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Amplitude-squared squeezing of the electromagnetic field

Mark Hillery
Department of Physics and Astronomy, Hunter College of the City University of New York, 695 Park Avenue, New York, New York 10021
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Operators are defined which correspond to the real and imaginary parts of the square of the field amplitude. These operators obey an uncertainty relation. Squeezing with respect to these operators is defined and minimum uncertainty states are discussed. A number of nonlinear optical processes are examined and shown to produce this effect. Interactions which couple one mode amplitude to the square of another convert this type of squeezing into normal squeezing. An example of such an interaction is given.

I. INTRODUCTION

The electric field in an electromagnetic wave can be represented by a complex amplitude which describes both the magnitude and the phase of the field. In the quantum-mechanical description the real and imaginary parts of this amplitude obey an uncertainty relation. If the uncertainties in the real and imaginary parts are equal, then the minimum uncertainty states are coherent states. If the requirement of equal uncertainties is dropped, then there is a whole new class of minimum uncertainty states which are said to be squeezed.[1–3] In general, a state is squeezed when the uncertainty in one of its field quadrature components is less than it would be in a coherent state.

By considering higher-order correlation functions of the field amplitude it is possible to define higher-order squeezing effects. Hong and Mandel defined a state to be squeezed to $2N$th order if the expectation value of the $2N$th power of the difference between a field quadrature component and its average value is less than it would be in a coherent state.[4,5] They found this type of squeezing in a number of nonlinear optical processes. Another approach has been explored by Braunstein and McLachlan.[6] They considered higher-order analogs of the squeeze operator[3] in order to define what they called generalized squeezed states. As is shown by their Q representations, these states have highly unusual noise properties. It is possible to define yet another form of higher-order squeezing in terms of the real and imaginary parts of the square (or higher powers) of the field amplitude.[7] This type of squeezing, amplitude-squared squeezing, arises in a natural way in second-harmonic generation. It should be noted that all of the different kinds of higher-order squeezed states are nonclassical.

In this paper further properties of amplitude-squared squeezing will be examined. In Sec. II minimum uncertainty states for the variables which characterize this kind of squeezing are discussed. Generalizations of the total noise of a state are defined and used to find a lower bound on the square of the number of photons necessary to achieve a given level of amplitude-square squeezing. Sections III and IV are concerned with the role of amplitude-squared squeezed states in nonlinear optics. The production of these states and their use in the production of normal squeezed states are both discussed.

II. DEFINITIONS AND MINIMIUM UNCERTAINTY STATES

Let us consider a single mode of the electromagnetic field with frequency ω and creation and annihilation operators $a^\dagger$ and a. When examining squeezing effects it is more useful to work with the slowly varying operators

$$A = e^{i\omega t}a, \quad A^\dagger = e^{-i\omega t}a^\dagger . \tag{2.1}$$

Standard squeezing is defined in terms of the operators

$$X_1 = (A^\dagger + A)/2, \quad X_2 = i(A^\dagger - A)/2 , \tag{2.2}$$

which correspond to the real and imaginary parts, respectively, of the mode amplitude. In a similar fashion one can define operators which represent the real and imaginary parts of the square of the amplitude[7]

$$Y_1 = (A^{\dagger 2} + A^2)/2, \quad Y_2 = i(A^{\dagger 2} - A^2)/2 . \tag{2.3}$$

These operators obey the commutation relation

$$[Y_1, Y_2] = i(2N+1) \tag{2.4}$$

and, as a result, satisfy the uncertainty relation

$$\Delta Y_1 \Delta Y_2 \geq \langle N + \tfrac{1}{2} \rangle , \tag{2.5}$$

where $N = A^\dagger A$. A state is squeezed in the Y_1 variable if

$$(\Delta Y_1)^2 < \langle N + \tfrac{1}{2} \rangle \tag{2.6}$$

and similarly for Y_2. As was mentioned in the Introduction, such states are nonclassical.[7]

A class of states which are minimum uncertainty states in the Y_1 and Y_2 variables are the even and odd coherent states.[8] These states play the same role for Y_1 and Y_2 that the coherent states do for X_1 and X_2. Even and odd coherent states are defined by

$$|\alpha\rangle_e=(\cosh|\alpha|^2)^{-1/2}\sum_{n=0}^{\infty}[\alpha^{2n}/\sqrt{(2n)!}]\,|2n\rangle\ ,$$
$$|\alpha\rangle_o=(\sinh|\alpha|^2)^{-1/2}\sum_{n=0}^{\infty}[\alpha^{2n+1}/\sqrt{(2n+1)!}]\,|2n+1\rangle \tag{2.7}$$

