar g_2 a A_b^\dagger A_c + \text{H.c.} , \tag{3}$$

where $a^\dagger(a)$ is the creation (annihilation) operator for the light field; $A_\alpha^\dagger(A_\alpha)$ is the creation (annihilation)

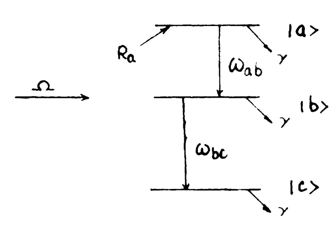

FIG. 1. Three-level atomic system.

operator for the αth level and $\hbar\omega_\alpha$ is the corresponding energy; g_1 and g_2 are atom-field coupling constants, and the rotating-wave approximation has been used.

In the interaction picture the interaction Hamiltonian becomes

$$V^I=\hbar g_1 a A_a^\dagger A_b e^{-i\Delta_1 t}+\hbar g_2 a A_b^\dagger A_c e^{i\Delta_2 t} , \tag{4}$$

where

$$\Delta_1=\Omega-\omega_{ab}=\Omega-(\omega_a-\omega_b) , \tag{5}$$

$$\Delta_2=\omega_{bc}-\Omega=(\omega_b-\omega_c)-\Omega , \tag{6}$$

Δ_1 and Δ_2 are single-photon detunings, and $\Delta=\Delta_1-\Delta_2$ is the two-photon detuning. The Wigner-Weisskopf theorem is assumed valid and γ is the decay constant. For simplicity, the same decay constant γ for all three levels and only a pumping to the upper level $|a\rangle$ with pumping rate R_a are assumed.

III. MASTER EQUATION

The state vector of the atom-field system at time t_0 can be expressed as

$$|\psi^I_{Af}(t_0)\rangle=\sum_n F_n(t_0)\,|n\rangle\,|a\rangle . \tag{7}$$

At time t, it develops into

$$|\psi^I_{Af}(t)\rangle=|\psi^I_{Af}(t_0+\tau)\rangle=\sum_n [a_n(t_0+\tau)\,|a,n\rangle+b_{n+1}(t_0+\tau)\,|b,n+1\rangle+c_{n+2}(t_0+\tau)\,|c,n+2\rangle] . \tag{8}$$

The development of the state vector obeys the following Schrödinger equation:

$$\frac{d}{dt}|\psi^I_{Af}(t)\rangle=-\frac{i}{\hbar}V^I|\psi^I_{Af}(t)\rangle . \tag{9}$$

Substituting Eq. (8) into Eq. (9), we obtain

$$i\frac{d}{dt}a_n(t)=V_1 e^{-i\Delta_1 t}b_{n+1}(t) , \tag{10}$$

$$i\frac{d}{dt}b_{n+1}(t)=V_1^* e^{i\Delta_1 t}a_n(t)+V_2 e^{i\Delta_2 t}c_{n+2}(t) , \tag{11}$$

$$i\frac{d}{dt}c_{n+2}(t)=V_2^* e^{-i\Delta_2 t}b_{n+1}(t) , \tag{12}$$

where

$$V_1=g_1\sqrt{n+1},\quad V_2=g_2\sqrt{n+2} .$$

Let the solution of Eqs. (10)–(12) be expressed as

$$\begin{aligned} a_n(t)&=\sum_\omega a_n(\omega)e^{-i\omega t} ,\\ b_{n+1}(t)&=\sum_\omega b_{n+1}(\omega)e^{-i\omega t} ,\\ c_{n+2}(t)&=\sum_\omega c_{n+2}(\omega)e^{-i\omega t} . \end{aligned} \tag{13}$$

Substituting Eq. (13) in Eqs. (10)–(12), we obtain

$$\begin{aligned} &\omega a_n(\omega)=V_1 b_{n+1}(\omega-\Delta_1) ,\\ &\omega b_{n+1}(\omega)=V_1^* a_n(\omega+\Delta_1)+V_2 c_{n+2}(\omega+\Delta_2) ,\\ &\omega c_{n+2}(\omega)=V_2^* b_{n+1}(\omega-\Delta_2) . \end{aligned} \tag{14}$$

From Eq. (14) we find

$$(\omega+\Delta_1)(\omega+\Delta_2)\omega=(\omega+\Delta_2)|V_1|^2+(\omega+\Delta_1)|V_2|^2 , \tag{15}$$

that is to say, the possible ω's are determined by the following cubic equation:

$$\omega^3+(\Delta_1+\Delta_2)\omega^2+(\Delta_1\Delta_2-|V_1|^2-|V_2|^2)\omega -\Delta_1|V_2|^2-\Delta_2|V_1|^2=0 . \tag{16}$$

