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Action of passive, lossless optical systems in quantum optics

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A quantum optical formulation of the action of passive, lossless optical instruments on quantum light fields is developed. The quantum Maxwell equations are formally solved and the commutation relations for various combinations of field operators at different times are studied. General relationships between field correlation functions and correlation functions of source-quantity operators and free-field operators are derived. Formulas are presented for the case when the free field is the vacuum field. Furthermore, the mixing of source-field light with coherent free-field light is treated. The theory is applied to the calculation of the photocount distribution. The theory presented renders it possible to take into account the effects of light propagation through the optical system. It is shown that the effect of the optical instruments may be taken into account by introducing an apparatus function in a way which formally corresponds to that of classical optics. However, the calculation of the corresponding convolution integrals is governed by operator ordering rules, which are essential in the case of quantum light fields and which can give rise to substantial differences between classical and quantum optics.

I. INTRODUCTION

Light sources producing light with nonclassical properties such as photon antibunching, sub-Poisson photon statistics, and squeezing have been of considerable theoretical and experimental interest.[1–9] From a more theoretical point of view, such nonclassical properties may be defined by the requirement that the normally ordered variances of appropriately chosen (multimode) field quantities are negative. Physically, they indicate that the fluctuations of these quantities are reduced below their standard quantum limits defined by the requirement that the normally ordered variances are equal to zero. For example, the fluctuations of the electric field strength of squeezed light are reduced (in certain space-time intervals) below the vacuum noise level.[10,11] For this reason nonclassical light fields are of interest for high-performance optical systems such as low-noise optical communication or high-precision interferometric measurements.[12–21] Several methods for generating such quantum light have been proposed and various experiments have successfully been performed.[1–9]

The generation of the light and its application require, of course, a more or less complicated experimental setup, in which the sources producing the light are embedded. In particular, the light must be transmitted through various kinds of passive optical instruments such as resonatorlike cavities, beam splitters, lenses, filters, interferometers, and so on. Clearly, these optical devices do not only modify the properties of the light in the usual sense of classical optics but they may also be expected to change the quantum features. For example, from a more intuitive quantum-mechanical treatment of a beam splitter it is known that dividing a monochromatic plane-wave squeezed light field into two beams of equal intensity reduces the squeezing effect in either beam to half of the squeezing effect in the incoming squeezed wave.[22] This increase of noise results from the vacuum fluctuations of the electric field strength in the unused input port of the beam splitter. In the case of more complex optical devices acting on complicated multimode quantum light fields the problem of the correct quantum-mechanical treatment of the action of the optical instruments is not trivial and its solution needs very careful consideration.

This may be illustrated by the following example. In many cases of practical interest the light is generated and/or amplified by sources that are situated inside a resonatorlike cavity, which is in contact with the environment via appropriately chosen mirrors, so that the field inside the cavity (internal field) is coupled to the field outside the cavity (external field). The experimental studies are performed, of course, on the external field, the properties of which are therefore the desired information. Recently attempts have been made to solve this problem by generalizing the well-known quantum-mechanical noise theories usually used for finding the properties of the internal field. In particular, the external field is decomposed into an input field and an output field and the output field is related to the internal field and/or the input field. This theory is based on quantum stochastic Markov approximations and its applicability is restricted to high-Q-value cavities and to input fields the spectra of which are sufficiently flat.[23–26] Apart from the fact that only a very particular experimental setup is considered, the theory does not take into account the full space-time structure of the field.

An alternative approach to the problem of the quantum-mechanical description of the action of passive optical systems on light fields has recently been developed on the basis of the concepts of quantum field theory and was applied to the problem of spectral filtering of light.[27,28] The idea is to relate the properties of the light to the properties of the sources that the light is attributed to. The properties of the light are defined in terms of field correlation functions and the relationships between these field correlation functions and the correlation functions of source operators are derived. In particular, these relationships also answer the question of how to describe the action of optical instruments in quantum optics. The advantage of such an approach is that it renders it possible to take into account the effects of light propagation and to find the light properties in arbitrary space-time intervals. The only assumptions are that the interaction between sources and light is linear in the vector potential and the optical system is lossless. The latter assumption, which is also made in the quantum stochastic treatments,[23–26] means that the absorption of light by the material of the optical devices can be neglected and the dispersion associated with it can also be disregarded. This renders it possible to model the optical system by a dielectric with appropriately chosen space-dependent refractive index $n(\mathbf{r})$. Now the concept of mode expansion can be formulated and the quantum-mechanical formulation of the action of the optical instruments can be developed.

The aim of this paper is to develop this quantum-mechanical theory from a more general point of view than in Ref. 28 and to give a closed and detailed derivation of the relations used in Ref. 28. This general formulation of the theory renders it possible to apply it to a wide class of passive, lossless optical systems and to understand their action on quantum light fields. For instance, the theory also allows the study of effects such as the diffraction of nonclassical light at gratings. In this case from the results of Ref. 29 the quantum fluctuations are expected to be space dependent. In Sec. II the classical Maxwell equations with sources and optical devices are formulated and solved by the procedure of mode expansion, and the quantized version is derived. In Sec. III the Heisenberg equations of motion for the field operators are derived and formally solved, so that the field operators may be expressed in terms of free-field and source-field operators. The commutation relations for various combinations of field operators at different times are studied in Sec. IV, and relationships between field commutators and source-quantity commutators are derived. In Sec. V these commutation relations are used in order to express field correlation functions in terms of correlation functions of source operators and free-field operators and to clarify the effect of the optical system on the properties of quantum light fields. In Sec. VI the theory is applied to the photocount statistics. A summary and some conclusions are given in Sec. VII.

II. QUANTIZATION OF LIGHT WITH SOURCES IN A DIELECTRIC WITH SPACE-DEPENDENT REFRACTIVE INDEX

A. Classical basic equations

As outlined in the Introduction, we shall be studying the action of optical instruments in the sense of linear, lossless filters, which may be modeled by a dielectric with space-dependent refractive index $n(\mathbf{r})$. The corresponding Maxwell equations with sources are

$$\nabla\cdot\mathbf{B}(\mathbf{r},t)=0\ , \tag{2.1}$$

$$\nabla\times\mathbf{E}(\mathbf{r},t)+\dot{\mathbf{B}}(\mathbf{r},t)=0\ , \tag{2.2}$$

$$\nabla\cdot\mathbf{D}(\mathbf{r},t)=\rho(\mathbf{r},t)\ , \tag{2.3}$$

$$\nabla\times\mathbf{H}(\mathbf{r},t)-\dot{\mathbf{D}}(\mathbf{r},t)=\mathbf{j}(\mathbf{r},t)\ , \tag{2.4}$$

where

$$\mathbf{B}(\mathbf{r},t)=\mu_0\mathbf{H}(\mathbf{r},t)\ , \tag{2.5}$$

$$\mathbf{D}(\mathbf{r},t)=\epsilon_0\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r},t)\ , \tag{2.6}$$

and

$$\epsilon(\mathbf{r})=n^2(\mathbf{r})\ . \tag{2.7}$$

From Eqs. (2.3) and (2.4) the charge density $\rho(\mathbf{r},t)$ and the current density $\mathbf{j}(\mathbf{r},t)$ are seen to satisfy the continuity equation

$$\dot{\rho}(\mathbf{r},t)+\nabla\cdot\mathbf{j}(\mathbf{r},t)=0\ . \tag{2.8}$$

We now assume that the sources embedded in the dielectric are point charges. Denoting the charge and the mass of the ath particle by Q_a and m_a, respectively, we may write

$$\rho(\mathbf{r},t)=\sum_a Q_a\delta[\mathbf{r}-\mathbf{r}_a(t)]\ , \tag{2.9}$$

$$\mathbf{j}(\mathbf{r},t)=\sum_a Q_a\dot{\mathbf{r}}_a(t)\delta[\mathbf{r}-\mathbf{r}_a(t)]\ . \tag{2.10}$$

In these equations, the position vectors $\mathbf{r}_a(t)$ obey the equations of motion

$$m_a\ddot{\mathbf{r}}_a(t)=-\nabla U_{\mathrm{ext}}(\mathbf{r}_a(t))+Q_a[\mathbf{E}(\mathbf{r}_a(t),t)+\dot{\mathbf{r}}_a(t)\times\mathbf{B}(\mathbf{r}_a(t),t)]\ , \tag{2.11}$$

where $U_{\mathrm{ext}}(\mathbf{r})$ is an appropriately chosen (external) potential which ensures that the charges might be localized in a certain range of space.

The Maxwell equations (2.1) and (2.2) are identically satisfied if we let

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t)\ , \tag{2.12}$$

$$\mathbf{E}(\mathbf{r},t)=-\dot{\mathbf{A}}(\mathbf{r},t)-\nabla V(\mathbf{r},t)\ , \tag{2.13}$$

where $V(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$, respectively, are the scalar and the vector potential. Substituting Eqs. (2.12) and (2.13) together with Eqs. (2.5) and (2.6) into the Maxwell equations (2.3) and (2.4), we obtain

$$\epsilon_0\nabla\cdot[\epsilon(\mathbf{r})\nabla V(\mathbf{r},t)]=-\rho(\mathbf{r},t)-\epsilon_0\nabla\cdot[\epsilon(\mathbf{r})\dot{\mathbf{A}}(\mathbf{r},t)]\ , \tag{2.14}$$

$$\nabla\times\nabla\times\mathbf{A}(\mathbf{r},t)+\mu_0\epsilon_0\epsilon(\mathbf{r})\ddot{\mathbf{A}}(\mathbf{r},t)=\mu_0[\mathbf{j}(\mathbf{r},t)-\epsilon_0\epsilon(\mathbf{r})\nabla\dot{V}(\mathbf{r},t)]\ . \tag{2.15}$$

If we put Eqs. (2.12) and (2.13) into Eq. (2.11) we arrive at

$$m_a\ddot{\mathbf{r}}_a(t)=-\nabla U_{\mathrm{ext}}(\mathbf{r}_a(t))-Q_a[\nabla V(\mathbf{r}_a(t),t)+\dot{\mathbf{A}}(\mathbf{r}_a(t),t)-\dot{\mathbf{r}}_a(t)\times\nabla\times\mathbf{A}(\mathbf{r}_a(t),t)]\ . \tag{2.16}$$

Remembering Eqs. (2.9) and (2.10) we can readily verify that Eqs. (2.14)–(2.16) may be derived from the following Lagrangian:

$$L=\sum_a \tfrac{1}{2}m_a\dot{\mathbf{r}}_a^2+\int d^3r\left[\tfrac{1}{2}\left[\epsilon_0\epsilon(\dot{\mathbf{A}}+\nabla V)^2-\frac{1}{\mu_0}(\nabla\times\mathbf{A})^2\right]+\mathbf{j}\,\mathbf{A}-\rho(V+U_{\rm ext})\right]. \quad (2.17)$$

The generalized momenta are defined in the usual way, viz.,

$$\frac{\partial L}{\partial\dot{\mathbf{r}}_a}=\mathbf{p}_a=m_a\dot{\mathbf{r}}_a+Q_a\,\mathbf{A}(\mathbf{r}_a)\;, \quad (2.18)$$

$$\frac{\delta L}{\delta\dot{\mathbf{A}}}=-\mathbf{D}=\epsilon_0\epsilon(\dot{\mathbf{A}}+\nabla V)\;, \quad (2.19)$$

$$\frac{\delta L}{\delta\dot{V}}=0\;. \quad (2.20)$$

Making use of Eqs. (2.10) and (2.14) the Hamiltonian

$$H=\sum_a\frac{\partial L}{\partial\dot{\mathbf{r}}_a}\dot{\mathbf{r}}_a+\int d^3r\frac{\delta L}{\delta\dot{\mathbf{A}}}\dot{\mathbf{A}}-L \quad (2.21)$$

may be written as

$$H=\tfrac{1}{2}\int d^3r\left[\frac{1}{\epsilon_0\epsilon(\mathbf{r})}\mathbf{D}^2+\frac{1}{\mu_0}(\nabla\times\mathbf{A})^2\right]+\sum_a\left[\frac{1}{2m_a}[\mathbf{p}_a-Q_a\,\mathbf{A}(\mathbf{r}_a)]^2+U_{\rm ext}(\mathbf{r}_a)\right]. \quad (2.22)$$

