Quasiclassical approach to the shifting parameter of the $1/N$ method

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Proof is given that $1/N$ expansions for energy levels can be interpreted in terms of the quasiclassical minimizations of corresponding Hamiltonian forms like $\mathcal{H}(r,p=\hbar d_0/r)$ and vice versa. This mutual connection enables us to show that the fixing condition for the expansion parameter $(k=N+2l-a)$ proposed previously is implied to first order only. In general, the a parameter should be chosen, order by order, so that the sum of corrections to the zeroth-order result vanishes. This requirement leads to algebraic equations which produce self-consistent $d_0=k/2$ evaluations. (d_0 denotes the underlying phase-space quantum.) Particular fixing conditions can also be proposed. First-order d_0 estimates are discussed in more detail by using as an example the linear-plus-Coulomb potential. We also find the covariance behavior of d_0 under quasiclassical symmetry transformations.

I. INTRODUCTION

Systematic expansions for the energy levels of the spherically symmetrical Schrödinger equation have been proposed.[1] As a first step, it has been assumed that the expansion parameter is $1/k_0=1/(N+2l)$, where N denotes the inflating number of space dimensions, whereas l is the angular momentum. More efficient expansions in inverse powers of k, where this time $k=N+2l-a$, have also been given.[2] Here a is a suitable shift, which has the meaning of an additional degree of freedom. This improved approach has been called the shifted-$1/N$ or -$1/k$ method. In general, both methods can be viewed as realizations of inherent quantum fluctuations in terms of certain corrections to the underlying potential.[3] These corrections lead to effective potentials exhibiting the minima needed. Then $1/k_0$ or $1/k$ expansions work in conjunction with Taylor-series expansions of the effective potentials around the locations of such minima.[4] Other useful points stem from the logarithmic perturbation theory.[5] Of course, the main problem is the suitable definition of a. So far, a choice which has proved to be quite useful in practice is that of invoking the agreement with the exact energy levels of the harmonic oscillator and of the Coulomb potential.[2] Accordingly, the a parameter has been chosen so as to make the first-order energy correction vanish. One would then obtain Eq. (15) of Ref. 2, which extrapolates the particular agreements mentioned above towards other potentials. However, a deeper point of view, relying on the full structure of the theory, would be desirable. This situation motivates one to look for a suitable approach in which the fixing condition for the a parameter could be justified theoretically. For this purpose we shall analyze the k-fixing problem via the mutual connection between the $1/N$ method and the quasiclassical minimization of the Hamiltonian form $\delta\mathcal{H}(r)=\mathcal{H}(r,\hbar d_0/r)$ discussed previously.[6] It turns out that such interconnections enable us to define the k parameter in a self-consistent manner. Proceeding order by order, the k parameter should be chosen so as to obtain the corresponding $1/N$ energy evaluation in terms of the zeroth-order result. These latter evaluations can then be improved successively. This criterion leads to order-dependent algebraic equations for k, or equivalently, for a. Now it becomes clear that Eq. (15) of Ref. 2 is reproduced to first order only. Of course, other particular fixing conditions can also be proposed. We shall then use this opportunity to perform comparisons between such concrete fixings and the self-consistent k evaluation mentioned above. Systematic higher-order expansions go, however, beyond the scope of this paper, so that subsequent contingent problems will be analyzed by using first-order calculations. The reason is that first-order results provide, up to certain degrees of accuracy, a general and comprehensive description of energy levels.

Above, the underlying phase-space quantum has been denoted by d_0.[7] The standard Hamiltonian reads $\mathcal{H}(r,p)=p^2/2m_0+V(r)$, where $r=|\mathbf{x}|$ and $p=|\mathbf{p}|$. The point of interest is that quasiclassical minimizations exhibit typical analytical forms and symmetry properties which are able to be interpreted in terms of $1/N$ results. These minima refer to ground-state energies (GSE's), but generalizations towards energy levels can also be done. Scaling properties of d_0 have also been established.[8] General classical attributes of $1/N$ expansions are also worthy of being mentioned.[9]

This paper is organized as follows. The quasiclassical approach to the fixing parameter of the $1/N$ method is presented in Sec. II. Next-order estimates for power potentials serve as illustrative examples. Particular fixing conditions have also been proposed. Considering as a further example the linear-plus-Coulomb potential, we have discussed first-order d_0 estimates in Sec. III. Comparisons between several possible choices have been done. Using the quasiclassical equivalence between the linear-plus-Coulomb potential and the quartic anharmonic oscillator, the covariance behavior of d_0 under quasiclassical symmetry transformations has also been established. Numerical data concerning the GSE's have been invoked. Conclusions are presented in Sec. IV.