Let the three roots of this equation be ω_1, ω_2, and ω_3. Thus the solution of Eqs. (10)–(12) can be written as

$$a_n(t)=a_n^{(1)}e^{-i(\omega_1+\Delta_1)t}+a_n^{(2)}e^{-i(\omega_2+\Delta_1)t} +a_n^{(3)}e^{-i(\omega_3+\Delta_1)t} , \tag{17}$$

$$b_{n+1}(t)=b_{n+1}^{(1)}e^{-i\omega_1 t}+b_{n+1}^{(2)}e^{-i\omega_2 t} +b_{n+1}^{(3)}e^{-i\omega_3 t} , \tag{18}$$

$$c_{n+2}(t)=c_{n+2}^{(1)}e^{-i(\omega_1+\Delta_2)t}+c_{n+2}^{(2)}e^{-i(\omega_2+\Delta_2)t} +c_{n+2}^{(3)}e^{-i(\omega_3+\Delta_2)t} , \tag{19}$$

where $a_n^{(i)}$, $b_{n+1}^{(i)}$, and $c_{n+2}^{(i)}$ are determined by Eq. (14) with $\omega=\omega_i$ and have the relations

$$(\omega_i+\Delta_1)a_n^{(i)}=V_1 b_{n+1}^{(i)}\quad(i=1,2,3) , \tag{20}$$

$$(\omega_i+\Delta_2)c_{n+2}^{(i)}=V_2^* b_{n+1}^{(i)}\quad(i=1,2,3) . \tag{21}$$

According to the initial condition, Eq. (7), we have

$$a_n^{(1)}+a_n^{(2)}+a_n^{(3)}=F_n(t_0) , \tag{22}$$

$$b_{n+1}^{(1)}+b_{n+1}^{(2)}+b_{n+1}^{(3)}=0 , \tag{23}$$

$$c_{n+2}^{(1)}+c_{n+2}^{(2)}+c_{n+2}^{(3)}=0 . \tag{24}$$

Using Eqs. (20)–(24), the coefficients $a_n^{(i)}$, $b_{n+1}^{(i)}$, and $c_{n+2}^{(i)}$ can be expressed as

$$\begin{aligned} a_n^{(1)}&=-\frac{|V_1|^2(\omega_2+\Delta_2)F_n(t_0)}{(\omega_1+\Delta_1)(\omega_1-\omega_2)(\omega_3-\omega_1)}=A_n^{(1)}F_n(t_0) ,\\ a_n^{(2)}&=-\frac{|V_1|^2(\omega_2+\Delta_2)F_n(t_0)}{(\omega_2+\Delta_1)(\omega_1-\omega_2)(\omega_2-\omega_3)}=A_n^{(2)}F_n(t_0) ,\\ a_n^{(3)}&=-\frac{|V_1|^2(\omega_3+\Delta_2)F_n(t_0)}{(\omega_3+\Delta_1)(\omega_2-\omega_3)(\omega_3-\omega_1)}=A_n^{(3)}F_n(t_0) , \end{aligned} \tag{25}$$

$$b_{n+1}^{(1)}=\frac{-V_1^*(\omega_1+\Delta_2)}{(\omega_1-\omega_2)(\omega_3-\omega_1)}F_n(t_0)=B_{n+1}^{(1)}F_n(t_0)\ ,$$

$$b_{n+1}^{(2)}=\frac{-V_1^*(\omega_2+\Delta_2)}{(\omega_1-\omega_2)(\omega_2-\omega_3)}F_n(t_0)=B_{n+1}^{(2)}F_n(t_0)\ , \tag{26}$$

$$b_{n+2}^{(3)}=\frac{-V_1^*(\omega_3+\Delta_2)}{(\omega_2-\omega_3)(\omega_3-\omega_1)}F_n(t_0)=B_{n+1}^{(3)}F_n(t_0)\ ,$$

$$c_{n+2}^{(1)}=\frac{-V_1^*V_2^*F_n(t_0)}{(\omega_1-\omega_2)(\omega_3-\omega_1)}=C_{n+2}^{(1)}F_n(t_0)\ ,$$

$$c_{n+2}^{(2)}=\frac{-V_1^*V_2^*F_n(t_0)}{(\omega_1-\omega_2)(\omega_2-\omega_3)}=C_{n+2}^{(2)}F_n(t_0)\ , \tag{27}$$

$$c_{n+2}^{(3)}=\frac{-V_1^*V_2^*F_n(t_0)}{(\omega_2-\omega_3)(\omega_3-\omega_1)}=C_{n+2}^{(3)}F_n(t_0)\ .$$

We must mention that ω_i is a function of n, so that $A_n^{(i)}$, $B_{n+1}^{(i)}$, and $C_{n+2}^{(i)}$ are all functions of n.