These states, and any linear combination of them, are eigenstates of the operator a^2. The expectation values and variances of Y_1 and Y_2 (at $t=0$) in the above states are

$$_e\langle\alpha|Y_1|\alpha\rangle_e=(\alpha^{*2}+\alpha^2)/2\ ,$$
$$_o\langle\alpha|Y_1|\alpha\rangle_o=(\alpha^{*2}+\alpha^2)/2\ ,$$
$$_e\langle\alpha|Y_2|\alpha\rangle_e=i(\alpha^{*2}-\alpha^2)/2\ ,$$
$$_o\langle\alpha|Y_2|\alpha\rangle_o=i(\alpha^{*2}-\alpha^2)/2\ ,$$
$$(\Delta Y_1)_e^2={}_e\langle\alpha|N+\tfrac{1}{2}|\alpha\rangle_e=|\alpha|^2\tanh(|\alpha|^2)+\tfrac{1}{2}\ ,$$
$$(\Delta Y_1)_o^2={}_o\langle\alpha|N+\tfrac{1}{2}|\alpha\rangle_o=|\alpha|^2\coth(|\alpha|^2)+\tfrac{1}{2}\ , \tag{2.8}$$
$$(\Delta Y_2)_e^2={}_e\langle\alpha|N+\tfrac{1}{2}|\alpha\rangle_e=|\alpha|^2\tanh(|\alpha|^2)+\tfrac{1}{2}\ ,$$
$$(\Delta Y_2)_o^2={}_o\langle\alpha|N+\tfrac{1}{2}|\alpha\rangle_o=|\alpha|^2\coth(|\alpha|^2)+\tfrac{1}{2}\ .$$

From Eq. (2.8) it can be seen that for both $|\alpha\rangle_e$ and $|\alpha\rangle_0$ Eq. (2.5) is satisfied with the inequality replaced by an equality, and that $(\Delta Y_1)^2=(\Delta Y_2)^2$. In fact, any linear combination of $|\alpha\rangle_e$ and $|\alpha\rangle_o$ has these properties as well.

The even and odd coherent states also minimize a generalized form of the total noise. The total noise of a state, as defined by Schumaker,[9] is

$$T_1=(\Delta X_1)^2+(\Delta X_2)^2\ . \tag{2.9}$$

This quantity is a minimum for coherent states. In an analogous fashion one can define

$$T_2=(\Delta Y_1)^2+(\Delta Y_2)^2\ . \tag{2.10}$$

In terms of creation and annihilation operators this becomes

$$T_2=\langle(A^{\dagger 2}-\langle A^{\dagger 2}\rangle)(A^2-\langle A^2\rangle)\rangle+\langle(2N+1)\rangle$$
$$=\langle(a^{\dagger 2}-\langle a^{\dagger 2}\rangle)(a^2-\langle a^2\rangle)\rangle+\langle(2N+1)\rangle\ . \tag{2.11}$$

For a fixed value of $\langle N\rangle$ the states which minimize T_2 will be those for which the first term in Eq. (2.11) is a minimum. Examination of this term shows that

$$\langle\Psi|(a^{\dagger 2}-\langle a^{\dagger 2}\rangle)(a^2-\langle a^2\rangle)|\Psi\rangle=\|(a^2-\langle a^2\rangle)\Psi\|^2\geq 0 \tag{2.12}$$

so that the minimum occurs when

$$(a^2-\langle a^2\rangle)\Psi=0\ . \tag{2.13}$$

The most general solution to this equation is

$$\Psi=c_1|\alpha\rangle_e+c_2|\alpha\rangle_o\ , \tag{2.14}$$

where $|c_1|^2+|c_2|^2=1$, and c_1 and α are chosen to give the specified value of $\langle N\rangle$. Therefore, for a fixed value of $\langle N\rangle$ the value of T_2 is a minimum for a linear combination of even and odd coherent states.

As was shown in a previous publication,[7] the type of squeezing defined in Eq. (2.6) arises naturally in second-harmonic generation. This form of squeezing, however, has the disadvantage that it depends upon the amplitude of the state, i.e., if $|\Psi\rangle$ is squeezed in the Y_1 variable this does not mean that $D(\alpha)|\Psi\rangle$ will be squeezed in Y_1. Here $D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a)$ is the coherent state displacement operator. It is sometimes useful to modify the definition of amplitude-squared squeezing to overcome this difficulty. In order to do so, we define the operators

$$Z_1=[(A^\dagger-\langle A^\dagger\rangle)^2+(A-\langle A\rangle)^2]/2\ ,$$
$$Z_2=i[(A^\dagger-\langle A^\dagger\rangle)^2-(A-\langle A\rangle)^2]/2\ , \tag{2.15}$$

which measure the square of the fluctuations of A about its mean value. The commutator of Z_1 and Z_2 is

$$[Z_1,Z_2]=i[2N+1-2(A^\dagger\langle A\rangle+\langle A^\dagger\rangle A)+2\langle A^\dagger\rangle\langle A\rangle]\ , \tag{2.16}$$

which yields the uncertainty relation

$$(\Delta Z_1)(\Delta Z_2)\geq\langle N\rangle-\langle A^\dagger\rangle\langle A\rangle+\tfrac{1}{2}=(\Delta X_1)^2+(\Delta X_2)^2\ . \tag{2.17}$$