In what follows we shall be working in the generalized Coulomb gauge

$$\nabla\cdot(\epsilon\,\mathbf{A})=0 \quad (2.23)$$

so that Eq. (2.14) simply reads as

$$\epsilon_0\nabla\cdot[\epsilon(\mathbf{r})\nabla V(\mathbf{r},t)]=-\rho(\mathbf{r},t)\;, \quad (2.24)$$

from which the scalar potential is seen to be a functional of the charge density,

$$V(\mathbf{r},t)=\phi[-\rho(\mathbf{r}',t)]\;, \quad (2.25)$$

and furthermore

$$\dot{V}(\mathbf{r},t)=\phi[-\dot{\rho}(\mathbf{r}',t)]\;. \quad (2.26)$$

Remembering Eq. (2.8) we may rewrite Eq. (2.26) as

$$\dot{V}(\mathbf{r},t)=\phi[\nabla\cdot\mathbf{j}(\mathbf{r}',t)]\;. \quad (2.27)$$

Inserting this form into Eq. (2.15) leads to

$$\nabla\times\nabla\times\mathbf{A}(\mathbf{r},t)+\mu_0\epsilon_0\epsilon(\mathbf{r})\ddot{\mathbf{A}}(\mathbf{r},t)=\mu_0\{\mathbf{j}(\mathbf{r},t)-\epsilon_0\epsilon(\mathbf{r})\nabla\phi[\nabla\cdot\mathbf{j}(\mathbf{r}',t)]\}\;. \quad (2.28)$$

Expressing in Eq. (2.22) the generalized momentum $-\mathbf{D}$ in terms of $\dot{\mathbf{A}}$ and ∇V and making use of Eqs. (2.9) and (2.24), we easily find that in the generalized Coulomb gauge [Eq. (2.23)] the Hamiltonian of the system under study may be decomposed in the following way:

$$H=H_s+H_r+H_{\rm int}\;, \quad (2.29)$$

where

$$H_s=\sum_a\left[\left[\frac{1}{2m_a}\mathbf{p}_a^2+\tfrac{1}{2}Q_a V(\mathbf{r}_a)\right]+U_{\rm ext}(\mathbf{r}_a)\right], \quad (2.30)$$

$$H_r=\tfrac{1}{2}\int d^3r\left[\epsilon_0\epsilon\,\dot{\mathbf{A}}^2+\frac{1}{\mu_0}(\nabla\times\mathbf{A})^2\right], \quad (2.31)$$

$$H_{\rm int}=-\sum_a\frac{Q_a}{2m_a}[2\,\mathbf{A}(\mathbf{r}_a)\mathbf{p}_a-Q_a\,\mathbf{A}^2(\mathbf{r}_a)]\;. \quad (2.32)$$

We therefore may regard H_s as the Hamiltonian of the point charges including their mutual interaction via the Coulomb coupling modified by the dielectric with space-dependent refractive index $n(\mathbf{r})$. Note that $V(\mathbf{r}_a)$ in Eq. (2.30) is determined by Eq. (2.25). Analogously, H_r may be regarded as the Hamiltonian of the electromagnetic radiation field in the dielectric. The interaction between the two systems is described by $H_{\rm int}$.

B. Quantization of the radiation field without sources

We first consider the simple case of light propagation through the dielectric without sources being present. The (classical) Hamiltonian reads as follows:

$$H=H_r=\tfrac{1}{2}\int d^3r\left[\epsilon_0\epsilon\,\dot{\mathbf{A}}^2+\frac{1}{\mu_0}(\nabla\times\mathbf{A})^2\right] \quad (2.33)$$

[see Eq. (2.31)], in which the vector potential is determined from Eq. (2.28) with $\mathbf{j}=\mathbf{0}$,

$$\nabla\times\nabla\times\mathbf{A}(\mathbf{r},t)+\frac{\epsilon(\mathbf{r})}{c^2}\ddot{\mathbf{A}}(\mathbf{r},t)=\mathbf{0}\;, \quad (2.34)$$

where $c^2=(\mu_0\epsilon_0)^{-1}$.

By the familiar procedure of separation of variables we may assume the solution of Eq. (2.34) of the form

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{\sqrt{\epsilon_0}}\sum_\lambda q_\lambda(t)\,\mathbf{A}_\lambda(\mathbf{r})\;, \quad (2.35)$$

where the (vectorial) mode functions $\mathbf{A}_\lambda(\mathbf{r})$ and the amplitudes $q_\lambda(t)$ obey the equations

$$\nabla\times\nabla\times\mathbf{A}_\lambda(\mathbf{r})-\epsilon(\mathbf{r})\frac{\omega_\lambda^2}{c^2}\mathbf{A}_\lambda(\mathbf{r})=0\;, \quad (2.36)$$

$$\ddot{q}_\lambda(t)+\omega_\lambda^2 q_\lambda(t)=0\;, \quad (2.37)$$

where ω_λ^2 is the separation constant for each λ. Note that the $\mathbf{A}_\lambda(\mathbf{r})$ must fulfill the gauge condition

$$\nabla\cdot[\epsilon(\mathbf{r})\,\mathbf{A}_\lambda(\mathbf{r})]=0\;. \quad (2.38)$$

At this point we note that in many practical applications the dielectric is composed of appropriately chosen dielectric layers each of which has constant refractive index. Clearly, in this case Eq. (2.36) can only be stated inside each layer and reads as

$$\Delta\,\mathbf{A}_\lambda(\mathbf{r})+\epsilon_l\frac{\omega_\lambda^2}{c^2}\mathbf{A}_\lambda(\mathbf{r})=\mathbf{0},\quad l=1,2,3,\;\cdots \quad (2.39)$$

where l labels the layers. The transition of the electromagnetic field across the discontinuity surfaces is then determined by the boundary conditions which follow from the integral formulation of Maxwell's theory in the standard way.

From Eq. (2.36) the mode functions $\mathbf{A}_\lambda(\mathbf{r})$ are easily verified to be orthogonal in the following sense:

$$\int d^3r\, \epsilon(\mathbf{r})\, \mathbf{A}_\lambda(\mathbf{r})\cdot \mathbf{A}_{\lambda'}(\mathbf{r})=0 \quad \text{for } \omega_\lambda \neq \omega_{\lambda'} . \tag{2.40}$$

Now we may assume that they give a complete set of normal modes in the space of vectorial functions satisfying the gauge condition (2.38) and that they are normalized to unity and orthogonal in the sense of Eq. (2.40),

$$\int d^3r\, \epsilon(\mathbf{r})\, \mathbf{A}_\lambda(\mathbf{r})\cdot \mathbf{A}_{\lambda'}(\mathbf{r})=\delta_{\lambda\lambda'} . \tag{2.41}$$

It should be noted that the mode index λ may also include continuous variables. In such a case parts of the summations in Eq. (2.35) must be understood as integrations, and the symbols $\delta_{\lambda\lambda'}$ in Eq. (2.41) represents, of course, the δ functions.

Inserting Eq. (2.35) into Eq. (2.33) and making use of Eq. (2.41), we may represent the Hamiltonian in the following form:

$$H=\tfrac{1}{2}\sum_\lambda (\dot{q}_\lambda^2+\omega_\lambda^2 q_\lambda^2) . \tag{2.42}$$

The electromagnetic radiation field in the dielectric is therefore equivalent to an infinite set of uncoupled harmonic oscillators, so that we have

$$H=\tfrac{1}{2}\sum_\lambda (p_\lambda^2+\omega_\lambda^2 q_\lambda^2) \tag{2.43}$$

and

$$\frac{\partial H}{\partial p_\lambda}=\dot{q}_\lambda=p_\lambda , \tag{2.44}$$

$$\frac{\partial H}{\partial q_\lambda}=-\dot{p}_\lambda=\omega_\lambda^2 q_\lambda . \tag{2.45}$$

Combining Eqs. (2.44) and (2.45) yields the equations of motions (2.37).

The quantization of the field is now straightforward. Making use of standard approaches we associate Hermitian operators $\hat{q}_\lambda$ and $\hat{p}_\lambda$ with the classical variables q_λ and p_λ and postulate the commutation relations

$$\begin{aligned}&[\hat{q}_\lambda,\hat{p}_{\lambda'}]=i\hbar\delta_{\lambda\lambda'} ,\\ &[\hat{q}_\lambda,\hat{q}_{\lambda'}]=0=[\hat{p}_\lambda,\hat{p}_{\lambda'}] .\end{aligned} \tag{2.46}$$

Defining the non-Hermitian photon destruction and creation operators $\hat{a}_\lambda$ and $\hat{a}_\lambda^\dagger$, respectively, in the usual way by

$$\hat{q}_\lambda=\left[\frac{\hbar}{2\omega_\lambda}\right]^{1/2}(\hat{a}_\lambda^\dagger+\hat{a}_\lambda) , \tag{2.47}$$

$$\hat{p}_\lambda=i\left[\frac{\hbar\omega_\lambda}{2}\right]^{1/2}(\hat{a}_\lambda^\dagger-\hat{a}_\lambda) , \tag{2.48}$$

we derive from Eq. (2.46) the familiar commutation relations

$$\begin{aligned}&[\hat{a}_\lambda,\hat{a}_{\lambda'}^\dagger]=\delta_{\lambda\lambda'} ,\\ &[\hat{a}_\lambda,\hat{a}_{\lambda'}]=0=[\hat{a}_\lambda^\dagger\hat{a}_{\lambda'}^\dagger] .\end{aligned} \tag{2.49}$$

In terms of $\hat{a}_\lambda$ and $\hat{a}_\lambda^\dagger$ the Hamiltonian, which follows from Eq. (2.43) by considering q_λ and p_λ as operators $\hat{q}_\lambda$ and $\hat{p}_\lambda$, may be written as

$$\hat{H}=\sum_\lambda \hbar\omega_\lambda(\hat{a}_\lambda^\dagger\hat{a}_\lambda+\tfrac{1}{2}) \tag{2.50}$$

and the Heisenberg equations of motion for $\hat{a}$ and $\hat{a}^\dagger$ are

$$\begin{aligned}&\dot{\hat{a}}_\lambda=\frac{1}{i\hbar}[\hat{a}_\lambda,\hat{H}]=-i\omega_\lambda\hat{a}_\lambda ,\\ &\dot{\hat{a}}_\lambda^\dagger=\frac{1}{i\hbar}[\hat{a}_\lambda^\dagger,\hat{H}]=i\omega_\lambda\hat{a}_\lambda^\dagger .\end{aligned} \tag{2.51}$$

Inserting Eq. (2.47) into Eq. (2.35), the operator of the vector potential may be written as follows:

$$\hat{\mathbf{A}}(\mathbf{r},t)=\sum_\lambda \mathbf{A}_\lambda(\mathbf{r})[\hat{a}_\lambda(t)+\hat{a}_\lambda^\dagger(t)] . \tag{2.52}$$

For notational convenience in Eq. (2.52) we omit the factor $(\hbar/2\epsilon_0\omega_\lambda)^{1/2}$ and write it in the normalization condition, which now becomes

$$\int d^3r\epsilon(\mathbf{r})\, \mathbf{A}_\lambda(\mathbf{r})\cdot \mathbf{A}_{\lambda'}(\mathbf{r})=\frac{\hbar}{2\epsilon_0\omega_\lambda}\delta_{\lambda\lambda'} \tag{2.53}$$

instead of Eq. (2.41). In many cases it might be convenient to choose complex mode functions $\mathbf{A}_\lambda(\mathbf{r})$ instead of real ones. It can be verified easily that in such a case the Hamiltonian also takes the form given in Eq. (2.50), but the operator of the vector potential now reads as

$$\hat{\mathbf{A}}(\mathbf{r},t)=\sum_\lambda [\,\mathbf{A}_\lambda(\mathbf{r})\hat{a}_\lambda(t)+\mathbf{A}_\lambda^*(\mathbf{r})\hat{a}_\lambda^\dagger(t)] . \tag{2.54}$$

It should be emphasized that Eqs. (2.50) and (2.54) also include the case of radiation in free space, that is, $\epsilon(\mathbf{r})\equiv 1$. Clearly, in this case the mode functions $\mathbf{A}_\lambda(\mathbf{r})$ are the well-known (complex) plane waves. In the case of optical instruments being present the actual mode structure must be calculated from Eq. (2.34) [or Eqs. (2.39) together with the corresponding boundary conditions]. The mode structure of the field may therefore be said to reflect the action of the optical system.