Let ρ_f and ρ represent the density matrices of the light field and the atom-field system; $\rho_{n,m}$ and $\rho_{\alpha,n;\beta,m}$ are their elements, respectively.

As the upper level is being pumped and there exists cavity loss, we can write

$$\dot\rho_f=\dot\rho_f^{(a)}+\dot\rho_f^{(l)}\ , \tag{28}$$

where $\rho_f^{(a)}$ stands for the change caused by the pumping and $\dot\rho_f^{(l)}$ caused by the cavity loss. Following the procedure of Sargent *et al.*,[18,19] they can be determined as

$$\dot\rho_{n,m}^{(a)}(t_0)=R_a\int_0^\infty d\tau\,\gamma e^{-\gamma\tau}\left[\sum_\beta \rho_{\beta,n;\beta,m}^{(a)}(t_0+\tau)-\rho_{n,m}(t_0)\right]\ , \tag{29}$$

$$\begin{aligned}\rho_{a,n;a,n}^{(a)}(t_0+\tau)&=\langle a,n\mid\psi_{Af}^I(t_0+\tau)\rangle\langle\psi_{Af}^I(t_0+\tau)\mid a,n\rangle=a_n^*(t)a_n(t)\\&=F_n^*(t_0)F_n(t_0)(2A_n^{(1)}A_n^{(2)}\{\cos[(\omega_1-\omega_2)\tau]-1\}+2A_n^{(1)}A_n^{(3)}\{\cos[(\omega_1-\omega_3)\tau]-1\}\\&\qquad+2A_n^{(1)}A_n^{(3)}\{\cos[(\omega_2-\omega_3)\tau]-1\}+1)\ ,\end{aligned} \tag{30}$$

$$\begin{aligned}\rho_{b,n;b,n}^{(a)}(t_0+\tau)&=\rho_{n-1,n-1}(t_0)(2B_n^{(1)}B_n^{(2)}\{\cos[(\omega_1'-\omega_2')\tau]-1\}\\&\qquad+2B_n^{(1)}B_n^{(2)}\{\cos[(\omega_1'-\omega_3')\tau]-1\}+2B_n^{(1)}B_n^{(3)}\{\cos[(\omega_2-\omega_3)]-1\})\ ,\end{aligned} \tag{31}$$

$$\begin{aligned}\rho_{c,n;c,n}^{(a)}(t_0+\tau)&=\rho_{n-2,n-2}(t_0)(2C_n^{(1)}C_n^{(2)}\{\cos[(\omega_1''-\omega_2'')\tau]-1\}+2C_n^{(1)}C_n^{(3)}\{\cos[(\omega_1''-\omega_3'')\tau]-1\}\\&\qquad+2C_n^{(2)}C_n^{(3)}\{\cos[(\omega_2''-\omega_3'')]\tau-1\})\ ,\end{aligned} \tag{32}$$

where $\omega_i'(\omega_i'')$ is also the root of Eq. (16) with V_1 and V_2 replaced by $V_1'=g_1\sqrt{n}\,(V_1''=g_1\sqrt{n-1})$ and $V_2'=g_2\sqrt{n+1}\,(V_2''=g_2\sqrt{n}\,)$, and $\rho_{n,n}(t_0)=F_{n,n}(t_0)F_{n,n}^*(t_0)$.