A state is squeezed in the Z_1 variable if

$$(\Delta Z_1)^2<\langle N\rangle-\langle A^\dagger\rangle\langle A\rangle+\tfrac{1}{2} \tag{2.18}$$

and similarly for Z_2.

The uncertainty relation, Eq. (2.17), is minimized by even and odd coherent states but, unlike Eq. (2.5), not by arbitrary linear combinations of them. In this case, however, states of the form $D(\beta)|\alpha\rangle_e$ and $D(\beta)|\alpha\rangle_o$, for any α and β, are minimum uncertainty states. This is because in going from Y to Z the operators A and $A^\dagger$ were replaced by $A-\langle A\rangle$ and $A^\dagger-\langle A^\dagger\rangle$, respectively. Note that because $|0\rangle_e=|0\rangle$ the normal coherent states are minimum uncertainty states of Z_1 and Z_2.

It is also possible to define a higher-order total noise in terms of the Z variables

$$T_2'=(\Delta Z_1)^2+(\Delta Z_2)^2\ . \tag{2.19}$$

This expression attains its minimum value only for coherent states. In order to see this it is useful to express T_2' as

$$T_2'=\langle[(A^\dagger-\langle A^\dagger\rangle)^2-\langle(A^\dagger-\langle A^\dagger\rangle)^2\rangle]\times[(A-\langle A\rangle)^2-\langle(A-\langle A\rangle)^2\rangle]\rangle+2(\langle N\rangle-\langle A^\dagger\rangle\langle A\rangle)+1\ . \tag{2.20}$$

The first two terms on the right-hand side of this equation are greater than or equal to zero so that $T_2'\geq 1$.

For coherent states T_2' is equal to 1. Coherent states are the only states for which this is true. This follows from the inequality

$$T_2' \geq 2(\langle N \rangle - \langle A^\dagger \rangle \langle A \rangle) + 1 , \quad (2.21)$$

and the fact that $\langle N \rangle - \langle A^\dagger \rangle \langle A \rangle > 0$ for all states other than coherent states.

It should also be mentioned that a state which is squeezed in the Z variables is nonclassical. For Z_1 this follows from

$$(\Delta Z_1)^2 = [(\Delta X_1)^2 + (\Delta X_2)^2] + \tfrac{1}{4}\int d^2\alpha\, P(\alpha)\{e^{-2i\omega t}[(\alpha^* - \langle a^\dagger \rangle)^2 - \langle (a^\dagger - \langle a^\dagger \rangle)^2 \rangle] + e^{2i\omega t}[(\alpha - \langle a \rangle)^2 - \langle (a - \langle a \rangle)^2 \rangle]\}^2 . \quad (2.22)$$

For a classical state $P(\alpha)$ is non-negative definite so that $(\Delta Z_1)^2 \geq [(\Delta X_1)^2 + (\Delta X_2)^2]$ which implies that a state which satisfies Eq. (2.18) must be nonclassical. The argument for Z_2 is similar.

Finally, let us discuss the number of photons which is necessary to attain a given squeezing effect.[10] For normal squeezing we have

$$(\Delta X_1)^2 + (\Delta X_2)^2 = \langle N \rangle - \langle A^\dagger \rangle \langle A \rangle + \tfrac{1}{2} , \quad (2.23)$$

which implies that

$$\langle N \rangle + \tfrac{1}{2} \geq (\Delta X_1)^2 + (\Delta X_2)^2 . \quad (2.24)$$

Let us consider squeezing in the X_1 direction. It is possible to eliminate ΔX_2 from the above equation by using the uncertainty relation $\Delta X_1 \Delta X_2 \geq \frac{1}{4}$. The result is

$$\langle N \rangle + \tfrac{1}{2} \geq (\Delta X_1)^2 + 1/(4\Delta X_1)^2 . \quad (2.25)$$

The right-hand side is a minimum for $\Delta X_1 = \frac{1}{2}$ and increases as ΔX_1 decreases. Equation (2.25) provides a lower bound on the number of photons necessary to achieve a given squeezing effect.