Finally, the operator of that part of the electric field strength which is associated with the radiation field is given by

$$\begin{aligned}\hat{\mathbf{E}}(\mathbf{r},t)&=-\frac{\partial\hat{\mathbf{A}}(\mathbf{r},t)}{\partial t}\\ &=-\frac{1}{i\hbar}[\,\hat{\mathbf{A}},\hat{H}\,]\\ &=\sum_\lambda i\omega_\lambda[\,\mathbf{A}_\lambda(\mathbf{r})\hat{a}_\lambda(t)-\mathbf{A}_\lambda^*(\mathbf{r})\hat{a}_\lambda^\dagger(t)]\end{aligned} \tag{2.55}$$

and the operator of the magnetic field strength reads as

$$\begin{aligned}\hat{\mathbf{B}}(\mathbf{r},t)&=\nabla\times\hat{\mathbf{A}}(\mathbf{r},t)\\ &=\sum_\lambda [\nabla\times\mathbf{A}_\lambda(\mathbf{r})\hat{a}_\lambda(t)+\nabla\times\mathbf{A}_\lambda^*(\mathbf{r})\hat{a}_\lambda^\dagger(t)] .\end{aligned} \tag{2.56}$$

It should be noted that the method of quantization applied above to the case of a dielectric with space-dependent refractive index was developed for particular cases in the past.[30,31]

C. Quantization of the radiation field with sources

We now turn to the problem of the quantum-mechanical formulation of the interaction of a radiation field with sources in a dielectric with space-dependent refractive index. For this purpose we must find a Hamiltonian such that the Heisenberg equations of motion for the field and source quantities, considered as operators, reduce to equations having the same form as the classical equations of motion given in Eqs. (2.15) and (2.16) together with Eqs. (2.25) and (2.26). Starting from the classical Hamiltonian in the form given in Eqs. (2.29)–(2.32), we may write the quantum theoretical Hamiltonian as follows:

$$\hat{H}=\hat{H}_r+\hat{H}_s+\hat{H}_{\rm int} \; . \tag{2.57}$$

In this equation $\hat{H}_r$ is the field Hamiltonian as given in Eq. (2.50),

$$\hat{H}_r=\sum_\lambda \hbar\omega_\lambda(\hat{a}^\dagger_\lambda\hat{a}_\lambda+\tfrac{1}{2}) \; . \tag{2.58}$$

According to Eq. (2.30), the source Hamiltonian $\hat{H}_s$ may be written as

$$\hat{H}_s=\sum_a\left[\frac{1}{2m_a}\hat{\mathbf{p}}_a^2+\tfrac{1}{2}Q_aV(\hat{\mathbf{r}}_a)+U_{\rm ext}(\hat{\mathbf{r}}_a)\right] , \tag{2.59}$$

where $\hat{\mathbf{r}}_a$ and $\hat{\mathbf{p}}_a$ must now be considered as operators, the Cartesian components of which satisfy the commutation relations

$$\begin{aligned}&[\hat{r}_{ka},\hat{p}_{k'a'}]=i\hbar\delta_{aa'}\delta_{kk'} \; ,\\ &[\hat{r}_{ka},\hat{r}_{k'a'}]=0=[\hat{p}_{ka},\hat{p}_{k'a'}] \; .\end{aligned} \tag{2.60}$$

Note that in Eq. (2.59) the effect of the optical system is included in the potential $V(\hat{\mathbf{r}}_a)$, which must be calculated from Eqs. (2.24) and (2.25).

Finally, from Eq. (2.32) the interaction Hamiltonian $\hat{H}_{\rm int}$ becomes

$$\hat{H}_{\rm int}=-\sum_a\frac{Q_a}{2m_a}\{\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)\cdot\hat{\mathbf{p}}_a+\hat{\mathbf{p}}_a\cdot\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)-Q_a[\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)]^2\} \; , \tag{2.61}$$

where $\hat{\mathbf{A}}$ is now the operator of the vector potential, and

$$\begin{aligned}&[\hat{r}_{ka},\hat{a}_\lambda]=[\hat{r}_{ka},\hat{a}^\dagger_\lambda]=0 \; ,\\ &[\hat{p}_{ka},\hat{a}_\lambda]=[\hat{p}_{ka},\hat{a}^\dagger_\lambda]=0 \; .\end{aligned} \tag{2.62}$$

We note that the operators $\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)$ and $\hat{\mathbf{p}}_a$ do not commute in general. By using the commutation relations given in Eqs. (2.60) and (2.62) the commutator becomes

$$[\hat{\mathbf{p}}_a,\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)]=-i\hbar\frac{\partial\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)}{\partial\hat{\mathbf{r}}_a} \; . \tag{2.63}$$

Remembering the gauge condition (2.23) we arrive at

$$[\hat{\mathbf{p}}_a,\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)]=i\hbar\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)\nabla[\ln\epsilon(\hat{\mathbf{r}}_a)] \; . \tag{2.64}$$

Introducing the Hermitian operator

$$\hat{\mathbf{J}}(\mathbf{r})=\sum_a\frac{Q_a}{2m_a}[\delta(\mathbf{r}-\hat{\mathbf{r}}_a)\hat{\mathbf{p}}_a+\hat{\mathbf{p}}_a\delta(\mathbf{r}-\hat{\mathbf{r}}_a) -Q_a\hat{\mathbf{A}}(\hat{\mathbf{r}}_a)\delta(\hat{\mathbf{r}}-\hat{\mathbf{r}}_a)] \; , \tag{2.65}$$

we may rewrite Eq. (2.61) as follows:

$$\hat{\mathbf{H}}_{\rm int}=-\int d^3r\,\hat{\mathbf{J}}\cdot\hat{\mathbf{A}}=-\int d^3r\,\hat{\mathbf{A}}\cdot\hat{\mathbf{J}} \; . \tag{2.66}$$

In Eq. (2.66) the operator of the vector potential $\hat{\mathbf{A}}$ is again given by Eq. (2.54), the photon destruction and creation operators $\hat{a}_\lambda,\hat{a}^\dagger_\lambda$ now obeying the Heisenberg equations of motion with the full Hamiltonian $\hat{H}$ in Eq. (2.57). The operator $\hat{\mathbf{A}}$ can be used to derive the operator of the electric field strength $\hat{\mathbf{E}}$ associated with the radiation field by means of Eq. (2.55) and to derive the operator of the magnetic field strength $\hat{\mathbf{B}}$ [Eq. (2.56)]. All of these field operators may be written in the form

$$\hat{\mathbf{F}}(\mathbf{r},t)=\sum_\lambda[\mathbf{F}_\lambda(\mathbf{r})\hat{a}_\lambda(t)+\mathbf{F}^*_\lambda(\mathbf{r})\hat{a}^\dagger_\lambda(t)] \; . \tag{2.67}$$

In dependence on the choice of $\hat{\mathbf{F}}$, the functions $\mathbf{F}_\lambda(\mathbf{r})$ can be readily derived from the mode functions of the vector potential $\mathbf{A}_\lambda(\mathbf{r})$.

It is often convenient to decompose a given field operator $\hat{\mathbf{F}}$ into two parts by

$$\hat{\mathbf{F}}(\mathbf{r},t)=\hat{\mathbf{F}}^{(+)}(\mathbf{r},t)+\hat{\mathbf{F}}^{(-)}(\mathbf{r},t) \; , \tag{2.68}$$

where

$$\hat{\mathbf{F}}^{(+)}(\mathbf{r},t)=\sum_\lambda\mathbf{F}_\lambda(\mathbf{r})\hat{a}_\lambda(t) \; , \tag{2.69}$$

$$\hat{\mathbf{F}}^{(-)}(\mathbf{r},t)=[\hat{\mathbf{F}}^{(+)}(\mathbf{r},t)]^\dagger \; . \tag{2.70}$$

In particular, applying this decomposition to the vector potential we may rewrite Eq. (2.66) as follows:

$$\hat{H}_{\rm int}=-\int d^3r(\hat{\mathbf{J}}^\dagger\cdot\hat{\mathbf{A}}^{(+)}+\hat{\mathbf{A}}^{(-)}\cdot\hat{\mathbf{J}}) \; . \tag{2.71}$$

For notational reasons in the first term in Eq. (2.71) we write $\hat{\mathbf{J}}^\dagger$. As long as $\hat{\mathbf{J}}$ is understood as the full operator given in Eq. (2.65), this is trivial because $\hat{\mathbf{J}}=\hat{\mathbf{J}}^\dagger$ is valid. However in the so-called rotating-wave approximation widely used in quantum optics, the actual operator $\hat{\mathbf{J}}$ in Eq. (2.71) only represents a non-Hermitian part of the full operator, and one must carefully distinguish $\hat{\mathbf{J}}$ from $\hat{\mathbf{J}}^\dagger$.

III. HEISENBERG EQUATIONS OF MOTION AND FORMAL REPRESENTATION OF THE RADIATION FIELD OPERATORS

In many cases when the interaction of the radiation field with the sources can be considered as an interaction with bound states of atomic systems, in the interaction Hamiltonian given in Eq. (2.61) the quadratic term in the vector potential may be disregarded. In what follows we shall be studying this case in more detail. Therefore let us assume that the operator J defined in

Eq. (2.65) may be approximated by

$$\hat{\mathbf{j}}(\mathbf{r})=\sum_a \frac{Q_a}{2m_a}[\delta(\mathbf{r}-\hat{\mathbf{r}}_a)\hat{\mathbf{p}}_a+\hat{\mathbf{p}}_a\delta(\mathbf{r}-\hat{\mathbf{r}}_a)] \ . \tag{3.1}$$

By making use of Eqs. (2.57)–(2.59) and Eqs. (2.71) and (3.1) together with the commutation relations given in Eqs. (2.49) and (2.62), the Heisenberg equations of motion for the photon destruction operators $\hat{a}_\lambda$ are derived to be

$$\dot{\hat{a}}_\lambda=\frac{1}{i\hbar}[\hat{a}_\lambda,\hat{H}]=-i\omega_\lambda\hat{a}_\lambda-\frac{1}{i\hbar}\int d^3r'\,A^*_{k'\lambda}(\mathbf{r}')\hat{J}_{k'}(\mathbf{r}') \ . \tag{3.2}$$

Here and in the following, vector components are labeled by the index k and repeated indices mean summation. The general, retarded solution of Eq. (3.2) is

$$\hat{a}_\lambda(t)=\hat{a}_{\lambda,\text{free}}(t)+\hat{a}_{\lambda,s}(t) \ , \tag{3.3}$$

where

$$\hat{a}_{\lambda,\text{free}}(t)=\hat{a}_{\lambda,\text{free}}(t')\exp[-i\omega_\lambda(t-t')] \tag{3.4}$$

and

$$\hat{a}_{\lambda,s}(t)=-\frac{1}{i\hbar}\int d^3r'\int dt'\Theta(t-t')\exp[-i\omega_\lambda(t-t')] \times A^*_{k'\lambda}(\mathbf{r}')\hat{J}_{k'}(\mathbf{r}',t') \ , \tag{3.5}$$

$\Theta(t)$ being the unit step function. At this point it should be noted that the operators $\hat{J}_k$ also obey Heisenberg equations of motion, which, of course, are coupled to the photonic equations of motion. The determination of the time evolution of the operators $\hat{J}_k$ therefore requires the solution of a system of coupled differential equations, which, in general, is hard to solve.