Substituting Eqs. (30)–(32) into Eq. (29) and finishing the integral, we obtain for the diagonal matrix element

$$\begin{aligned}\dot\rho_{n,n}^{(a)}(t_0)=&-2R_a\frac{|g_1|^2}{\gamma^2}(n+1)F_1(n)\rho_{n,n}(t_0)-2R_a\frac{|g_1|^2}{\gamma^2}(n+1)F_2(n)\rho_{n,n}(t_0)\\&+2R_a\frac{|g_1|^2}{\gamma^2}nF_1(n-1)\rho_{n-1,n-1}(t_0)+2R_a\frac{|g_1|^2}{\gamma^2}(n-1)F_2(n-2)\rho_{n-2,n-2}(t_0)\ ,\end{aligned} \tag{33}$$

where

$$F_1(n)=\frac{(\mu_1+\delta_2)(\mu_2+\delta_2)}{(\mu_2-\mu_3)(\mu_1-\mu_3)[(\mu_1-\mu_2)^2+1]}+\frac{(\mu_1+\delta_2)(\mu_3+\delta_2)}{(\mu_1-\mu_2)(\mu_3-\mu_2)[(\mu_1-\mu_3)^2+1]}+\frac{(\mu_2+\delta_2)(\mu_3+\delta_2)}{(\mu_2-\mu_1)(\mu_3-\mu_1)[(\mu_2-\mu_3)^2+1]}\ , \tag{34}$$

$$F_2(n)=\frac{|V_2|^2}{(\mu_2-\mu_3)(\mu_1-\mu_3)[(\mu_1-\mu_2)^2+1]}+\frac{|V_2|^2}{(\mu_1-\mu_2)(\mu_3-\mu_2)[(\mu_1-\mu_3)^2+1]}+\frac{|V_2|^2}{(\mu_2-\mu_1)(\mu_3-\mu_1)[(\mu_2-\mu_3)^2+1]}\ , \tag{35}$$

$$\mu_i=\omega_i/\gamma,\quad \delta_i=\Delta_i/\gamma\ . \tag{36}$$

Thus the master equation of the two-photon laser is obtained as

$$\begin{aligned}\dot p(n)=&-A(n+1)F_1(n)p(n)-A(n+1)F_2(n)p(n)+AnF_1(n-1)p(n-1)\\&+A(n-1)F_2(n-2)p(n-2)-Cnp(n)+C(n+1)p(n+1)\ ,\end{aligned} \tag{37}$$

where

$$A = 2R_a \frac{|g_1|^2}{\gamma^2}, \quad p(n) = \rho_{n,n}(t_0) ,$$

and the cavity loss has been included through[18,19]

$$\dot{\rho}^{l}_{n,n} = -Cnp(n) + C(n+1)p(n+1) .$$

On the right-hand side of the master equation there are six terms, and they can be interpreted as probability flows which can be expressed by arrows in a probability flow diagram, as shown in Fig. 2. This diagram can be extended to infinity. The number attached to each arrow indicates which term in Eq. (37) it represents. The physical meaning of each term in Eq. (37) (or arrow in Fig. 2) is very clear. The first and third terms stand for the single-photon gain and the second and fourth terms for the two-photon gain, while the fifth and sixth for the loss.

IV. STEADY-STATE EQUATION OF MOTION

In steady-state operation, $p(n)$ is independent of time, $\dot{p}(n) = 0$. From Eq. (37) the following equation can be derived as

$$\begin{aligned} C(n+1)p(n+1) = {} & A(n+1)F_1(n)p(n) \\ & + A(n+1)F_2(n)p(n) \\ & + AnF_2(n-1)p(n-1) , \end{aligned} \tag{38}$$

or

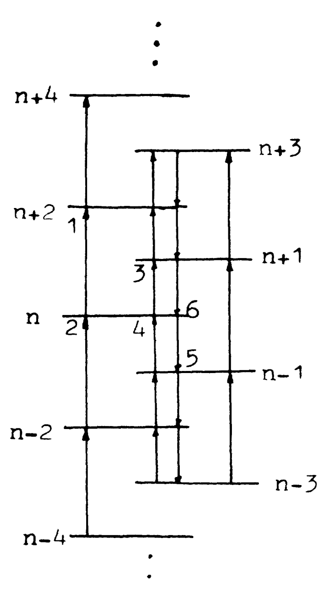

FIG. 2. Diagram of probability flow for the two-photon laser.

$$\begin{aligned} p(n+1) = {} & \frac{A}{C}[F_1(n) + F_2(n)]p(n) \\ & + \frac{A}{C}\frac{n}{n+1}F_2(n-1)p(n-1) . \end{aligned} \tag{38a}$$

If $g_2 = 0$, then $F_2(n) = 0$, Eqs. (37) and (38) are deduced to the corresponding equations for the usual single-photon laser.

From Eq. (38) and the normalization condition

$$\sum_{n=0}^{\infty} p(n) = 1 , \tag{39}$$

all $p(n)$ can be obtained, that is to say, the photon statistical distribution is known.