For amplitude-squared squeezing one does not get a restriction on $\langle N + \frac{1}{2} \rangle$ but on $\langle (N + \frac{1}{2})^2 \rangle$. In order to see this note that (working with the Y variables)

$$(\Delta Y_1)^2 + (\Delta Y_2)^2 = \langle N^2 + N + 1 \rangle - \langle A^{\dagger 2} \rangle \langle A^2 \rangle . \quad (2.26)$$

Considering amplitude-squared squeezing in the Y_1 direction we find

$$\langle (N + \tfrac{1}{2})^2 \rangle + \tfrac{3}{4} \geq (\Delta Y_1)^2 + (\Delta Y_2)^2 \geq (\Delta Y_1)^2 + [\langle N + \tfrac{1}{2} \rangle / (\Delta Y_1)]^2 , \quad (2.27)$$

where we have used Eq. (2.5). The right-hand side is a minimum when $(\Delta Y_1)^2 = \langle N + \frac{1}{2} \rangle$ and increases as $(\Delta Y_1)^2$ becomes smaller. This, in turn, means that $\langle (N + \frac{1}{2})^2 \rangle$ must increase. Therefore, as the amount of amplitude-squared squeezing increases, the square of the number of photons must grow. There does not seem to be a simple relation of this form for the Z variables.

III. LOWER-ORDER PROCESSES

It has previously been shown that amplitude-squared squeezing can arise in second-harmonic generation. In this section two other nonlinear processes will be discussed which lead to this kind of squeezing. Degenerate parametric amplification, which produces squeezed states [1,2], also produces amplitude-squared squeezed states. Two-photon absorption can also produce states of this type.

Let us first consider degenerate parametric amplification. In this process a pump wave at frequency 2ω gives rise to a signal at ω through an interaction mediated by a nonlinear medium.[11] If the pump wave is strong, we can represent it as a c number. In this approximation the Hamiltonian for the system is

$$H = \omega a^\dagger a + i\kappa_2(\beta e^{-2i\omega t} a^{\dagger 2} - \beta^* e^{2i\omega t} a^2) , \quad (3.1)$$

where κ_2 is the coupling constant, $\beta = |\beta| e^{i\theta}$ is the amplitude of the pump field, and $a^\dagger$ and a are the creation and annihilation operators for the signal mode. Under the action of this Hamiltonian the evolution of the slowlying varying operators of the signal mode is given by

$$A(t) = A(0)\cosh(2|\beta|\kappa_2 t) + A^\dagger(0)e^{i\theta}\sinh(2|\beta|\kappa_2 t) ,$$
$$A^\dagger(t) = A^\dagger(0)\cosh(2|\beta|\kappa_2 t) + A(0)e^{-i\theta}\sinh(2|\beta|\kappa_2 t) . \quad (3.2)$$

These results allow one to calculate the variances which are needed to examine amplitude-squared squeezing.

Let us first consider what happens to ΔY_1. If the system is initially in a coherent state with amplitude α, then at time t we find that

$$[\Delta Y_1(t)]^2 = \cos(2\theta)\cosh^2 r \sinh^2 r + (\cosh^4 r + \sinh^4 r)/2 + [(\alpha \cosh r + \alpha^* e^{i\theta}\sinh r)^2 e^{i\theta} + (\alpha^* \cosh r + \alpha e^{-i\theta}\sinh r)^2 e^{-i\theta}]\cosh r \sinh r + |\alpha \cosh r + \alpha^* e^{i\theta}\sinh r|^2(\cosh^2 r + \sinh^2 r) , \quad (3.3)$$

where $r = 2|\beta|\kappa_2 t$. In order to determine whether there is any amplitude-squared squeezing, it is necessary to subtract $\langle N(t)+\frac{1}{2}\rangle$ from the above expression:

$$[\Delta Y_1(t)]^2-\langle N(t)+\tfrac{1}{2}\rangle=[1+\cos(2\theta)]\cosh^2 r\,\sinh^2 r-\sinh^2 r$$
$$+[(\alpha\cosh r+\alpha^* e^{i\theta}\sinh r)^2 e^{i\theta}+(\alpha^*\cosh r+\alpha e^{-i\theta}\sinh r)^2 e^{-i\theta}]\cosh r\,\sinh r+2\,|\,\alpha\cosh r+\alpha^* e^{i\theta}\sinh r\,|^2\sinh^2 r. \quad (3.4)$$

With the proper choice of parameters this expression can become negative. For example, if $\alpha=0$ and $\theta=\pi/2$, then Eq. (3.4) becomes

$$[\Delta Y_1(t)]^2-\langle N(t)+\tfrac{1}{2}\rangle=-\sinh^2 r\ . \quad (3.5)$$

From this one can conclude that the vacuum is amplitude-squared squeezed by a degenerate parametric amplifier. The optimal amount of amplitude-squared squeezing is obtained if $\theta=\pi/2$ and $\alpha=|\,\alpha\,|\,e^{i\pi/4}$. In this case one has

$$[\Delta Y_1(t)]^2-\langle N(t)+\tfrac{1}{2}\rangle=-(2\,|\,\alpha\,|^2 e^r+\sinh r)\sinh r\ , \quad (3.6)$$

which is an increase over the vacuum value by a factor of $4\,|\,\alpha\,|^2+1$ (when r is large).