Combining Eqs. (2.69), (3.3), and (3.5) we may decompose any field operator $\hat{F}_k^{(+)}$ into a free-field operator and a source-field operator as follows:

$$\hat{F}_k^{(+)}(\mathbf{r},t)=\hat{F}_{k,\text{free}}^{(+)}(\mathbf{r},t)+\hat{F}_{k,s}^{(+)}(\mathbf{r},t) \ , \tag{3.6}$$

where

$$\hat{F}_{k,\text{free}}^{(+)}(\mathbf{r},t)=\sum_\lambda F_{k\lambda}(\mathbf{r})\hat{a}_{\lambda,\text{free}}(t) \ , \tag{3.7}$$

$$\hat{F}_{k,s}^{(+)}(\mathbf{r},t)=\int d^3r'\int dt'\Theta(t-t')K_{kk'}(\mathbf{r},t;\mathbf{r}',t') \times\hat{J}_k{}'(\mathbf{r}',t') \ . \tag{3.8}$$

In Eq. (3.8), the kernel $K_{kk'}$ is defined by

$$K_{kk'}(\mathbf{r},t;\mathbf{r}',t')=-\frac{1}{i\hbar}\sum_\lambda F_{k\lambda}(\mathbf{r})A^*_{k'\lambda}(\mathbf{r}') \times\exp[-i\omega_\lambda(t-t')] \ . \tag{3.9}$$

Inserting Eq. (3.8) into Eq. (3.6) yields the the following representation of $\hat{F}_k^{(+)}$:

$$\hat{F}_k^{(+)}(\mathbf{r},t)=\int d^3r'\int dt'\Theta(t-t')K_{kk'}(\mathbf{r},t;\mathbf{r}',t') \times\hat{J}_{k'}(\mathbf{r}',t')+\hat{F}_{k,\text{free}}^{(+)}(\mathbf{r},t) \ . \tag{3.10}$$

In particular, if we identify $\hat{F}_k^{(+)}$ with the vector potential $\hat{A}_k^{(+)}$ we have, according to Eqs. (2.54) and (2.67), $F_{k\lambda}\equiv A_{k\lambda}$, and the kernel $K_{kk'}$ takes the form

$$K_{kk'}(\mathbf{r},t;\mathbf{r}',t')=-\frac{1}{i\hbar}\sum_\lambda A_{k\lambda}(\mathbf{r})A^*_{k'\lambda}(\mathbf{r}') \times\exp[-i\omega_\lambda(t-t')] \ . \tag{3.11}$$

Analogously, if we are interested in the electric field strength of the radiation $\hat{E}_k^{(+)}$, from Eqs. (2.55), (2.67), and (3.9) we obtain

$$K_{kk'}(\mathbf{r},t;\mathbf{r}',t')=-\frac{1}{\hbar}\sum_\lambda \omega_\lambda A_{k\lambda}(\mathbf{r})A^*_{k'\lambda}(\mathbf{r}') \times\exp[-i\omega_\lambda(t-t')] \ . \tag{3.12}$$

Note that the symmetry relations

$$K^*_{kk'}(\mathbf{r},t;\mathbf{r}',t')=\mp K_{k'k}(\mathbf{r}',t';\mathbf{r},t) \tag{3.13}$$

are valid.

Equation (3.10) together with Eqs. (2.68) and (2.70) may be regarded as basic equation for describing the action of a passive lossless optical system in quantum optics. Since it is simply the general solution of the inhomogeneous Maxwell equations, it is clear that Eq. (3.10) formally looks like the equation known for the case without optical instruments. The difference between the two cases consists in the mode functions $F_{k\lambda}$ to be chosen. In particular, the information about the action of the optical instruments on the source field is contained in the space-time structure of the kernel $K_{kk'}$, which may be regarded as the apparatus function also used in classical optics. In quantum optics, however, this function also determines the radiation field commutation relations (cf. Sec. IV) and, in consequence, the effect of the optical instruments on the quantum noise properties of the field.

It should be noted that in classical optics any light field may be thought to be attributed to sources; hence in classical optics the free-field term in Eq. (3.6) may be omitted. In quantum optics, however, the situation is changed drastically. The operator $\hat{F}_k^{(+)}$ of a given field quantity cannot be related to the source-field operator $\hat{F}_{k,s}^{(+)}$ solely, but it must also be related to the free-field operator $\hat{F}_{k,\text{free}}^{(+)}$, which is obviously needed for the correct description of the effects of quantum noise, at least of the vacuum. From a more general point of view, the free-field operator, which, in general, does not commute with the source-field operator [cf. Eq. (4.13)], ensures the quantum-mechanical consistence of the theory. To demonstrate this, let us consider the field commutation relations in more detail.

IV. COMMUTATION RELATIONS

The commutation relation for any combination of the field operators $\hat{F}_k^{(+)}$, $\hat{F}_{k'}^{(-)}$ at equal times can easily be constructed by means of Eqs. (2.69) and (2.70) together with the basic commutation relations (2.49) for the photonic operators $\hat{a}_\lambda$, $\hat{a}^\dagger_\lambda$. We renounce the calculations of such commutators because they may readily be performed in a straightforward way. Note that the results apparently are the same as in the case of the free-field operators $\hat{F}^{(+)}_{k,\text{free}}$, $\hat{F}^{(-)}_{k,\text{free}}$. Clearly, the commutators of the free-field operators at different times may also be constructed in the way outlined above, by including the time exponentials given in Eq. (3.4) in the calculation.

In contrast to this, the calculation of the commutators of the field operators $\hat{F}^{(+)}_k$, $\hat{F}^{(-)}_{k'}$ or combinations of them at different times is nontrivial, because the solution of the interaction problem with sources is needed. Therefore, the aim of this section can only consist in expressing such commutators in terms of free-field commutators and commutators of source quantities.

For the sake of notational clearness it will be convenient to use the following abbreviations and definitions:

$$x=\{\mathbf{r},t\}, \quad \int dx\ldots = \int d^3r \int dt\ldots, \tag{4.1}$$

$$\hat{J}^{(+)}_k(x)=\hat{J}_k(\mathbf{r},t), \quad \hat{J}^{(-)}_k(x)=[\hat{J}^{(+)}_k(x)]^\dagger , \tag{4.2}$$

$$K^{(+)}_{kk'}(x,x')=K_{kk'}(\mathbf{r},t;\mathbf{r}',t') ,$$
$$K^{(-)}_{kk'}(x,x')=[K_{kk'}(x,x')]^* , \tag{4.3}$$

where $K_{kk'}(\mathbf{r},t;\mathbf{r}',t')$ is defined in Eq. (3.9). Thus we may write Eqs. (2.68), (2.70), and (3.6) as

$$\hat{F}_k(x)=\hat{F}^{(+)}_k(x)+\hat{F}^{(-)}_k(x) , \tag{4.4}$$

$$\hat{F}^{(-)}_k(x)=[\hat{F}^{(+)}_k(x)]^\dagger , \tag{4.5}$$

$$\hat{F}^{(j)}_k(x)=\hat{F}^{(j)}_{k,\text{free}}(x)+\hat{F}^{(j)}_{k,s}(x) , \tag{4.6}$$

where (j) may be $(+)$ or $(-)$. In Eq. (4.6), $\hat{F}^{(j)}_{k,\text{free}}(x)$ is given by Eq. (3.7) together with Eq. (4.5), and, according to Eq. (3.8), Eqs. (4.1)–(4.3), and Eq. (4.5), $\hat{F}^{(j)}_{k,s}(x)$ is given by

$$\hat{F}^{(j)}_{k,s}(x)= \int dx'\Theta(t-t')K^{(j)}_{kk'}(x,x')\hat{J}^{(j)}_{k'}(x') . \tag{4.7}$$

Now let us consider the commutator of $\hat{F}^{(j)}_{k_1\text{free}}(\mathbf{r}_1,t_1)$ and an arbitrarily chosen source-quantity operator $\hat{Q}(t_2)$. Combining Eqs. (3.3), (3.4), and (3.5) we readily find

$$\hat{a}_{\lambda,\text{free}}(t_1)=\hat{a}_\lambda(t_2)\exp[i\omega_\lambda(t_2-t_1)] + \frac{1}{i\hbar}\int d^3r'_1 \int dt'_1\Theta(t_2-t'_1) \times \exp[-i\omega_\lambda(t_1-t'_1)] \times A^*_{k'_1\lambda}(\mathbf{r}'_1)\hat{J}_{k'_1}(\mathbf{r}'_1,t'_1) . \tag{4.8}$$

According to the commutation relations (2.62), for equal times photonic and source-quantity operators commute and therefore the commutation rule

$$[\hat{a}_\lambda(t_2),\hat{Q}(t_2)]=0 \tag{4.9}$$

holds. Hence, we derive

$$[\hat{a}_{\lambda,\text{free}}(t_1),\hat{Q}(t_2)] = \frac{1}{i\hbar}\int d^3r'_1 \int dt'_1\Theta(t_2-t'_1)\exp[-i\omega_\lambda(t_1-t'_1)] \times A^*_{k'_1\lambda}(\mathbf{r}'_1)[\hat{J}_{k'_1}(\mathbf{r}'_1,t'_1),\hat{Q}(t_2)] , \tag{4.10}$$

and analogously

$$[\hat{a}^\dagger_{\lambda,\text{free}}(t_1),\hat{Q}(t_2)] = -\frac{1}{i\hbar}\int d^3r'_1 \int dt'_1\Theta(t_2-t'_1)\exp[i\omega_\lambda(t_1-t'_1)] \times A_{k'_1\lambda}(\mathbf{r}'_1)[\hat{J}^\dagger_{k'_1}(\mathbf{r}'_1,t'_1),\hat{Q}(t_2)] . \tag{4.11}$$

We now multiply Eqs. (4.10) and (4.11) with $F_{k_1\lambda}(\mathbf{r}_1)$ and $F^*_{k_1\lambda}(\mathbf{r}_1)$, respectively, and sum over λ. Remembering the definitions of the free-field operators $\hat{F}^{(j)}_{k,\text{free}}$ [Eq. (3.7) together with Eq. (4.5)] and the kernel $K_{kk'}$ [Eq. (3.9)] and using the abbreviating denotations and definitions given above, we arrive at the following representation of the commutator sought:

$$[\hat{F}^{(j)}_{k_1,\text{free}}(x_1),\hat{Q}(t_2)]=-\int dx'_1\Theta(t_2-t'_1)K^{(j)}_{k_1k'_1}(x_1,x'_1) \times[\hat{J}^{(j)}_{k'_1}(x'_1),\hat{Q}(t_2)] . \tag{4.12}$$

Note that Eq. (4.12) may be regarded as the general formulation of the results derived for special cases in Refs. 32 and 33. The commutator rule (4.12) now enables us to express commutators of the form $[\hat{F}^{(j_1)}_{k_1,\text{free}}(x_1),\hat{F}^{(j_2)}_{k_2s}(x_2)]$ in terms of source-quantity commutators. Combining Eqs. (4.12) and (4.7) yields

$$[\hat{F}^{(j_1)}_{k_1,\text{free}}(x_1),\hat{F}^{(j_2)}_{k_2,s}(x_2)] = -\int\int dx'_1dx'_2\Theta(t_2-t'_2)\Theta(t'_2-t'_1) \times K^{(j_1)}_{k_1k'_1}(x_1,x'_1)K^{(j_2)}_{k_2k'_2}(x_2,x'_2) \times[\hat{J}^{(j_1)}_{k'_1}(x'_1),\hat{J}^{(j_2)}_{k'_2}(x'_2)] . \tag{4.13}$$