II. THE SELF-CONSISTENT FIXING OF THE SHIFTING PARAMETER

Defining the dimensionless coordinate as $x=m_0 r/\hbar$, the quasiclassical minimization of the Hamiltonian form

$$\delta\mathcal{H}(x)=\frac{1}{m_0}\delta\mathcal{H}(r)=\frac{d_0^2}{2x^2}+V(x)\ , \tag{2.1}$$

where $V(x)=V(r)/m_0$ gives the energy

$$\epsilon(y)=V(y)+\tfrac{1}{2}yV'(y)\ , \tag{2.2}$$

provided that

$$g(y)\equiv\frac{3}{y}V'(y)+V''(y)>0\ . \tag{2.3}$$

This inequality expresses the concavity behavior needed. The above minimum is located at $x=y$, which comes from the quasiclassical virial equation

$$d_0^2=f(y)=y^3V'(y)>0\ . \tag{2.4}$$

The primes denote differentiations with respect to the radial coordinate. Next let us assume that y subjects itself to vanishingly small variations around a certain point $x=x_0$, which will be specified later. Then Eqs. (2.2) and (2.4) yield the expansions

$$\epsilon(y)=\epsilon(x_0)+(y-x_0)\epsilon'(x_0)+\cdots \tag{2.5}$$

and

$$d_0^2=f(x_0)+(y-x_0)f'(x_0)+\cdots\ , \tag{2.6}$$

where

$$f'(x_0)=2x_0^2\epsilon'(x_0)=x_0^3g(x_0)\ . \tag{2.7}$$

On the other hand, the minimization of the $1/N$ effective potential

$$V^{(\mathrm{eff})}(x)=\frac{k^2}{8x^2}+V(x) \tag{2.8}$$

proceeds by using the substitutions $d_0\to k/2$ and $y\to x_0$, owing to the similarity between Eqs. (2.1) and (2.8). Then the minimum of $V^{(\mathrm{eff})}(x)$ is located at $x=x_0$, so that $\min V^{(\mathrm{eff})}(x)=\epsilon(x_0)$, in so far as

$$g(x_0)>0\ , \tag{2.9}$$

which corresponds to Eq. (2.3). In addition, $k^2/4=f(x_0)$. Then the $1/N$ energy level is given by the expansion

$$\epsilon^{(N)}(l,n_r)=\epsilon(x_0)+\epsilon^{(1)}(x_0)+\epsilon^{(2)}(x_0)+\cdots\ , \tag{2.10}$$

in which $\epsilon(x_0)=\epsilon^{(0)}(x_0)=O(k^2)$ expresses the zeroth-order result, whereas $\epsilon^{(i)}(x_0)=O(k^{2-i})$, $i=1,2,\ldots$, are the higher-order corrections. The radial quantum number has been denoted by n_r. More definitely, the s-order result reads

$$\epsilon_s^{(N)}(l,n_r)=\epsilon(x_0)+\sum_{i=1}^{s}\epsilon^{(i)}(x_0)\ . \tag{2.11}$$

The first-order correction takes the simple form

$$\epsilon^{(1)}(x_0)=k\left[\frac{a-2}{4x_0^2}+\frac{1}{2k}[g(x_0)]^{1/2}\right]\ , \tag{2.12}$$

but the $\epsilon^{(i)}(x_0)$'s become progressively much more elaborate at higher orders. Hereafter we put $N=3$, so that $a-2=1-k+2l$. Next we see that the parametrization

$$y=x_0+\frac{1}{k}\beta(k) \tag{2.13}$$

gives a suitable realization of the mutual interconnection between the quasiclassical and $1/N$ methods emphasized above. Accordingly, Eq. (2.5) reproduces Eq. (2.11) if

$$\beta(k)=\beta_s(k)=\frac{2k}{x_0g(x_0)}\sum_{i=1}^{s}\epsilon^{(i)}(x_0)\ , \tag{2.14}$$

where $\beta(k)=O(1)$, which proceeds to s order. Now we would like to stress that we are looking for a mutual interconnection between the $1/N$ and quasiclassical methods which acts irrespective of the concrete form of $V(x)$, thereby preserving the same results. Under such conditions Eq. (2.3) is implied by Eq. (2.9), and conversely, if $y=x_0$ only. In particular, this result is also given immediately to zeroth order, by virtue of the simultaneous validity of the inequalities $\epsilon(y)\le\epsilon(x_0)$ and $\epsilon(x_0)\le\epsilon(y)$. Consequently, the mutual interconnection criterion reads $\beta(k)=\beta_s(k)=0$, so that