The situation is somewhat simpler if we look at Z_1. Again assuming that the initial state is a coherent state with amplitude α, we find for $(\Delta Z_1)^2$

$$[\Delta Z_1(t)]^2=[1+\cos(2\theta)]\cosh^2 r\,\sinh^2 r+\tfrac{1}{2}\ . \quad (3.7)$$

From this it is necessary to subtract

$$[\Delta X_1(t)]^2+[\Delta X_2(t)]^2=\sinh^2 r+\tfrac{1}{2} \quad (3.8)$$

in order to see if Z_1 is squeezed. The result is

$$[\Delta Z_1(t)]^2-\{[\Delta X_1(t)]^2+[\Delta X_2(t)]^2\}$$
$$=[1+\cos(2\theta)]\cosh^2 r\,\sinh^2 r-\sinh^2 r\ . \quad (3.9)$$

Note that this result is independent of the amplitude of the initial coherent state. If $\cos(2\theta)<0$, i.e., $\pi/4<\theta<3\pi/4$ or $5\pi/4<\theta<7\pi/4$, then Z_1 will be squeezed, at least initially. However, unless θ equals $\pi/2$ or $3\pi/2$ the quantity on the right-hand side of Eq. (3.9) will become positive for sufficiently large times. If θ equals $\pi/2$ or $3\pi/2$, then the squeezing of Z_1 will continue indefinitely.

Two-photon absorption can also produce amplitude-squared squeezing. Consider a collection of M two-level atoms with a two-photon transition between the upper and lower levels. The energy difference of the levels is 2ω and ω is the angular frequency of the field mode which interacts with them. The phenomenological Hamiltonian which describes this system is

$$H'=\omega a^\dagger a+\sum_{n=1}^{M}\omega\sigma_n^{(3)}+\sum_{n=1}^{M}g(e^{-i\mathbf{k}\cdot\mathbf{x}_n}a^{\dagger 2}\sigma_n^{(-)}+e^{i\mathbf{k}\cdot\mathbf{x}_n}a^2\sigma_n^{(+)})\ , \quad (3.10)$$

where $\mathbf{k}$ is the wave vector of the field mode, $\mathbf{x}_n$ is the position of the nth atom, and g is the coupling constant. The operators $\sigma_n^{(+)}$ and $\sigma_n^{(-)}$ are the raising and lowering operators for the nth atom, and if $|\,a_n\rangle$ and $|\,b_n\rangle$ are the upper and lower states, respectively, of the nth atom, then the operator $\sigma_n^{(3)}$ is given by

$$\sigma_n^{(3)}=|\,a_n\rangle\langle a_n\,|-|\,b_n\rangle\langle b_n\,|\ . \quad (3.11)$$

As is discussed in Ref. 12, it is possible to eliminate the phase factors in the interaction term by means of a unitary transformation. The Hamiltonian can be further simplified if we define the total spin operators[13]

$$S^{(3)}=\tfrac{1}{2}\sum_{n=1}^{M}\sigma_n^{(3)},\quad S^{(+)}=\sum_{n=1}^{M}\sigma_n^{(+)},\quad S^{(-)}=S^{(+)\dagger}\ . \quad (3.12)$$

As a result we can use the Hamiltonian

$$H=\omega a^\dagger a+2\omega S^{(3)}+g(a^{\dagger 2}S^{(-)}+a^2 S^{(+)}) \quad (3.13)$$

to describe the system. The total spin operator

$$\mathbf{S}^2=(S^{(3)})^2+\tfrac{1}{2}(S^{(+)}S^{(-)}+S^{(-)}S^{(+)}) \quad (3.14)$$

commutes with the Hamiltonian so that the total spin is conserved. We will be interested in the situation in which all of the atoms are initially in their ground states. This means that we need to consider the subspace of the total atomic Hilbert space on which $\mathbf{S}^2$ has the eigenvalue $(M/2)[(M/2)+1]$. The operators $S^{(3)}$, $S^{(+)}$, and $S^{(-)}$ can, then, be taken to be spin operators for a spin-$M/2$ particle.