Making use of the relation $\Theta(t)+\Theta(-t)=1$, we may rewrite Eq. (4.13) as follows:

$$[\hat{F}^{(j_1)}_{k,\text{free}}(x_1),\hat{F}^{(j_2)}_{k_2,s}(x_2)]=\hat{C}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2)+\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2) , \tag{4.14}$$

where

$$\hat{C}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2) = -\int\int dx'_1dx'_2\Theta(t_2-t'_2)\Theta(t'_2-t'_1)\Theta(t_1-t'_1) \times K^{(j_1)}_{k_1k'_1}(x_1,x'_1)K^{(j_2)}_{k_2k'_2}(x_2,x'_2) \times[\hat{J}^{(j_1)}_{k'_1}(x'_1),\hat{J}^{(j_2)}_{k'_2}(x'_2)] , \tag{4.15}$$

and

$$\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2) = -\int\int dx'_1 dx'_2 \Theta(t_2-t'_2)\Theta(t'_2-t'_1)\Theta(t'_1-t_1) \times K^{(j_1)}_{k_1k'_1}(x_1,x'_1)K^{(j_2)}_{k_2k'_2}(x_2,x'_2) \times [\hat{J}^{(j_1)}_{k'_1}(x'_1),\hat{J}^{(j_2)}_{k'_2}(x'_2)] \ . \tag{4.16}$$

From an inspection of Eq. (4.16) we readily verify that

$$\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2)=0 \quad \text{if } t_1>t_2 \ . \tag{4.17}$$

We now turn to the problem of expressing commutators of the form $[\hat{F}^{(j_1)}_{k_1}(x_1),\hat{F}^{(j_2)}_{k_2}(x_2)]$ in terms of free-field and source-quantity commutators. For this purpose we note that the operator product $\hat{F}^{(j_1)}_{k_1}(x_1)\hat{F}^{(j_2)}_{k_2}(x_2)$ may be decomposed, according to Eq. (4.6), as follows:

$$\hat{F}^{(j_1)}_{k_1}(x_1)\hat{F}^{(j_2)}_{k_2}(x_2)=\hat{F}^{(j_1)}_{k_1,\text{free}}(x_1)\hat{F}^{(j_2)}_{k_2,\text{free}}(x_2)+\hat{F}^{(j_1)}_{k_1,s}(x_1)\hat{F}^{(j_2)}_{k_2,\text{free}}(x_2) +\hat{F}^{(j_2)}_{k_2,s}(x_2)\hat{F}^{(j_1)}_{k_1,\text{free}}(x_1)+\hat{F}^{(j_1)}_{k_1,s}(x_1)\hat{F}^{(j_2)}_{k_2,s}(x_2)+[\hat{F}^{(j_1)}_{k_1,\text{free}}(x_1),\hat{F}^{(j_2)}_{k_2,s}(x_2)] \ . \tag{4.18}$$

Making use of Eq. (4.7) and Eqs. (4.13)–(4.15) and remembering the relation $\Theta(t)+\Theta(-t)=1$, we may easily prove that combining the last two terms on the right-hand side in Eq. (4.18) yields

$$\hat{F}^{(j_1)}_{k_1}(x_1)\hat{F}^{(j_2)}_{k_2}(x_2)=\hat{F}^{(j_1)}_{k_1,\text{free}}(x_1)\hat{F}^{(j_2)}_{k_2,\text{free}}(x_2)+\hat{F}^{(j_1)}_{k_1,s}(x_1)\hat{F}^{(j_2)}_{k_2,\text{free}}(x_2)+\hat{F}^{(j_2)}_{k_2,s}(x_2)\hat{F}^{(j_1)}_{k_1,\text{free}}(x_1) + \int\int dx'_1 dx'_2 \Theta(t_1-t'_1)\Theta(t_2-t'_2)K^{(j_1)}_{k_1k'_1}(x_1,x'_1)K^{(j_2)}_{k_2k'_2}(x_2,x'_2) \times T_+\hat{J}^{(j_1)}_{k'_1}(x'_1)\hat{J}^{(j_2)}_{k'_2}(x'_2)+\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2) \ . \tag{4.19}$$

Here and in the remainder of this paper the time-ordering symbols T_+ and T_- are used. They are defined as follows. Let us consider any operator product $\hat{A}_1(t_1)\hat{A}_2(t_2)\cdots\hat{A}_n(t_n)$. The symbol T_+ introduces time ordering of the operators $\hat{A}_i(t_i)$ with the latest time to the far left,

$$T_+\hat{A}_1(t_n)\hat{A}_2(t_2)\cdots\hat{A}_n(t_n) =\hat{A}_{i_1}(t_{i_1})\hat{A}_{i_2}(t_{i_2})\cdots\hat{A}_{i_n}(t_{i_n}) \quad \text{with } t_{i_1}>t_{i_2}>\cdots>t_{i_n} \ , \tag{4.20}$$

and the symbol T_- introduces time ordering of the operators $A_i(t_i)$ with the latest time to the far right,

$$T_-\hat{A}_1(t_1)\hat{A}_2(t_2)\cdots\hat{A}_n(t_n) =\hat{A}_{i_1}(t_{i_1})\hat{A}_{i_2}(t_{i_2})\cdots\hat{A}_{i_n}(t_{i_n}) \quad \text{with } t_{i_1}<t_{i_2}<\cdots t_{i_n} \ . \tag{4.21}$$

From Eq. (4.19) we easily arrive at the commutator relation

$$[\hat{F}^{(j_1)}_{k_1}(x_1),\hat{F}^{(j_2)}_{k_2}(x_2)]=[\hat{F}^{(j_1)}_{k_1,\text{free}}(x_1),\hat{F}^{(j_2)}_{k_2,\text{free}}(x_2)] +\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2) -\hat{D}^{(j_2,j_1)}_{k_2k_1}(x_2,x_1) \ . \tag{4.22}$$

We note that the field commutation relations derived above are generalizations of the results found by Cresser for the particular case of a field radiated by a single atom in vacuum[34] to the case of more complicated source distributions and optical instruments being present.

Especially, the terms $\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2)$ and $\hat{D}^{(j_2,j_1)}_{k_2k_1}(x_2,x_1)$ defined according to Eq. (4.16) represent the so-called time-delayed contributions[34] in the very general case under study. From an inspection of Eq. (4.22) we see that the commutators of fields that are attributed to sources differ from the corresponding free-field commutators in the time-delayed contributions $\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2)$ and $\hat{D}^{(j_2,j_1)}_{k_2k_1}(x_2,x_1)$. It is worth noting that, according to Eq. (4.16), in the integrals over x'_1 and x'_2 the terms that may contribute to $\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2)$ are time ordered in such a way that $t_1<t'_1<t'_2<t_2$. This is just the time ordering necessary for the propagation of light from the space-time point $\mathbf{r}_1,t_1$ to the space-time point $\mathbf{r}_2,t_2$ via the sources. In the particular case when the source is a single atom in vacuum the time-delayed term $\hat{D}^{(j_1,j_2)}_{k_1k_2}(x_1,x_2)$ can be nonzero only if the condition $t_2-t_1>(l_1+l_2)/c$ is fulfilled, where l_1 and l_2, respectively, are the distances between the point of source and the points $\mathbf{r}_1$ and $\mathbf{r}_2$. The difference $\hat{D}^{(j_1,j_2)}_{k_1,k_2}(x_1,x_2)-\hat{D}^{(j_2,j_1)}_{k_2k_1}(x_2,x_1)$ may therefore represent a nonzero contribution to the commutator $[\hat{F}^{(j_1)}_{k_1}(x_1),\hat{F}^{(j_2)}_{k_2}(x_2)]$ provided that the times t_1 and t_2 satisfy the inequality $|t_2-t_1|>(l_1+l_2)/c$. In the more general case when the field of the radiating atom passes through an optical instrument the geometrical paths l_1 and l_2 must be replaced by the optical paths.[28] In either case it is easily seen that nonzero time-delayed contributions to the commutator in Eq. (4.22) may be expected only if the two times t_1 and t_2 are chosen in such a way that during the delay time $|t_2-t_1|$ a light signal has the chance to travel from one of the space points to the other via the source atom. Clearly, the farther away from the space points $\mathbf{r}_1$ and $\mathbf{r}_2$ the radiating atom is situated, the larger becomes the time interval $|t_2-t_1|$ in which the field commutators $[\hat{F}^{(j_1)}_{k_1}(x_1),\hat{F}^{(j_2)}_{k_2}(x_2)]$ are simply given by the free-field commutators $[\hat{F}^{(j_1)}_{k,\mathrm{free}}(x_1),\hat{F}^{(j_2)}_{k_2,\mathrm{free}}(x_2)]$. As mentioned at the beginning of this section, the latter may readily be constructed by means of Eqs. (2.49), (3.4), and (3.7). The result is

$$[\hat{F}^{(\pm)}_{k_1,\mathrm{free}}(x_1),\hat{F}^{(\pm)}_{k_2,\mathrm{free}}(x_2)]=0 , \quad (4.23)$$

$$[\hat{F}^{(+)}_{k_1,\mathrm{free}}(x_1),\hat{F}^{(-)}_{k_2,\mathrm{free}}(x_2)]=F_{k_1k_2}(x_1,x_2) , \quad (4.24)$$

where

$$F_{k_1k_2}(x_1,x_2)=\sum_{\lambda}F_{k_1\lambda}(\mathbf{r}_1)F^{*}_{k_2\lambda}(\mathbf{r}_2)\exp[-i\omega_\lambda(t_1-t_2)] . \quad (4.25)$$

Note that the effect of the optical instruments is included in $F_{k_1k_2}(x_1,x_2)$.

V. CORRELATION FUNCTIONS OF FIELD OPERATORS

In this section we turn to the problem of expressing quantum optical correlation functions of field operators $\hat{F}^{(j)}_{k}(x)$ in terms of correlation functions of source-quantity operators $\hat{J}^{(j)}_{k}(x')$ and free-field operators $\hat{F}^{(j)}_{k,\mathrm{free}}(x)$. Since in many cases of practical interest the field operators are subjected to normal ordering and certain time ordering, we will demonstrate the method for the following important class of correlation functions:

$$G^{(m,n)}_{k_1\cdots k_{m+n}}(x_1,\ldots,x_{m+n}) =\left\langle \left[T_-\prod_{j=1}^{m}\hat{F}^{(-)}_{k_j}(x_j)\right]\left[T_+\prod_{j=m+1}^{m+n}\hat{F}^{(+)}_{k_j}(x_j)\right]\right\rangle . \quad (5.1)$$

In Eq. (5.1), the time-ordering symbols T_+ and T_- are defined according to Eqs. (4.20) and (4.21).

For example, from Glauber's theory of light detection the photocount distribution function is determined by correlation functions of the type given in Eq. (5.1) with $n=m$, $m=1,2,3,\ldots$ and the identification $\hat{F}^{(\pm)}_{k}(x)\equiv\hat{E}^{(\pm)}_{k}(x)$ (cf. Sec. VI).[35,36] Correlation functions with $n\neq m$ may be observed in photon detection experiments after homodyne mixing the light under study with a reference beam. The homodyne detection scheme for observing squeezed light is an example. In this case correlation functions $G^{(2,0)}_{k_1,k_2}$ and $G^{(0,2)}_{k_1k_2}$ also contribute to the detection signal.[37]

Practically, in Eq. (5.1) the field operators $\hat{F}^{(\pm)}_{k}$ are decomposed in source-field operators $\hat{F}^{(\pm)}_{k,s}$ and free-field operators $\hat{F}^{(\pm)}_{k,\mathrm{free}}$, and by means of the commutation relations given in Eq. (4.14) the resulting, mixed operator products are rearranged in such a way that the operators $\hat{F}^{(+)}_{k,\mathrm{free}}$ are on the right of the operators $\hat{F}^{(+)}_{k,s}$ and, correspondingly, the operators $\hat{F}^{(-)}_{k,\mathrm{free}}$ are on the left of the operators $\hat{F}^{(-)}_{k,s}$.

