Delay-induced instability in a pendular Fabry-Perot cavity

J. M. Aguirregabiria
Física Teórica, Facultad de Ciencias, Universidad del País Vasco, Apartado Postal 644, 48080 Bilbao, Spain

L. Bel
Institut H. Poincaré, Laboratoire de Physique Théorique, 11 rue Pierre et Marie Curie, 75231 Paris, France
(Received 4 May 1987)

It is shown that if time delays are not neglected, a pendular Fabry-Perot cavity is no longer stable in some cases of actual interest, including some configurations of each arm of the interferometers designed to detect gravitational radiation.

Over the past several years considerable attention has been devoted to nonlinear effects in optics, including bistability and chaos.[1] In particular, the multistability of Fabry-Perot cavities has been studied, both experimentally and theoretically.[2,3] These devices might have practical interest in different applications[2] and their use is under consideration for constructing interferometers to detect gravitational radiation.[4]

One of us (L.B.) has shown[5] by numerical integration that the equilibrium points of a very small cavity which appear to be stable if analyzed in the adiabatic approximation, are actually unstable in certain cases if the time delays are considered. In the following we shall analyze in what cases this instability appears due to the hereditary nature of the cavity governing equations.

Let us consider the pendular Fabry-Perot cavity of Fig. 1. We shall assume that the mirror M_1 has no losses and that the reflection and transmission coefficients are given by $R=\cos\theta e^{-i\mu}$ and $T=i\sin\theta\, e^{-i\sigma}$. Similarly, we shall suppose that on mirror M_2 the reflection is metallic and without losses. Let us assume that the incoming light field at A is $\phi_A(t)=\sqrt{P}\exp[-i(2\pi/\lambda)(ct+\alpha)$, P being the laser power. In these conditions, the equations of motion for the mirror M_2 are given by the differential equation for the x position,

$$\ddot{x}+\frac{\Omega}{Q}\dot{x}+\Omega^2 x=\frac{2\,|\,\phi\,|^2}{Mc}\ , \tag{1}$$

and the functional equation for the light field ϕ at the mirror,

$$\phi(t)=T\phi_A\left[t-\frac{D_s+x}{c}\right]-R\phi(\hat{t})\ . \tag{2}$$

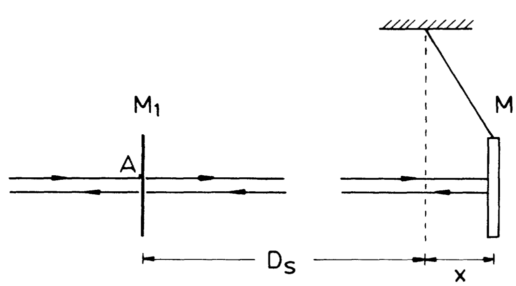

FIG. 1. Pendular Fabry-Perot cavity.

In Eq. (1) Q and Ω are, respectively, the quality factor and the proper angular frequency of the mirror suspension, M is the mirror mass, and $2\,|\,\phi\,|^2/c$ is the force due to the radiation pressure. The retarded time $\hat{t}$ is defined by the following equation:

$$c(t-\hat{t})=2D_s+x(t)+x(\hat{t})\ . \tag{3}$$

If we use the magnitude

$$f=-i\,\exp i\,[2\pi/\lambda(ct-D_s-x+\alpha)+\sigma]\phi$$

instead of the light field ϕ, the equation of motion becomes the following functional-differential system:

$$\ddot{x}+\frac{\Omega}{Q}\dot{x}+\Omega^2 x=\frac{2\,|\,f\,|^2}{Mc}\ , \tag{4a}$$

$$f(t)=\sqrt{P}\,\sin\theta-\cos\theta\exp\left[i\left[\frac{4\pi}{\lambda}(D_s+\hat{x})-\mu\right]\right]f(\hat{t})\ . \tag{4b}$$

Taking $x(t)=x(\hat{t})=x(\hat{\hat{t}})=\cdots$ and using the fact that $|\cos\theta\,|<1$, the resulting equation of motion in the adiabatic approximation is an ordinary differential equation:[6]

$$\ddot{x}+\frac{\Omega}{Q}\dot{x}+\Omega^2 x =\frac{2P}{Mc}\frac{\sin^2\theta}{1+\cos^2\theta+2\cos\theta\cos\left[\dfrac{4\pi}{\lambda}(D_s+x)-\mu\right]}\ . \tag{5}$$