Because of the complicated form of $F_1(n_1)$ and $F_2(n)$, it is difficult to discern the character of the photon statistics by inspection of Eqs. (38a), (34), and (35). With the aid of a computer we obtain the curves of the photon-number statistical distribution, as shown in Fig. 3. It can be seen that some of the curves have two peaks, one of which is at the origin, peak 1, and represents the single-photon process below threshold, while the other, peak 2, reflects the two-photon process. In fact, there always exist two peaks when $26 > A/C > 11$ for $\delta_1 = \delta_2 = 5$, but when $A/C > 12$, peak 2, caused by the two-photon process, is much higher than peak 1, caused by the single-photon process below threshold $[p(\bar{n})/p(0) \gg 1]$, so that only peak 2 shows its existence. As A/C increases, peak 2 becomes higher and its position shifts towards the direction of increasing n, while peak 1 at the origin lowers its height.

Figure 4 shows the effect of the detuning between the lower and the intermediate levels, δ_2. The change of δ_2 brings about the variations of single-photon detuning and two-photon detuning. When the detuning δ_2 increases from two-photon resonance ($\delta_1 = \delta_2$), both single-photon gain and two-photon gain decrease, so that peak 2 declines and shifts to the origin while peak 1 arises, as shown by the curve with $\delta_2 = 5.1$ and 5.2 in Fig. 4. When δ_2 decreases from the two-photon resonance, the single-photon gain increases while the two-photon gain decreases because of the departure from the two-photon resonance, so that the net effect is the result of the competition between the two opposite influences. For small departure from the two-photon resonance the influence of the increase of the single-photon gain is larger, while for large departure from the two-photon resonance the influence of the decrease of the two-photon gain becomes larger. Therefore when δ_2 decreases from 5, i.e., from the two-photon resonance, peak 2 first arises and shifts towards the direction of increasing n, and then declines and shifts to the origin as shown by the curves of $\delta_2 = 4.5$ to 4.9 in Fig. 4.

V. THE CASE OF TWO-PHOTON RESONANCE $\delta = \delta_1 = \delta_2$

In the two-photon resonance, Eq. (37) can be simplified and reads

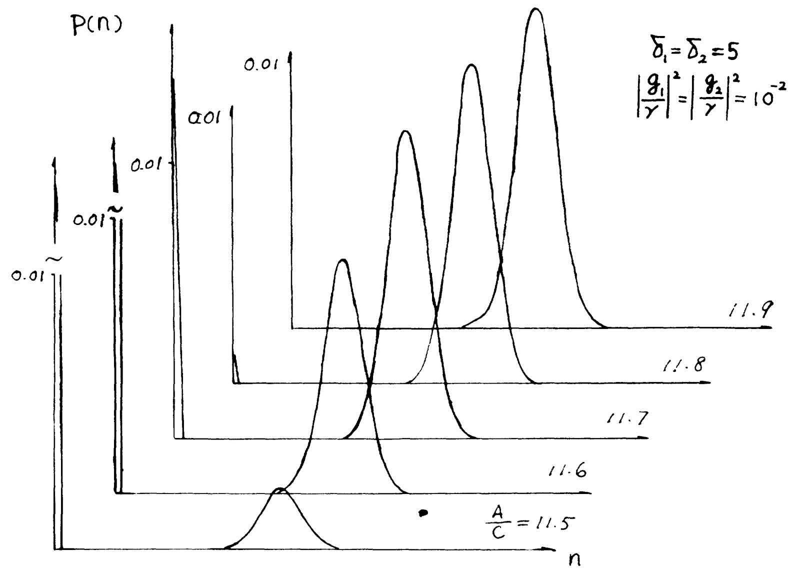

FIG. 3. Photon-number distributions for different A/C with $\delta_1=\delta_2=5$ and $|g_2/\gamma|^2=|g_1/\gamma|^2=10^{-2}$.