To perform this procedure let us begin with the operator product $\hat{F}^{(+)}_{k_1}(x_1)\hat{F}^{(+)}_{k_2}(x_2)$, which, according to Eq. (4.19) with $(j_1)=(j_2)=(+)$ and according to Eqs. (4.2) and (4.3), may be written as

$$\hat{F}^{(+)}_{k_1}(x_1)\hat{F}^{(+)}_{k_2}(x_2)=\hat{F}^{(+)}_{k_1,\mathrm{free}}(x_1)\hat{F}^{(+)}_{k_2,\mathrm{free}}(x_2)+\hat{F}^{(+)}_{k,s}(x_1)\hat{F}^{(+)}_{k_2,\mathrm{free}}(x_2)+\hat{F}^{(+)}_{k_2,s}(x_2)\hat{F}^{(+)}_{k_1,\mathrm{free}}(x_1)$$
$$+\int\int dx'_1dx'_2\Theta(t_1-t'_1)\Theta(t_2-t'_2)K_{k_1k'_1}(x_1,x'_1)K_{k_2k'_2}(x_2,x'_2)$$
$$\times T_+\hat{J}_{k'_1}(x'_1)\hat{J}_{k'_2}(x'_2)+\hat{D}^{(+,+)}_{k_1k_2}(x_1,x_2) . \quad (5.2)$$

Writing down the analogous expression for the operator product $\hat{F}^{(+)}_{k_2}(x_2)\hat{F}^{(+)}_{k_1}(x_1)$ and remembering that, according to Eq. (4.17), for $t_1>t_2$ the time-delayed term $\hat{D}^{(+,+)}_{k_1k_2}(x_1,x_2)$ vanishes and, accordingly, for $t_2>t_1$ the time-delayed term $D^{(+,+)}_{k_2k_1}(x_2,x_1)$ vanishes, we readily derive

$$T_+\hat{F}^{(+)}_{k_1}(x_1)\hat{F}^{(+)}_{k_2}(x_2)=\hat{F}^{(+)}_{k_1\mathrm{free}}(x_1)\hat{F}^{(+)}_{k_2,\mathrm{free}}(x_2)+\hat{F}^{(+)}_{k_1s}(x_1)\hat{F}^{(+)}_{k_2,\mathrm{free}}(x_2)+\hat{F}^{(+)}_{k_2,s}(x_2)\hat{F}^{(+)}_{k_1,\mathrm{free}}(x_1)$$
$$+\int\int dx'_1dx'_2\Theta(t_1-t'_1)\Theta(t_2-t'_2)K_{k_1k'_1}(x_1,x'_1)K_{k_2k'_2}(x_2,x'_2)T_+\hat{J}_{k'_1}(x'_1)\hat{J}_{k'_2}(x'_2) . \quad (5.3)$$

Note that in the first term on the right-hand side in Eq. (5.3) the time-ordering symbol T_+ is left because the free-field operators $\hat{F}^{(+)}_{k,\mathrm{free}}$ commute [cf. Eq. (4.23)]. We see that the T_+ time ordering of the operator product $\hat{F}^{(+)}_{k_1}(x_1)\hat{F}^{(+)}_{k_2}(x_2)$ obviously rules out any time-delayed effect. The T_+ time ordering may therefore be said to pick out the commuting parts of $\hat{F}^{(+)}_{k_1}(x_1)$ and $\hat{F}^{(+)}_{k_2}(x_2)$ in the product $\hat{F}^{(+)}_{k_1}(x_1)\hat{F}^{(+)}_{k_2}(x_2)$.

Remembering Eqs. (4.6) and (4.7) we may write Eq. (5.3) in a more compact form as follows:

$$T_+\hat{F}^{(+)}_{k_1}(x_n)\hat{F}^{(+)}_{k_2}(x_2)=O_+[\hat{F}^{(+)}_{k_1,\mathrm{free}}(x_1)+\hat{F}^{(+)}_{k_1,s}(x_1)]\times[\hat{F}^{(+)}_{k_2,\mathrm{free}}(x_2)+\hat{F}^{(+)}_{k_2,s}(x_2)] . \quad (5.4)$$

In Eqs. (5.4) and the following ones the ordering symbols O_+ and O_- are used. The symbol O_+ introduces the following operator ordering in products of operators

$\hat{F}^{(+)}_{k_i,s}(x_i)$, $\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)$:

(i) Ordering of the operators $\hat{F}^{(+)}_{k_i,s}(x_i)$, $\hat{F}^{(+)}_{k_j\mathrm{free}}(x_j)$ with the operators $\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)$ to the right of the operators $\hat{F}^{(+)}_{k_i,s}(x_i)$.

(ii) Substituting of Eq. (4.7) for the operators $\hat{F}^{(+)}_{k_i,s}(x_i)$ and T_+ time ordering of the source-quantity operators $\hat{J}_{k_i'}(x_i')$ in the resulting source-quantity operator products before performing the integrations with respect to t_i'.

The symbol O_- introduces the following operator ordering in products of operators $\hat{F}^{(-)}_{k_i,s}(x_i)$, $\hat{F}^{(-)}_{k_j,\mathrm{free}}(x_j)$:

(i) Ordering of the operators $\hat{F}^{(-)}_{k_i,s}(x_i)$, $\hat{F}^{(-)}_{k_j,\mathrm{free}}(x_j)$ with the operators $\hat{F}^{(-)}_{k_j,\mathrm{free}}(x_j)$ to the left of the operators $\hat{F}^{(-)}_{k_i,s}(x_i)$.

(ii) Substituting of Eq.(4.7) for the operators $\hat{F}^{(-)}_{k_i,s}(x_i)$ and T_- time ordering of the source-quantity operators $\hat{J}^{\dagger}_{k_i'}(x_i')$ in the resulting source-quantity operator products before performing the integrations with respect to t_i'.

Equation (5.4) may now be generalized to the case of higher-order products of operators $F^{(+)}_k(x)$. As shown in the Appendix, the result is

$$T_+\prod_{j=1}^{n}\hat{F}^{(+)}_{k_j}(x_j)=O_+\prod_{j=1}^{n}[\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\ . \tag{5.5}$$

Taking the Hermitian conjugate of Eq. (5.5) and remembering the definition of the ordering symbols T_- and O_-, we obtain

$$T_-\prod_{j=1}^{n}\hat{F}^{(-)}_{k_j}(x_j)=O_-\prod_{j=1}^{n}[\hat{F}^{(-)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(-)}_{k_j,s}(x_j)]\ . \tag{5.6}$$

Finally, combining Eqs. (5.1), (5.5), and (5.6) we may represent the correlation functions $G^{(m,n)}_{k_1\cdots k_{m+n}}$ as follows:

$$G^{(m,n)}_{k_1\cdots k_{m+n}}(x_1,\ldots,x_{m+n})=\left\langle\left[O_-\prod_{j=1}^{m}[\hat{F}^{(-)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(-)}_{k_j,s}(x_j)]\right]\left[O_+\prod_{j=m+1}^{m+n}[\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\right]\right\rangle\ . \tag{5.7}$$

In practice, the field properties that are closely related to the source field are often desired to be observed. This implies an observational scheme that guarantees that at the points of observation the following conditions are fulfilled:

$$\langle\cdots\hat{F}^{(+)}_{k,\mathrm{free}}\rangle=0=\langle\hat{F}^{(-)}_{k,\mathrm{free}}\cdots\rangle\ . \tag{5.8}$$

In particular, when the radiating sources are optically pumped and the pump field can be treated in the sense of a free field, the range of observation must be outside the pump beam. At this point it should be noted that apart from the vacuum field any real light field may be thought to be attributed to sources. That is, we may regard any real pump field as part of the source field and therefore include it in the source field originally considered. From this point of view the only free field is the vacuum field so that the conditions (5.8) are always fulfilled.

Assuming that the conditions given in Eq. (5.8) are fulfilled and remembering the definitions of the ordering symbols $O_\pm$, we may omit the free-field operators $\hat{F}_{k_j,\mathrm{free}}$ in Eq. (5.7). This enables us to express the correlation functions $G^{(m,n)}_{k_1\cdots k_{m+n}}$ in terms of source-quantity correlation functions solely:

$$G^{(m,n)}_{k_1\cdots k_{m+n}}(x_1,\ldots,x_{m+n})=\left\langle\left[O_-\prod_{j=1}^{m}\hat{F}^{(-)}_{k_j,s}(x_j)\right]\left[O_+\prod_{j=m+1}^{m+n}\hat{F}^{(+)}_{k_j,s}(x_j)\right]\right\rangle\ , \tag{5.9}$$

which, according to the definitions of the ordering symbols $O_\pm$, may be written in more detail as

$$\begin{aligned}G^{(m,n)}_{k_1\cdots k_{m+n}}&(x_1,\ldots,x_{m+n})\\ &=\int dx_1'\Theta(t_1-t_1')K^*_{k_1k_1'}(x_1,x_1')\cdots\int dx_m'\Theta(t_m-t_m')K^*_{k_mk_m'}(x_m,x_m')\\ &\qquad\times\int dx_{m+1}'\theta(t_{m+1}-t_{m+1}')\mathbf{K}_{k_{m+1}k_{m+1}'}(x_{m+1},x_{m+1}')\\ &\qquad\times\cdots\int dx_{m+n}'\Theta(t_{m+n}-t_{m+n}')K_{k_{m+n}k_{m+n}'}(x_{m+n},x_{m+n}')\\ &\qquad\times\langle[T_-\hat{J}^{\dagger}_{k_1'}(x_1')\cdots\hat{J}^{\dagger}_{k_m'}(x_m')]\\ &\qquad\times[T_+\hat{J}'_{k_{m+1}'x_{m+1}'}\cdots\hat{J}_{k_{m+n}'}(x_{m+n}')]\rangle\ .\end{aligned} \tag{5.10}$$

Equations (5.9) and (5.10) establish that in the case of the conditions (5.8) being fulfilled, in the calculation of field correlation functions of the type defined in Eq. (5.1) the total field operators $\hat{F}^{(\pm)}_{k_j}$ may formally be replaced by the source-field operators $\hat{F}^{(\pm)}_{k_j,s}$, the T_+ and T_- time ordering originally concerning the operators $\hat{F}^{(+)}_{k_j}$ and $\hat{F}^{(-)}_{k_j}$, respectively, being transferred to the corresponding source-quantity operators $\hat{J}_{k_j'}$ and $\hat{J}^{\dagger}_{k_j'}$. Clearly, this ordering rule is a consequence of using the quantum mechanically correct basic equation (3.10). Similar results were also found by Apanasevich and Kilin studying correlation functions of (plane-wave) photon destruction and creation operators for the case of radiating sources in vacuum.[32] We emphasize that the results derived above render it possible, at first, to describe the full space-time structure of (multimode) field correlation functions and, at second, to take into account the action of optical systems. As known from Sec. III, the effect of optical systems is included in the actual structure of the kernel $K_{kk'}$, which is simply the apparatus function known from classical optics. However, there is an essential difference between classical and quantum optics, namely, that the (multitime) convolution integrals in Eq. (5.10) cannot be performed independently from each other as is possible in classical optics. Hence, in the case of quantum light fields the result of the integrations in Eq. (5.10) is expected to be different from that predicted from classical optics. This implies that in the case of quantum light fields the effect of the optical instruments on the field may be changed drastically (cf. Ref. 28).