$$\sum_{i=1}^{s}\epsilon^{(i)}(x_0)=0 \tag{2.15}$$

to s order, which represents an algebraic equation for the determination of the k parameter. Accounting for Eqs. (2.11) and (2.4), one finds that the quasiclassical d_0 parameter is given self-consistently by

$$d_0=\frac{k}{2}=[f(x_0)]^{1/2}\ , \tag{2.16}$$

so that

$$\epsilon_s^{(N)}(l,n_r)=\epsilon(x_0)\ , \tag{2.17}$$

to each order. So Eq. (2.17) works, order by order, in terms of selected k values, as given by Eq. (2.15). It is understood that Eq. (2.17) can also be used with respect to the basic Hamiltonian form $\epsilon(x)=\delta\mathcal{H}(x)$, provided that the $x\to x_0$ limit is supplemented by $d_0=k/2$. Equivalently, we found a self-consistent solution to the fixing problem of the k parameter.

Setting $s=1$, we get the first-order fixing condition

$$k(k-2l-1)=2x_0^2(1+2n_r)[g(x_0)]^{1/2}\ , \tag{2.18}$$

which reproduces identically Eq. (15) of Ref. 2, as one might expect. Next-order calculations are of further interest. For the sake of simplicity, we shall restrict ourselves to the attractive power potentials $V_n(x)=\gamma(n)/x^n$ where $-n\gamma(n)>0$. Then the second-order fixing condition reads

$$\epsilon^{(1)}(x_0)+\epsilon^{(2)}(x_0)=0 \ , \tag{2.19}$$

in which

$$\epsilon^{(1)}(x_0)=\frac{1}{4x_0^2}[a-2+(1+2n_r)(2-n)^{1/2}] \tag{2.20}$$

represents the first-order correction, whereas $\epsilon^{(2)}(x_0)$ takes the form

$$\epsilon^{(2)}(x_0)=\frac{1}{x_0^2}\left[\frac{(2+n)(1+2n_r)}{2(2-n)^{1/2}}-\frac{3n+10}{8(2-n)}+\frac{1}{32}(n^2+8n+27)(1+2n_r+2n_r^2)-\frac{(n+5)^2}{288}(11+30n_r+30n_r^2)\right] . \tag{2.21}$$

This agrees with the last higher-order term from the right-hand side of Eq. (7) of Ref. 10. Considering the GSE ($l=n_r=0$), we see that Eq. (2.19) gives the solution

$$d_0=d_0^{(2)}(n)=\frac{4d_0^{(1)}(n)+[(2-n)F_2(n)]^{1/2}}{6-n} \ , \tag{2.22}$$

where

$$d_0=d_0^{(1)}(n)=\tfrac{1}{2}[1+(2-n)^{1/2}] \tag{2.23}$$

expresses the first-order result relying on Eq. (2.18) and where

$$F_2(n)=\frac{1}{72}(n^3+13n^2-98n+120)+\tfrac{1}{2}(2-n)^{3/2} \ . \tag{2.24}$$

We remark that $F_2(n)$ vanishes for $n=n^{(+)}\cong 1.724\,417$ and for $n=n^{(-)}\cong -25.141\,154$, so that $d_0^{(2)}(n)$ exhibits complex values for $n^{(+)}<n<2$ and $n<n^{(-)}$. But it is easy to verify that the imaginary contributions for $n^{(+)}<n<2$ take quite small values, like $\mathrm{Im}d_0^{(2)}(1.75)\cong 0.018\,746$. Moreover, these contributions vanish at $n=2$ and $n^{(+)}$. This situation motivates us to approach $d_0^{(2)}(n)$ by its real part if $n^{(+)}<n<2$. Then $\mathrm{Im}d_0^{(2)}(n)$ can be seen as a small oscillation.

The next-order correction to the leading energy contribution has also been calculated using the first-order fixing condition (2.18) instead of Eq. (2.19).[11] Then the GSE obtained in this manner can be fitted in terms of the effective d_0 parameter

$$d_0=d_0^*(n)=\left[[d_0^{(1)}(n)]^2-\frac{(n+2)(n-1)}{144}\right]^{1/2} , \tag{2.25}$$

which vanishes for $n=n^*_{(-)}\cong -49.639117$. This means that Eq. (2.25) becomes meaningless for $n<n^*_{(-)}$. Let us now compare Eq. (2.22) with Eq. (2.25). First, we remark that both $d_0^{(2)}(n)$ and $d_0^{(1)}(n)$ fulfil the exact boundary condition

$$d_0^{(2)}(2)=d_0^{(1)}(2)=\tfrac{1}{2} \ , \tag{2.26}$$

at $n=2$, which has the meaning of a symmetry condition.[8] In contradistinction, $d_0^*(2)\cong 0.471404$, which