The stationary solutions of system (4) and Eq. (5) are the same and given by $x(t)=x_0$ and $f(t)=f_0$, where x_0 is the solution of the transcendental equation

$$\Omega^2 x_0=\frac{2P}{Mc}\frac{\sin^2\theta}{1+\cos^2\theta+2\cos\theta\cos\left[\dfrac{4\pi}{\lambda}(D_s+x_0)-\mu\right]}\ , \tag{6}$$

and f_0 is defined by

$$f_0=\frac{\sqrt{P}\,\sin\theta}{1+\cos\theta\exp\left[i\left[\frac{4\pi}{\lambda}(D_s+x_0)-\mu\right]\right]} . \tag{7}$$

In many cases, Eq. (6) has many different solutions and, thus, the system displays multistability.[2,3] In fact, to every peak of the radiation pressure there corresponds two (or none) equilibrium points, as depicted in Fig. 2, the second one being stable *in the adiabatic approximation*. From now on we select one stationary solution of this last type that it will be characterized by a parameter ϵ defined by

$$\frac{4\pi}{\lambda}(D_s+x_0)-\mu=(2N+1)\pi+\epsilon \quad (0\lesssim\epsilon\lesssim\pi) . \tag{8}$$

If we introduce the deviation from this equilibrium point, $y=x-x_0$, and the magnitude $g=f/f_0-1$, the equation of motion (4) reads

$$\ddot y+\frac{\Omega}{Q}\dot y+\Omega^2 y=S(\,|g\,|^2+2\,\mathrm{Re}g\,) , \tag{9a}$$

$$g=\zeta\left[\exp\left[i\frac{4\pi}{\lambda}\hat y\right](\hat g+1)-1\right] , \tag{9b}$$

where $\hat y=y(\hat t)$, $\hat g=g(\hat t)$, $c(t-\hat t)=2(D_s+x_0)+y+\hat y$, $\zeta=\cos\theta\, e^{i\epsilon}$,

$$S=\frac{2\,|f_0\,|^2}{Mc}=\frac{2P}{Mc}\Delta\sin^2\theta , \tag{10}$$

and

$$\Delta^{-1}=1+\cos^2\theta-2\cos\theta\cos\epsilon . \tag{11}$$

To analyze the linear stability of the equilibrium point at $y=0(x=x_0)$, we must linearize the system (9) to obtain

$$\ddot y+\frac{\Omega}{Q}\dot y+\Omega^2 y=2S\,\mathrm{Re}g , \tag{12a}$$

$$g=\zeta\left[i\frac{4\pi}{\lambda}\hat y+\hat g\right] . \tag{12b}$$

By iterating the last equation and using the fact that $|\zeta|<1$, the functional-differential system (12) appears as the following retarded-differential equation:

$$\ddot y+\frac{\Omega}{Q}\dot y+\Omega^2 y=-\frac{8\pi}{\lambda}S\sum_{k=1}^{\infty}\mathrm{Im}\zeta^k\hat y(k) . \tag{13}$$

Here we have used the notation $\hat y^{(k)}$ to indicate the value $y(\hat t^{(k)})$, where we have defined $\hat t^{(1)}\equiv\hat t$ and $\hat t^{(k+1)}\equiv\hat t^{(k)}$, recursively.

If we put into Eq. (13) $y\propto e^{zt}$, we obtain the characteristic equation

$$z^2+\frac{\Omega}{Q}z+\Omega^2=-\frac{8\pi}{\lambda}S\Delta\cos\theta\sin\epsilon[1+D(r,z)]^{-1} , \tag{14}$$

where

$$D(r,z)=\Delta[(e^{zr}-1)+(e^{-zr}-1)\cos^2\theta] , \tag{15}$$

and $r=2(D_s+x_0)/c$ is the delay corresponding to the equilibrium point. Any solution of (14) with Re $z>0$ corresponds to an unstable equilibrium point of the system (9) [Eq. (14) is probably meaningless for Re $z<0$ and stable equilibrium points of (9) because on the right-hand side of Eq. (13) would appear arbitrarily great values e^{zt} that would not satisfy the linear approximation hypothesis].

If the equilibrium point exactly corresponds to the maximum of the radiation pressure, the parameter ϵ and the right-hand side of Eq. (14) are zero and, therefore there is no solution of Eq. (14) with Re$z>0$. This particular case has been analyzed by Tourrenc and Deruelle.[6] We shall consider only the nonexceptional cases $\epsilon\neq0$.