Suppose, now, that the intensity of the light is such that the number of excited atoms is small compared to M. This allows us to make use of an approximation for the spin operators based upon the Holstein-Primakoff representation.[14] This is a representation for spin operators in terms of boson creation and annihilation operators, $\xi^\dagger$ and ξ, and is explicitly given by

$$S^{(+)}=\xi^\dagger(2s-\xi^\dagger\xi)^{1/2}\ ,$$
$$S^{(-)}=(2s-\xi^\dagger\xi)^{1/2}\xi\ , \quad (3.15)$$
$$S^{(3)}=-s+\xi^\dagger\xi\ ,$$

where s is the total spin. When the excitation number is small, the square roots can be expanded and only the first terms retained. This yields for $S^{(+)}$ and $S^{(-)}$ the approximations

$$S^{(+)}\cong\sqrt{2s}\,\xi^\dagger,\quad S^{(-)}\cong\sqrt{2s}\,\xi\ . \quad (3.16)$$

Substitution of these expressions into the Hamiltonian gives

$$H=\omega a^\dagger a+2\omega\xi^\dagger\xi-\omega M+\lambda(a^{\dagger 2}\xi+a^2\xi^\dagger)\ , \quad (3.17)$$

where $\lambda=\sqrt{M}g$ and s has been set equal to $M/2$.

This Hamiltonian is, up to an additive constant, formally identical to the Hamiltonian which describes second-harmonic generation. This means that we can take over directly the results for amplitude-squared squeezing in second-harmonic generation which were derived in Ref. 7. In particular, one finds for an initial state in which the atoms are all in their ground states and the field is initially in a coherent state with amplitude $\alpha = |\alpha| e^{i\theta}$ that to second order in λt

$$[\Delta Y_1(t)]^2 - \langle N(t) + \tfrac{1}{2} \rangle = -2(\lambda t)^2 |\alpha|^4 \cos(4\theta) . \quad (3.18)$$

When $\cos(4\theta) > 0$, amplitude-squared squeezing is present in the field.

The results of this section show that amplitude-squared squeezing is a relatively common feature of two-photon processes. In Sec. IV it will be shown that it occurs in four-photon processes as well.

IV. HIGHER-ORDER PROCESSES

In this section we will examine two different processes. The first, a form of four-wave mixing, provides a way of converting amplitude-squared squeezing into normal squeezing. As such it provides a method of detecting amplitude-squared squeezing. The second process, fourth-subharmonic generation, is one in which one pump photon produces four signal photons. In the parametric approximation this process produces what have been called by Braunstein and McLachlan generalized squeezed states.[6] Here it will be shown that the states which are produced from the vacuum are not squeezed in the normal sense but are squeezed in the amplitude-squared sense.

Let us first examine the four-wave interaction described by the following Hamiltonian:

$$H = \omega_A a^\dagger a + \omega_B b^\dagger b + \omega_C c^\dagger c + \kappa_3 (a^2 b^\dagger c^\dagger + a^{\dagger 2} b c) , \quad (4.1)$$

where $2\omega_A = \omega_B + \omega_C$. We will consider the a mode to be the input, the b mode the output, and the c mode, which will be in a large-amplitude coherent state, will act as a pump.

This system will be solved using perturbation theory. It is first convenient to define the slowly varying operators

$$\begin{aligned} A(t) &= e^{i\omega_A t} a(t) , \\ B(t) &= e^{i\omega_B t} b(t) , \\ C(t) &= e^{i\omega_C t} c(t) . \end{aligned} \quad (4.2)$$

The equations of motion for these operators are

$$\begin{aligned} dA/dt &= -2i\kappa_3 A^\dagger B C , \\ dC/dt &= -i\kappa_3 B^\dagger A^2 , \\ dB/dt &= -i\kappa_3 A^2 C^\dagger . \end{aligned} \quad (4.3)$$

To first order in the coupling one finds

$$A(t) = A - 2i(\kappa_3 t) A^\dagger B C, \quad C(t) = C - i(\kappa_3 t) B^\dagger A^2 , \quad (4.4)$$

and to second order

$$\begin{aligned} B(t) = B &- i(\kappa_3 t) A^2 C^\dagger \\ &- [(\kappa_3 t)^2/2] B (4 N_A N_C + 2 N_C - A^{\dagger 2} A^2) , \end{aligned} \quad (4.5)$$

where operators without arguments are assumed to be evaluated at $t=0$, and $N_A = A^\dagger A$ and $N_C = C^\dagger C$.