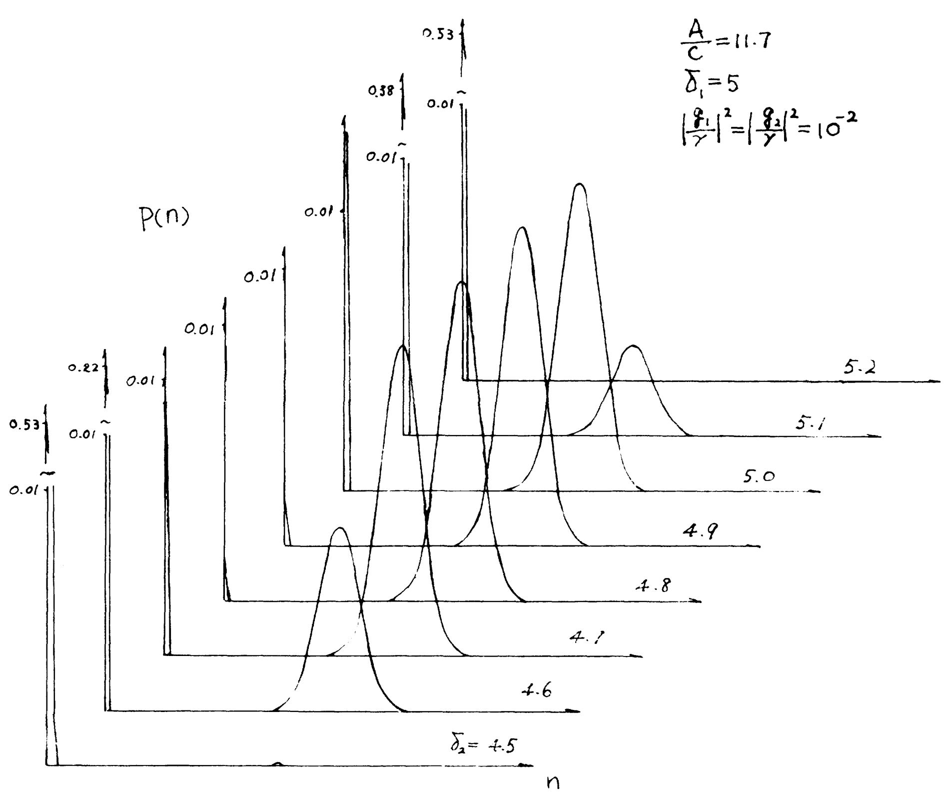

FIG. 4. Photon-number distributions for different δ_2 with $A/C=11.7$ and $\delta_1=5$.

$$\dot{p}(n)=\frac{-A(n+1)p(n)}{1+\delta^2+\frac{B_1}{A_1}(n+1)+\frac{B_2}{A_2}(n+2)}$$
$$+\frac{-\frac{3}{4}A(n+1)\frac{B_2}{A_2}(n+2)\left[1+\frac{1}{3}\delta^2+\frac{1}{4}\frac{B_1}{A_1}(n+1)+\frac{1}{4}\frac{B_2}{A_2}(n+2)\right]p(n)}{\left[1+\delta^2+\frac{B_1}{A_1}(n+1)+\frac{B_2}{A_2}(n+2)\right]\left[\left[1+\frac{1}{4}\frac{B_1}{A_1}(n+1)+\frac{1}{4}\frac{B_2}{B_2}(n+2)\right]^2+\delta^2\right]}$$
$$+\frac{Anp(n-1)}{1+\delta^2+\frac{B_1}{A_1}n+\frac{B_2}{A_2}(n+1)}+\frac{\frac{3}{4}A(n-1)\frac{B_2}{A_2}n\left[1+\frac{1}{3}\delta^2+\frac{1}{4}\frac{B_1}{A_1}(n-1)+\frac{1}{4}\frac{B_2}{A_2}n\right]p(n-2)}{\left[1+\delta^2+\frac{B_1}{A_1}(n-1)+\frac{B_2}{A_2}n\right]\left[\left[1+\frac{1}{4}\frac{B_1}{A_1}(n-1)+\frac{1}{4}\frac{B_2}{A_2}n\right]^2+\delta^2\right]}$$
$$-Cnp(n)+C(n+1)p(n+1)\,, \tag{40}$$

where $B_1/A_1=4\,|g_1/\gamma\,|^2$, $B_2/A_2=4\,|g_2/\gamma\,|^2$ and Eq. (38) becomes

$$Cp(n+1)=\frac{Ap(n)}{1+\delta^2+\frac{B_1}{A_1}(n+1)+\frac{B_2}{A_2}(n+2)}$$
$$+\frac{\frac{3}{4}A\frac{B_2}{A_2}(n+2)\left[1+\frac{1}{3}\delta^2+\frac{1}{4}\frac{B_1}{A_1}(n+1)+\frac{1}{4}\frac{B_2}{A_2}(n+2)\right]p(n)}{\left[1+\delta^2+\frac{B_1}{A_1}(n+1)+\frac{B_2}{A_2}(n+2)\right]\left[\left[1+\frac{1}{4}\frac{B_1}{A_1}(n+1)+\frac{1}{4}\frac{B_2}{A_2}(n+2)\right]^2+\delta^2\right]}$$
$$+\frac{\frac{3}{4}A\frac{B_2}{A_2}n\left[1+\frac{1}{3}\delta^2+\frac{1}{4}\frac{B_1}{A_1}n+\frac{1}{4}\frac{B_2}{A_2}(n+1)\right]p(n-1)}{\left[1+\delta^2+\frac{B_1}{A_1}n+\frac{B_2}{A_2}(n+1)\right]\left[\left[1+\frac{1}{4}\frac{B_1}{A_1}n+\frac{1}{4}\frac{B_2}{A_2}(n+1)\right]^2+\delta^2\right]}\,. \tag{41}$$