It should be noted that the more general equation (5.7) must be used, for example, for describing (homodyne or heterodyne) mixing of source light with a reference light beam provided that the latter can be described in terms of a free field. Commonly, such a mixing is experimentally performed by means of a beam splitter. In our theory the effect of the beam splitter is described by the structure of the mode functions and the structure of the integral kernel $K_{kk'}$, the latter giving rise to the prescription of how to relate the source quantities to the field quantities.

Let us consider the case when the reference beam can be regarded as a free field which may be approximated by a multimode coherent field. For this purpose we rewrite Eq. (5.7) as follows:

$$G^{(m,n)}_{k_1\ldots k_{m+n}}(x_1,\ldots,x_{m+n})=\mathrm{Tr}\left[\left[O_+\prod_{j=m+1}^{m+n}[\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\right]\hat{\rho}\left[O_-\prod_{j=1}^{m}[\hat{F}^{(-)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\right]\right], \tag{5.11}$$

where $\hat{\rho}$ is the density operator at the initial time, which may be assumed to be the time of the beginning of the interaction between the light field and the sources. Now we assume that $\hat{\rho}$ is given in the factored form

$$\hat{\rho}=\hat{\rho}_{\mathrm{field}}\hat{\rho}_{\mathrm{sources}} \tag{5.12}$$

with

$$\hat{\rho}_{\mathrm{field}}=|\{\alpha_\lambda\}\rangle\langle\{\alpha_\lambda\}|\ , \tag{5.13}$$

where $|\{\alpha_\lambda\}\rangle$ is the multimode coherent state of the free field, viz.,

$$\hat{a}_{\lambda,\mathrm{free}}(t)|\{\alpha_\lambda\}\rangle=\alpha_\lambda\exp\{-i\omega_\lambda t\}|\{\alpha_\lambda\}\rangle\ . \tag{5.14}$$

Making use of Eq. (3.7) we therefore arrive at

$$\hat{F}^{(+)}_{k,\mathrm{free}}(x)|\{\alpha_\lambda\}\rangle=\mathcal{F}_k(x)|\{\alpha_\lambda\}\rangle\ , \tag{5.15}$$

where the (complex) c-number function $\mathcal{F}_k$ is given by the relation

$$\mathcal{F}_k(x)=\sum_\lambda F_{k\lambda}(\mathbf{r})\alpha_\lambda\exp(-i\omega_\lambda t)\ . \tag{5.16}$$

Combining Eqs. (5.12), (5.13), and (5.15), we obtain

$$\hat{F}^{(+)}_{k,\mathrm{free}}(x)\hat{\rho}\hat{F}^{(-)}_{k',\mathrm{free}}(x')=\mathcal{F}_k(x)\mathcal{F}^*_{k'}(x')\hat{\rho}\ . \tag{5.17}$$

Remembering the definitions of the ordering symbols $O_\pm$, we therefore may rewrite Eq. (5.11) as

$$G^{(m,n)}_{k_1\ldots k_{m+n}}(x_1,\ldots,x_{m+n})=\left\langle\left[O_-\prod_{j=1}^{m}[\mathcal{F}^*_{k_j}(x_j)+\hat{F}^{(-)}_{k_j,s}(x_j)]\right]\times\left[O_+\prod_{j=m+1}^{m+n}[\mathcal{F}_{k_j}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\right]\right\rangle\ . \tag{5.18}$$

We see that in this case in Eq. (5.7) the free-field operators are simply replaced by c-number functions, so that the ordering symbols $O_\pm$ only affect the source-field operators. After multiplying out the products in Eq. (5.18), the result is an expansion of $G^{(m,n)}_{k_1\ldots k_{m+n}}$ in terms of source-quantity correlation functions of the type given in Eq. (5.10); the coefficients are determined by the c-number functions $\mathcal{F}_k$. We emphasize that Eq. (5.18) is the correct quantum-mechanical formulation of mixing source-field light with a coherent free-field reference beam. Clearly, if the range of observation is outside the reference beam, Eq. (5.18) simply reduces to Eq. (5.10).

VI. PHOTOCOUNT DISTRIBUTION

We now turn to the application of the results derived in Sec. V to the problem of determining the photocount statistics of a quantum light field in the presence of a passive optical system. Following Glauber's theory of photo detection,[35,36] the probability of observing precisely n events in a counting time interval $t,t+\Delta t$ is given by

$$p_n(t,\Delta t)=\left\langle \Omega \frac{1}{n!}[\hat{\Gamma}(t,\Delta t)]^n \exp[-\hat{\Gamma}(t,\Delta t)]\right\rangle , \quad (6.1)$$

where

$$\hat{\Gamma}(t,\Delta t)=\sum_i \int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 S(t_1-t_2) \times \hat{E}_k^{(-)}(\mathbf{r}_i,t_1)\hat{E}_k^{(+)}(\mathbf{r}_i,t_2) \quad (6.2)$$

may be interpreted as the operator of the number of photoelectrons which are produced by the absorption of light in the time interval $t,t+\Delta t$. in Eq. (6.2) the sum runs over the detector atoms, each of which is assumed to give rise to an equal, isotropic response described by the response function $S(t)$. Note that in Eq. (6.2) the rotating-wave approximation is used. In Eq. (6.1) the ordering symbol Ω introduces the following operator ordering: (i) normal ordering of the operators $\hat{E}_k^{(-)}(x),\hat{E}_k^{(+)}(x)$ with the operators $\hat{E}_k^{(-)}(x)$ to the left of the operators $\hat{E}_k^{(+)}(x)$, (ii) T_+ time ordering of the operators $\hat{E}_k^{(+)}(x)$ and T_- time ordering of the operators $\hat{E}_k^{(-)}(x)$.

Expanding the operator exponential in Eq. (6.1) we see that the photocount distribution is just determined by correlation function of the type studied in Sec. V. Identifying in Eq. (4.6) the field operators $\hat{F}$ with the operators of the electric field strength $\hat{E}$ and inserting this equation into Eq. (6.2) yields

$$\hat{\Gamma}(t,\Delta t)=\sum_i \int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 S(t_1-t_2)[\hat{E}_{k,\mathrm{free}}^{(-)}(\mathbf{r}_i,t_1)+\hat{E}_{k,s}^{(-)}(\mathbf{r}_i,t_1)][\hat{E}_{k,\mathrm{free}}^{(+)}(\mathbf{r}_i,t_2)+\hat{E}_{k,s}^{(+)}(\mathbf{r}_i,t_2)] . \quad (6.3)$$

Making use of the results given in Sec. V, we may identifically perform Eq. (6.1) as follows:

$$p_n(t,\Delta t)=\left\langle O \frac{1}{n!}[\hat{\Gamma}(t,\Delta t)]^n \exp[-\hat{\Gamma}(t,\Delta t)]\right\rangle , \quad (6.4)$$

where the Ω ordering is simply replaced by the O ordering defined as follows:

(i) Normal ordering of the operators $\hat{E}_{k,s}^{(-)}(x)$, $\hat{E}_{k,\mathrm{free}}^{(-)}(x)$, $\hat{E}_{k,s}^{(+)}(x)$, $\hat{E}_{k,\mathrm{free}}^{(+)}(x)$ with the operators $\hat{E}_{k,s}^{(-)}(x)$, $\hat{E}_{k,\mathrm{free}}^{(-)}(x)$ to the left of the operators $\hat{E}_{k,s}^{(+)}(x)$, $\hat{E}_{k,\mathrm{free}}^{(+)}(x)$.

(ii) O_+ ordering of the operators $\hat{E}_{k,s}^{(+)}(x)$, $\hat{E}_{k,\mathrm{free}}^{(+)}(x)$ and O_- ordering of the operators $\hat{E}_{k,s}^{(-)}(x)$, $\hat{E}_{k,\mathrm{free}}^{(-)}(x)$.

In particular, when the conditions given in Eq. (5.8) are fulfilled, the photocount statistics may directly be related to the statistics of the source quantities because in this case Eq. (6.3) may be simplified as follows:

$$\hat{\Gamma}(t,\Delta t)=\sum_i \int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 S(t_1-t_2)\hat{E}_{k,s}^{(-)}(\mathbf{r}_i,t_1) \times \hat{E}_{k,s}^{(+)}(\mathbf{r}_i,t_2) . \quad (6.5)$$

Note that the O ordering in Eq. (6.4) requires that, after expanding the operator exponential $\exp[-\hat{\Gamma}(t,\Delta t)]$ into a power series, the source-field operators $\hat{E}_{k,s}^{(-)}(\mathbf{r}_i,t_1)$, $\hat{E}_{k,s}^{(+)}(\mathbf{r}_i,t_2)$ must be expressed, according to Eq. (3.8), in terms of source-quantity operators $\hat{J}_{k'}^{\dagger}(\mathbf{r}_i',t_1),\hat{J}_{k'}(\mathbf{r}_i',t_2)$. The operators $\hat{J}_{k'}^{\dagger}(\mathbf{r}_i',t_1)$, which stand to the left of the operators $\hat{J}_{k'}(\mathbf{r}_i',t_2)$, are T_- time ordered and the operators $\hat{J}_{k'}(\mathbf{r}_i',t_2)$ are T_+ time ordered. Finally, we point out that in the case of mixing the source-field light with a coherent free-field reference beam, as considered at the end of Sec. V, in Eq. (6.3) the free-field operators can simply be replaced by c-number functions, viz.,

$$\hat{\Gamma}(t,\Delta t)=\sum_i \int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 S(t_1-t_2)[\mathcal{E}_k^*(\mathbf{r}_i,t_1)+\hat{E}_{k,s}^{(-)}(\mathbf{r}_i,t_1)][\mathcal{E}_k(\mathbf{r}_i,t_2)+\hat{E}_{k,s}^{(+)}(\mathbf{r}_i,t_2)] , \quad (6.6)$$

provided that Eq. (6.4) is used.

VII. SUMMARY AND CONCLUSIONS

We have presented a quantum-mechanical description of the interaction of sources with light for the very general case when passive, lossless optical devices such as beam splitters, pin holes, Fabry-Perot spectral filters, or other cavitylike filters, gratings, and optical fibers are present. Such kinds of optical instruments may be modeled by a dielectric with space-dependent refractive index and/or a composition of dielectrica with stepwise constant refractive index and appropriately chosen boundary conditions. The particular case when the sources are in the free space comes out by setting the refractive index equal to 1. Starting from a mode expansion, where the mode functions reflect the presence of the optical instruments, we have presented the multimode field operator of a given field quantity of the light in terms of free-field quantities and source-field quantities. The latter are related to source quantities by a linear integral transformation, the information on the optical instruments being involved in the integral kernel.

Apart from the neglect of the term proportional to the square of the vector potential in the interaction Hamiltonian, this result is exact. It should be noted that in most cases of practical interest the term quadratic in the vector potential may be neglected. Insofar as such a term does not appear in a relativistic theory, the conclusions drawn in this paper may be expected to be valid also under more general conditions.

A more serious restriction on validity is the assumption that the optical devices are lossless. In consequence of the neglect of any absorption of light by the material of the optical instruments, the corresponding dispersion of light is also disregarded. That is, the theory in the formulation given in this paper is therefore applicable to cases when within the bandwidth of the light the dispersion caused by certain elements of the optical system is sufficiently small. If the dispersion cannot be omitted, these disperse elements must be described microscopically. In this case the disperse elements give rise to additional source distributions, and the construction of an appropriate apparatus function requires further considerations.

The representation of multimode field operators (as functions of space and time) in terms of free-field and source-field operators renders it possible to study the space-time behavior of the light field, including the effects of light propagation. We therefore need not distinguish between operators describing the field inside the optical instruments and operators for the field outside the instruments, and in the case of frequency-sensitive devices we need not restrict ourselves to small transition bandwidths.