The general quadrature component of the b mode is defined to be

$$X_{B\varphi}(t) = [e^{i\varphi} B^\dagger(t) + e^{-i\varphi} B(t)]/2 . \quad (4.6)$$

We want to calculate its variance. Using the expressions in the preceding paragraph and assuming the modes to be uncorrelated at $t=0$ gives

$$\begin{aligned} [\Delta X_{B\varphi}(t)]^2 = {} & [\Delta X_{B\varphi}(0)]^2 - (\kappa_3 t)^2 [\Delta X_{B\varphi}(0)]^2 \langle 4 N_A N_C + 2 N_C - A^{\dagger 2} A^2 \rangle \\ & - [(\kappa_3 t)^2/4] [\langle (e^{i\varphi} A^{\dagger 2} C - e^{-i\varphi} A^2 C^\dagger)^2 \rangle - \langle (e^{i\varphi} A^{\dagger 2} C - e^{-i\varphi} A^2 C^\dagger) \rangle^2] . \end{aligned} \quad (4.7)$$

If the b mode is initially in a coherent state with amplitude β and the c mode is in a coherent state with amplitude $\xi = |\xi| e^{i\theta}C$, then

$$\begin{aligned} [\Delta X_{B\varphi}(t)]^2 = {} & \tfrac{1}{4} + [(\kappa_3 t)^2/2] \langle A^{\dagger 2} A^2 \rangle \\ & + (\kappa_3 t)^2 |\xi|^2 \{ [\Delta Y(\varphi + \theta_C + \pi/2)]^2 \\ & - \langle N_A + \tfrac{1}{2} \rangle \} , \end{aligned} \quad (4.8)$$

where

$$Y(\theta) = (e^{i\theta} A^{\dagger 2} + e^{-i\theta} A^2)/2 \quad (4.9)$$

is the general quadrature component of the square of the amplitude of the a mode.

An examination of Eq. (4.8) shows that if $|\xi|$ is sufficiently large, then amplitude-squared squeezing in the a mode will be converted into normal squeezing in the b mode. This is similar to the situation in second-harmonic generation. In that case amplitude-squared squeezing of the fundamental results in normal squeezing of the harmonic. Note that in this process, unlike in second-harmonic generation, the amount of squeezing in the b mode can be controlled. The larger the amplitude of the wave in the c mode, the larger is the squeezing in the b mode.

We now turn to a process which produces states which are amplitude-squared squeezed but not squeezed in the normal sense. In fourth-subharmonic generation a single pump photon at frequency 4ω is converted into four signal photons at frequency ω. The Hamiltonian which describes this is

$$H = 4\omega b^\dagger b + \omega a^\dagger a + \kappa_4 (b a^{\dagger 4} + b^\dagger a^4) , \quad (4.10)$$

where $b^\dagger$ and b are the creation and annihilation operators for the pump, and $a^\dagger$ and a are those for the signal mode. Again we define the slowly varying operators

$$A(t)=e^{i\omega t}a(t), \quad B(t)=e^{4i\omega t}b(t) , \tag{4.11}$$

which obey the equations of motion

$$dA/dt=-4i\kappa_4 BA^{\dagger 3}, \quad dB/dt=-i\kappa_4 A^4 . \tag{4.12}$$

Solving these equations to first order, one finds for the general quadrature component of the square of the amplitude

$$\begin{aligned}Y_\varphi(t)&=[e^{i\varphi}A^{\dagger 2}(t)+e^{-i\varphi}A^2(t)]/2\\&=Y_\varphi+2i\kappa_4 t[e^{i\varphi}B^\dagger(A^3A^\dagger+A^\dagger A^3)\\&\quad -e^{-i\varphi}B(AA^{\dagger 3}+A^{\dagger 3}A)] .\end{aligned} \tag{4.13}$$

Let us now suppose that the initial state of the system is the product of a coherent state in the a mode of amplitude α and a coherent state in the b mode of amplitude $\beta=|\beta|e^{i\theta_B}$. Using Eq. (4.13) and its square, we find that

$$[\Delta Y_\varphi(t)]^2=|\alpha|^2+\tfrac{1}{2}-4|\beta|(\kappa_4 t)[\sin(2\varphi-\theta_B)(6|\alpha|^4+12|\alpha|^2+3)-i(e^{-i\theta_B}\alpha^4-e^{i\theta_B}\alpha^{*4})] . \tag{4.14}$$

The number of photons in the signal mode is

$$\langle N_A(t)\rangle=|\alpha|^2+4i|\beta|(\kappa_4 t)(e^{-i\theta_B}\alpha^4-e^{i\theta_B}\alpha^{*4}) , \tag{4.15}$$

which in combination with Eq. (4.14) gives

$$\begin{aligned}&[\Delta Y_\varphi(t)]^2-\langle N_A(t)+\tfrac{1}{2}\rangle\\&\quad=-4|\beta|(\kappa_4 t)\sin(2\varphi-\theta_B)(6|\alpha|^4+12|\alpha|^2+3) .\end{aligned} \tag{4.16}$$

From this equation it is clear that if $\sin(2\varphi-\theta_B)>0$, then there is amplitude-squared squeezing in the φ direction. The existence of this squeezing is independent of the initial amplitude of the signal mode though its magnitude is not. In particular, the signal mode vacuum will be amplitude-squared squeezed by this process. The resulting state is, however, not squeezed in the normal sense. This is demonstrated in the Appendix where it is shown to be true to all orders and not just to first order. This shows that normal squeezing and amplitude-squared squeezing are independent effects.