If $B_2=0$, Eqs. (40) and (41) become the maser equations for a single-mode laser in a two-level system. The peak (or the valley) positions of the curve of the photon-number distribution can be found by setting

$$p(n-1)=p(n)=p(n+1)\,.$$

Then we obtain the following equation under the approximation of $|\,n\,|\gg 1$,

$$\left[1+\delta^2+\left[\frac{B_1}{A_1}+\frac{B_2}{A_2}\right]n\right]\left\{\left[1+\frac{1}{4}\left[\frac{B_1}{A_1}+\frac{B_2}{A_2}\right]n\right]^2+\delta^2\right\}=\frac{A}{C}\left\{\left[1+\frac{1}{4}\left[\frac{B_1}{A_1}+\frac{B_2}{A_2}\right]n\right]^2+\delta^2\right\}$$
$$+\frac{3}{2}\frac{A}{C}\frac{B_2}{A_2}n\left[1+\frac{1}{3}\delta^2+\frac{1}{4}\left[\frac{B_1}{A_1}+\frac{B_2}{A_2}\right]n\right]\,. \tag{42}$$

This is a cubic algebraic equation. Its constant term is $(1+\delta^2)(1+\delta^2-A/C)$. If $1+\delta^2-A/C>0$, the solution (three roots) of Eq. (42) may have the following situations: (i) one negative root, (ii) three negative roots, and (iii) one negative and two positive roots. In the first two cases there is no peak in the curve except the one at the origin, and in the third case there is one peak (and one valley) besides that at the origin, as shown in Fig. 3.

It is difficult to define a criterion for the threshold of the two-photon laser operation even for the simple case of two-photon resonance, because there may exist two peaks. We cannot define the appearance of the second peak as the threshold, because the second peak may be very small such as the curve with $A/C=11.5$ in Fig. 3 (in fact, it is so small that it cannot be seen in the curves of the photon-number distribution when $A/C<11.5$).

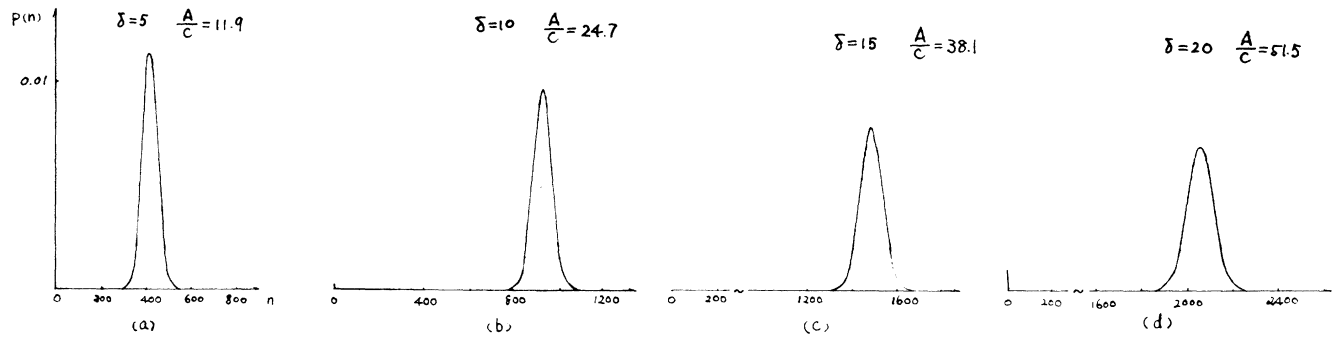

FIG. 5. The photon statistics of the two-photon laser at the two-photon resonance where peak 2 dominates ($|g_1/\gamma|^2=|g_2/\gamma|^2=10^{-2}$): (a) $\delta=5$, $A/C=11.9$, (b) $\delta=10$, $A/C=24.7$, (c) $\delta=15$, $A/C=38.1$, (d) $\delta=20$, $A/C=51.5$.