The examination of the commutation relations of the field operators at different space-time points shows that they differ from the corresponding free-field commutation relations in the time-delayed contributions.

The time-delayed contributions, which represent a pure quantum effect, can only be omitted if the two space-time points in the commutators are chosen in such a way that the time difference is smaller than the time required for the light to travel from one of the two space points to the other via radiating sources. When we identify the two space points with points of observation, we see that under standard experimental conditions the distances between a given light source and the points of observation are not large enough to exclude time-delayed effects from the consideration.

The examination of light by means of usual photo detection experiments allows the determination of the photocount distribution, which itself is determined by all orders of a specific type of normally and time-ordered field correlation functions. We have answered the question of how to express correlation functions of this type in terms of free-field and source-quantity correlation functions. We have shown that the time ordering originally concerning the field operators is transferred to the source-quantity operators. In particular, when the free field is the vacuum field the field correlation functions under study can be expressed in terms of time-ordered source-quantity correlation functions solely. Formulas are also presented for the case of mixing a source field with a multimode coherent reference beam. Furthermore, the theory is applied to the calculation of the photocount statistics.

It is worth noting that the normal and time orderings of the field operators in the correlation functions examined in usual photo detection experiments prevent the observation of time-delayed effects because time-delayed terms do not contribute to this type of field correlation function. From the commutation relations time-delayed effects are seen to be typical for fields attributed to sources. Except the vacuum field any real light field is attributed to sources; hence the question is raised, whether a quantum light field can (approximately) be treated as a free field (and what conditions must be fulfilled) or must it be described by taking into account the contact to its sources. Clearly, the answer cannot be given on the basis of usual photo detection experiments. It might be given, for example, from photon detection experiments with Mandel's detectors allowing the determination of antinormally and time-ordered field correlation functions. In correlation functions of this type the time-delayed terms may give rise to nonzero contributions. This problem will be the subject of a forthcoming paper. At this point we only note that the application of Eq. (5.7) to cases when the free field is not the vacuum field but a real quantum field requires a careful consideration. In such a case it might be more promising to attribute the total field under study to its sources and to use Eq. (5.7) in the form of Eq. (5.9).

Finally, it should be pointed out that Eqs. (5.7) and (5.9) may be used for the practical calculation of normally and time-ordered correlation functions of light produced by various kinds of light scattering and observed after having passed through certain optical instruments. In cases when the excitation of the radiating sources may be assumed to be independent of the radiation produced, the calculation is straightforward. If nonlinear optical processes must be taken into account, the formulas can also be used. Since in these cases the equations of motion for the source-quantity correlation functions (or appropriately chosen source-quantity Green's functions) are coupled to equations of motions for field correlation functions (or field Green's functions), a more or less complicated system of equations must be tried to be solved.

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APPENDIX: PROOF OF EQ. (5.5)

Let us consider the time-ordered operator product

$$\hat{P}=\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})\hat{J}_{k_n'}(x_n')\cdots\hat{J}_{k_j'}(x_j')\cdots\hat{J}_{k_1'}(x_1') \tag{A1}$$

with

$$t_{n+1}>t_n'>\cdots>t_j'>\cdots>t_1' \;. \tag{A2}$$

By means of standard commutator algebra we may write Eq. (A1) as follows:

$$\hat{P}=[\hat{F}^{(+)}_{k_{n+1}}(x_{n+1}),\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)]+\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})$$

$$=\sum_{j=1}^{n}\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_{j+1}}(x'_{j+1})[\hat{F}^{(+)}_{k_{n+1}}(x_{n+1}),\hat{J}_{k'_j}(x'_j)]\hat{J}_{k'_{j-1}}(x'_{j-1})\cdots\hat{J}_{k'_1}(x'_1)$$

$$+\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})\ . \qquad \text{(A3)}$$

Expressing the commutators $[\hat{F}^{(+)}_{k_{n+1}}(x_{n+1}),\hat{J}_{k'_j}(x'_j)]$ in Eq. (A3) in terms of source quantities, according to Eqs. (4.6), (4.7), and (4.12) we arrive at

$$\hat{P}=\int dx'_{n+1}K_{k_{n+1}k'_{n+1}}(x_{n+1},x'_{n+1})\left\{\sum_{j=1}^{n}[\Theta(t_{n+1}-t'_{n+1})-\Theta(t'_j-t'_{n+1})]\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_{j+1}}(x'_{j+1})\right.$$

$$\left.\times[\hat{J}_{k'_{n+1}}(x'_{n+1}),\hat{J}_{k'_i}(x'_j)]\hat{J}_{k'_{j-1}}(x'_{j-1})\cdots\hat{J}_{k'_1}(x'_1)\right\}$$

$$+\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})\ . \qquad \text{(A4)}$$

Rearranging the sum in the large parentheses and substituting Eq. (4.6) together with Eq. (4.7) for $\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})$, we obtain

$$\hat{P}=\int dx'_{n+1}K_{k_{n+1}k'_{n+1}}(x_{n+1},x'_{n+1})$$

$$\times\left\{[\Theta(t_{n+1}-t'_{n+1})-\Theta(t'_n-t'_{n+1})]\hat{J}_{k'_{n+1}}(x'_{n+1})\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\right.$$

$$+\sum_{j=2}^{n}[\Theta(t'_j-t'_{n+1})-\Theta(t'_{j-1}-t'_{n+1})]\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\hat{J}_{k'_{n+1}}(x'_{n+1})\hat{J}_{k'_{j-1}}(x'_{j-1})\cdots\hat{J}_{k'_1}(x'_1)$$

$$\left.+\Theta(t'_1-t'_{n+1})\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\hat{J}_{k'_{n+1}}(x'_{n+1})\right\}$$

$$+\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\hat{F}^{(+)}_{k_{n+1},\mathrm{free}}(x_{n+1})\ . \qquad \text{(A5)}$$

Making use of the relation $\Theta(t)+\Theta(-t)=1$ and remembering the conditions (A2), we may write

$$\Theta(t_{n+1}-t'_{n+1})-\Theta(t'_n-t'_{n+1})=\Theta(t_{n+1}-t'_{n+1})\Theta(t'_{n+1}-t'_n)\ , \qquad \text{(A6)}$$

$$\Theta(t'_j-t'_{n+1})-\Theta(t'_{j-1}-t'_{n+1})=\Theta(t_{n+1}-t'_{n+1})\Theta(t'_j-t'_{n+1})\Theta(t'_{n+1}-t'_{j-1})\ , \qquad \text{(A7)}$$

$$\Theta(t'_1-t'_{n+1})=\Theta(t_{n+1}-t'_{n+1})\Theta(t'_1-t'_{n+1})\ . \qquad \text{(A8)}$$

Inserting Eqs. (A6)–(A8) into Eq. (A5) yields

$$\hat{P}=\int dx'_{n+1}K_{k_{n+1}k'_{n+1}}(x_{n+1},x'_{n+1})\Theta(t_{n+1}-t'_{n+1})$$

$$\times\left\{\Theta(t'_{n+1}-t'_n)\hat{J}_{k'_{n+1}}(x'_{n+1})\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\right.$$

$$+\sum_{j=2}^{n}\Theta(t_j-t'_{n+1})\Theta(t'_{n+1}-t'_{j-1})\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\hat{J}_{k'_{n+1}}(x'_{n+1})\hat{J}_{k'_{j-1}}(x'_{j-1})\cdots\hat{J}_{k'_1}(x'_1)$$

$$\left.+\Theta(t'_1-t'_{n+1})\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\hat{J}_{k'_{n+1}}(x'_{n+1})\right\}$$

$$+\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)\hat{F}^{(+)}_{k_{n+1},\mathrm{free}}(x_{n+1})\ . \qquad \text{(A9)}$$

From an inspection of Eq. (A9) it is seen that the expression in the large parentheses is just the T_+ time-ordered product of the operator $\hat{J}_{k'_{n+1}}(x'_{n+1})$ with the T_+ time-ordered operator product $\hat{J}_{k'_n}(x'_n)\cdots\hat{J}_{k'_j}(x'_j)\cdots\hat{J}_{k'_1}(x'_1)$. We therefore arrive at

$$\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})T_+\prod_{j=1}^{n}\hat{J}_{k'_j}(x'_j)=\left[T_+\prod_{j=1}^{n}\hat{J}_{k'_j}(x'_j)\right]\hat{F}^{(+)}_{k_{n+1},\mathrm{free}}(x_{n+1})$$
$$+\int dx'_{n+1}\Theta(t_{n+1}-t'_{n+1})K_{k_{n+1}k'_{n+1}}(x_{n+1},x'_{n+1})T_+\prod_{j=1}^{n+1}\hat{J}_{k'_j}(x'_j)\,,$$
$$t_{n+1}>t'_1,t'_2,\ldots,t'_n\,. \quad \text{(A10)}$$

Multiplying Eq. (A10) with

$$\int dx'_1\cdots\int dx'_j\cdots\int dx'_n\Theta(t_1-t'_1)K_{k_1k'_1}(x_1,x'_1)\cdots\Theta(t_n-t'_n)K_{k_nk'_n}(x_n,x'_n)\,,$$

remembering Eq. (4.7), and introducing the ordering symbol O_+ defined in Sec. V, we obtain the result

$$\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})O_+\prod_{j=1}^{n}\hat{F}^{(+)}_{k_j,s}(x_j)=O_+[\hat{F}^{(+)}_{k_{n+1},\mathrm{free}}(x_{n+1})+\hat{F}^{(+)}_{k_{n+1},s}(x_{n+1})]\times\prod_{j=1}^{n}\hat{F}^{(+)}_{k_j,s}(x_j),\quad t_{n+1}>t_1,t_2,\ldots,t_n\,. \quad \text{(A11)}$$

Since the ordering rule O_+ includes ordering of the free-field operators $\hat{F}^{(+)}_{k,\mathrm{free}}(x)$ to the right of the source-field operators $\hat{F}^{(+)}_{k,s}(x)$, the equation (A11), of course, remains valid when we complement the operators $\hat{F}^{(+)}_{k_j,s}(x_j)$ by the operators $\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)$. We therefore derive

$$\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})O_+\prod_{j=1}^{n}[\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]=O_+\prod_{j=1}^{n+1}[\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\,,\quad t_{n+1}>t_1,t_2,\ldots,t_n\,. \quad \text{(A12)}$$

Now, we assume that Eq. (5.5), which is valid for $n=1$ and which has been shown to be valid for $n=2$ [see Eq. (5.4)], is valid for arbitrary integer n ($n\geq1$), viz.,

$$T_+\prod_{j=1}^{n}\hat{F}^{(+)}_{k_j}(x_j)=O_+\prod_{j=1}^{n}[\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\,. \quad \text{(A13)}$$

Inserting Eq. (A13) into the left-hand side of Eq. (A12) yields

$$\hat{F}^{(+)}_{k_{n+1}}(x_{n+1})T_+\prod_{j=1}^{n}\hat{F}^{(+)}_{k_j}(x_j)=O_+\prod_{j=1}^{n+1}[\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\,,\quad t_{n+1}>t_1,t_2,\ldots,t_n\,. \quad \text{(A14)}$$

At this point we see that the left-hand side of Eq. (A14) is just the T_+ time-ordered product of $n+1$ operators $\hat{F}^{(+)}_{k_j}(x_j)$, so that we arrive at the result

$$T_+\prod_{j=1}^{n+1}\hat{F}^{(+)}_{k_j}(x_j)=O_+\prod_{j=1}^{n+1}[\hat{F}^{(+)}_{k_j,\mathrm{free}}(x_j)+\hat{F}^{(+)}_{k_j,s}(x_j)]\,. \quad \text{(A15)}$$

That is, if Eq. (A13) is valid in the case of the product of n operators $\hat{F}^{(+)}_{k_j}(x_j)$, then it is also valid for $n+1$ operators; hence Eq. (A13) is proved.

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