V. CONCLUSION

Amplitude-squared squeezing is a nonclassical effect which occurs in a number of nonlinear optical processes. It is present in such well-studied situations as second-harmonic generation, degenerate parametric amplification, and two-photon absorption, as well as in higher-order processes. Amplitude-squared squeezing can be converted into normal squeezing by interactions in which the square of the field amplitude of one mode is coupled to the amplitude of a second mode. Second-harmonic generation and certain kinds of four-wave mixing are examples of such interactions. This means that amplitude-squared squeezed states can be of use in obtaining noise reduction in the output of certain nonlinear optical devices.

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APPENDIX

Here it will be shown that in fourth-subharmonic generation the state which results from the signal mode vacuum is not squeezed. This has already been shown to be true in the parametric approximation where the pump mode is replaced by a classical field.[15] Here we will consider a quantized pump, but the argument will be quite similar to that in Ref. 15. It is included here for the sake of completeness. Finally, note that in contrast to the body of the paper the Schrödinger picture will be used throughout this appendix.

The proof is based upon the fact that the operator

$$M=4b^\dagger b+a^\dagger a \tag{A1}$$

commutes with the Hamiltonian given in Eq. (4.10). This means that the operator

$$V(\theta)=e^{i\theta M}=e^{i\theta a^\dagger a}e^{4i\theta b^\dagger b} \tag{A2}$$

commutes with the time development transformation $U(t)=e^{-itH}$. The operator $V(\theta)$ has the following properties:

$$V(\theta)aV^{-1}(\theta)=e^{-i\theta}a, \quad V(\theta)bV^{-1}(\theta)=e^{-4i\theta}b . \tag{A3}$$

Now let the initial state of the system be $|\Psi\rangle$. We then have that

$$\begin{aligned}&\langle\Psi|U^{-1}(t)V(\theta)A^n(t)V^{-1}(\theta)U(t)|\Psi\rangle\\&\quad=e^{-in\theta}\langle\Psi|U^{-1}(t)A^n(t)U(t)|\Psi\rangle\\&\quad=\langle\Psi|V(\theta)U^{-1}(t)A^n(t)U(t)V^{-1}(\theta)|\Psi\rangle ,\end{aligned} \tag{A4}$$

where, again, $A(t)=e^{i\omega t}a$. Let us consider the case in which the initial state is of the form $|\Psi\rangle=|0\rangle_A\otimes|\Psi\rangle_B$, i.e., the product of the vacuum state in the a mode and a general state in the b mode. The operator $V(\pi/2)$ takes such a state into itself. This is true because this state can be expanded as

$$|\Psi\rangle=\sum c_n|0\rangle_A\otimes|n\rangle_B , \tag{A5}$$

where $|n\rangle_B$ is a b mode number state and $V(\pi/2)|0\rangle_A \otimes |n\rangle_B = |0\rangle_A \otimes |n\rangle_B$. If we combine this result with Eq. (A4), we get that

$$(e^{-in\pi/2}-1)\langle \Psi | U^{-1}(t)A^n(t)U(t)|\Psi\rangle = 0 \ , \qquad \text{(A6)}$$

which implies

$$\langle \Psi | U^{-1}(t)A^n(t)U(t)|\Psi\rangle = 0 \qquad \text{(A7)}$$

for n not a multiple of 4, and in particular for $n=1$ or 2.

The properties of the matrix elements of $A^n(t)$ which are summarized in Eq. (A7) can be used to calculate $(\Delta X_{1A})^2$ and $(\Delta X_{2A})^2$, i.e., the fluctuations in the quadrature components of the mode. As a result of the property expressed in Eq. (A7), one finds that

$$(\Delta X_{1A})^2 = (\Delta X_{2A})^2 = \tfrac{1}{4}\langle \Psi | U^{-1}(t)(2a^\dagger a + 1)U(t)|\Psi\rangle \ . \qquad \text{(A8)}$$

Because $\Delta X_{1A}\Delta X_{2A} \geq \frac{1}{4}$, Eq. (A8) implies that both ΔX_{1A} and ΔX_{2A} are greater than or equal to $\frac{1}{2}$ so that neither is squeezed. These arguments apply to squeezing in any direction so that, in fact, the state which evolves from $|\Psi\rangle$ is not squeezed.

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