If we neglect the delay, $D(0,z)=0$ and the characteristic equation (14) becomes exactly the one corresponding to the adiabatic approximation. In this case, also, the equilibrium point that we have selected is stable. But if we keep the delay, the transcendental equation (14) might show among the new solutions one (or some) with Re$z>0$. To examine further the last possibility, we can study the bifurcation, that is when Re$z=0$ in Eq. (14).

If we assume that all the parameters are known except, for instance, the power P, and we put $z=i\beta$ we obtain from (14) and (15)

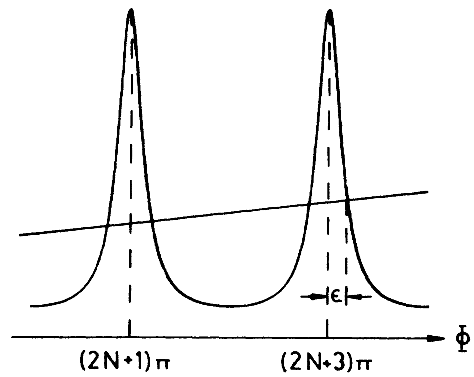

FIG. 2. The radiation pressure and the restoring force. In each peak there are two equilibrium points, but only the one on the right is stable in this adiabatic approximation. Φ is $(4\pi/\lambda)(D_s+x)-\mu$.

$$(\beta^2-\Omega^2)\sin^2\theta\sin(\beta r)=\frac{\Omega}{Q}\beta[(1+\cos^2\theta)\cos(\beta r)-2\cos\theta\cos\epsilon] , \tag{16a}$$

$$P=\frac{\Omega}{Q}\beta\frac{Mc\lambda}{16\pi}\frac{[(1+\cos^2\theta)\cos(\beta r)-2\cos\theta\cos\epsilon]^2+\sin^4\theta\sin^2(\beta r)}{\Delta\sin^4\theta\cos\theta\sin\epsilon\sin(\beta r)} . \tag{16b}$$

By solving Eq. (16a) we can know the angular frequency β at which the bifurcation occurs, and substituting this value for β in (16b) we obtain the laser power at which that happens, that is, the threshold power to make unstable the equilibrium point.

We have applied a method similar to the one just described to a very small interferometer that had been found[5] to be unstable for certain values of the parameters. It has been confirmed by numerical integration that the bifurcation occurs very likely for the parameter values predicted by the characteristic equation.

In the following we shall assume that $\theta \ll 1$, as in the case of each arm of the very long interferometers projected for detecting gravitational waves.[4] In this case the width of each peak of the radiation pressure is $\theta^2\lambda/4\pi$. We shall assume that the radiation pressure is not negligible at the equilibrium point and that we are not exactly in the maximum. So, we shall characterize the ϵ parameter by the value δ defined by the equation $\epsilon=\frac{1}{2}\delta\theta^2$, with $0<\delta\lesssim 1$.

If we take, for instance, $D_s=3$ km, $\lambda/2\pi=10^{-7}$ m, $\theta=10^{-2}$, $\Omega=6\ \mathrm{s}^{-1}$, $M=100$ kg, and $Q=10^6$, we obtain from (14) and (15) a bifurcation which occurs for $\beta\approx\Omega$ and $P\sim 1/\delta\times 10^{-10}$ W. Thus, this cavity in this configuration with non-negligible radiation pressure will be unstable except for laser powers much smaller than the one desired (~ 1 kW) or when the selected point is extremely close to the maximum of the radiation pressure. This instability, predicted by the characteristic equation, has also been confirmed by numerical integration and would presumably coexist with the effects due to thermal noise[6] that we have neglected here.

ACKNOWLEDGMENTS

One of us (J.M.A.) would greatly acknowledge the kind hospitality of the Laboratoire de Physique Théorique, Institut Henri Poincaré, where this work has been performed, and the financial support from Grupo Interuniversitario de Física Teórica de Altas Energias.

[1] For a recent review see J. R. Ackerhalt, P. W. Milonni, and M.-L. Shih, Phys. Rep. **128**, 205 (1985).

[2] A. Dorsel, J. D. McCullen, P. Meystre, E. Vignes, and H. Walther, Phys. Rev. Lett. **51**, 1550 (1983).

[3] N. Deruelle and Ph. Tourrenc, in *Gravitation, Geometry and Relativistic Physics* (Springer, Berlin, 1984).

[4] A. Brillet, Ref. 3.

[5] L. Bel (unpublished).

[6] Ph. Tourrenc and N. Deruelle, Ann. Phys. (Paris) **10**, 241 (1985).