We also cannot define the disappearance of the first peak at the origin as the threshold, because it may be so small that the second peak dominates, such as the curves with $A/C=11.8$ and 11.9 in Fig. 3.

It can be defined as an approximate threshold when the light field of the two-photon laser may be approximately be considered a coherent state. Figure 5 shows the photon-number distributions in different single-photon detunings at the two-photon resonance, where the coherence dominates. Though we cannot give a value of A/C for this threshold, it can be seen in Fig. 5 that the following inequalities are valid at this threshold,

$$1+\delta^2 \gg \frac{A}{C} \gg 1 . \tag{43}$$

That is to say, this threshold is much higher than that for the resonant single-photon laser and much lower than that for the nonresonant one.

It is very clear from Eq. (41), by putting $B_2=0$, that $1+\delta^2-A/C>0$ means we are below the single-photon threshold. So the second peak is caused by two-photon gain [the second and fourth terms on the right-hand side in Eq. (40)]. Therefore it can be viewed that the first peak at the origin is the result of single-photon processes, while the second is the result of two-photon processes. The property of the light field can be approximately considered as a mixture of a thermal light (laser below threshold) and a laser light above threshold. Figure 6 is a comparison of the photon-number distributions for three cases: (i) the Poisson distribution, (ii) single-mode

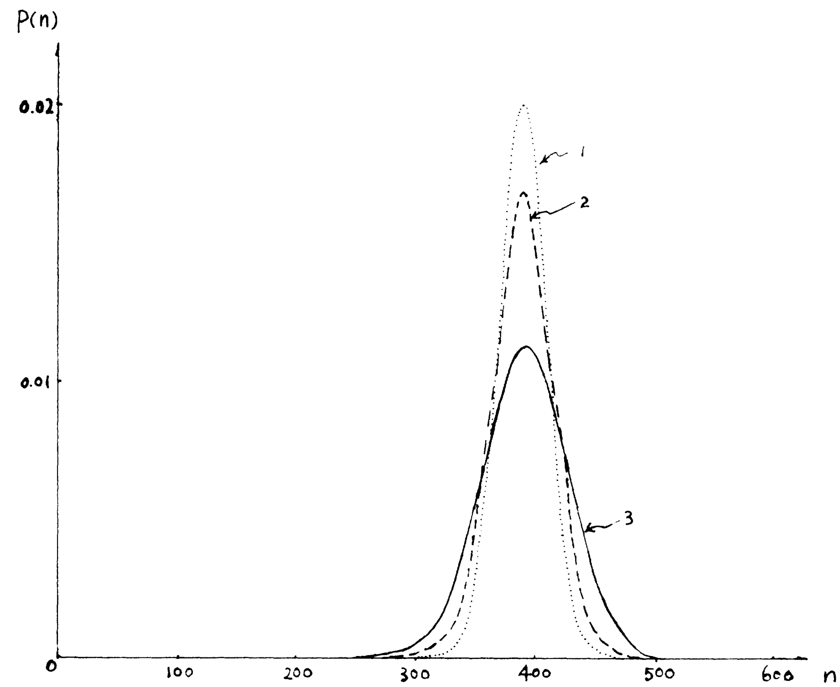

FIG. 6. A comparison of the photon-number distributions with the same peak position for the (i) Poisson distribution (dotted curve), (ii) single-photon laser (dashed curve), and (iii) two-photon laser (solid curve).

laser at resonance with a two-level system, and (iii) two-photon laser, with the same peak position at $n = 390$. It is very obvious that the coherence of a two-photon laser is less than that of a single-mode laser at resonance.

VI. CONCLUSION

We have studied the two-photon laser problems starting from the original Hamiltonian, not from the effective Hamiltonian. The master equation is derived, where the two-photon and single-photon processes can be clearly seen. The photon-number distribution is presented, which may have two peaks, one of which comes mainly from the single-photon process and stands for the thermal property of the two-photon laser, and the other mainly from the two-photon process and for the coherent property of the two-photon laser. For a certain value of A/C, which is much larger than the threshold for a resonant single-mode laser and is much smaller than that for a nonresonant one, the output light of the two-photon laser may approximately be considered a coherent state.

The property of the two-photon laser derived from the exact Hamiltonian may be quite different from that obtained from the effective one.[4] The photon statistics are not equivalent between the results of the exact Hamiltonian and the effective one.

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