TABLE I. Comparison of $d_0^{(2)}(n)$ with $d_0^*(n)$ for several n values. First-order $d_0^{(1)}(n)$ values have also been inserted. Except for $n=-14$, the confrontations with the numerical $d_0(n)$ evaluations favor clearly $d_0^{(2)}(n)$.

n	$d_0(n)$	$d_0^{(1)}(n)$	$d_0^*(n)$	$d_0^{(2)}(n)$
2	0.5	0.5	0.471 404	0.5
$\frac{7}{4}$	0.708 234	0.75	0.736 864	0.705 882[a]
$\frac{14}{9}$	0.808 107	0.833 333	0.825 062	0.814 815
1.5	0.832 116	0.853 553	0.846 404	0.838 048
$\frac{10}{7}$	0.860 876	0.877 964	0.872 134	0.865 651
$\frac{4}{3}$	0.896 147	0.908 248	0.903 990	0.899 455
1.2	0.940 818	0.947 213	0.944 864	0.942 449
1	1	1	1	1
$\frac{2}{3}$	1.084 051	1.077 350	1.080 211	1.083 020
0.4	1.142 145	1.132 455	1.136 862	1.141 168
0	1.218 629	1.207 107	1.212 846	1.218 453
−0.5	1.302 681	1.290 569	1.296 609	1.302 532
−1	1.376 083	1.366 025	1.371 100	1.376 106
−2	1.5	1.5	1.5	1.5
−3	1.602 044	1.618 034	1.609 427	1.600 699
−4	1.688 432	1.724 745	1.704 494	1.683 666
−5	1.763 068	1.822 876	1.788 260	1.752 146
−6	1.828 466	1.914 213	1.862 731	1.808 233
−7	1.886 481	2	1.929 306	1.853 344
−14	2.165 871	2.5	2.236 068	1.914 213

[a]This comes from $\mathrm{Re}d_0^{(2)}(n)$.

means that $d_0^*(n)$ crosses the exact d_0 evaluation at a point located in a small neighborhood of $n=2$. Furthermore, Eqs. (2.22), (2.23), and (2.25) meet, by construction, the exact d_0 evaluation at $n=1$ and -2. Choosing, for instance, $n\in[-7,2]$, one sees that the "exact" numerical estimates $d_0(n)$ (Ref. 12) are approached more accurately by $d_0^{(2)}(n)$ than by $d_0^*(n)$, as shown in Table I. The lower bound $n\geq -7$ mentioned above can also be moved towards $n\geq n_2$, where $n_2\approx -10$, as is crudely obtained by extrapolation. It is understood that accuracy comparisons should be performed, from the very beginning, within restricted n regions. Indeed, excessively large-n regions are inappropriate for such comparisons, since increasing degrees of accuracy may imply decreasing regions of validity. The above results then enable us to say that the most accurate solution acting within the interval $n\in[n_2,2]$ is given by $d_0^{(2)}(n)$. This interval encompasses, in practice, all the physically relevant cases. Such agreements confirm the actual relevance of the fixing condition (2.15) proposed above. Complementarily, $d_0^{(1)}(n)$ exhibits the largest region of validity.

We would like to remark that one has often dominant $V(x)$ constituents, such as Coulomb terms, which can be selected within corresponding regions of the coupling space. This allows us a description of energy levels using concrete $k=k_1$ values characterizing such constituents. These latter values come up by performing suitable $x_0\rightarrow x_1$ limits. Then Eq. (2.6) yields

$$d_0=\left[\frac{k_1^2}{4}+\frac{\beta(k_1)}{k_1}x_0^3 g(x_0)\right]^{1/2}, \tag{2.27}$$

in which, this time, $d_0^2\neq k_1^2/4=f(x_0)$. In view of this, $\beta(k_1)\neq 0$, so that Eq. (2.17) ceases to be valid. We need to specify that we are looking for general $x_0\neq x_1$ solutions. Then the energy levels come up by combining Eq. (2.27) with Eqs. (2.2) and (2.4). Such particular fixing conditions are able to give better results in specific cases. In general, this latter fixing approach seems to be much more appropriate for potentials exhibiting a more complex structure. Choosing as an illustrative example the linear-plus-Coulomb potential, we shall then perform comparisons between Eqs. (2.27) and (2.18) in Sec. III.

III. THE COVARIANCE BEHAVIOR OF d_0 UNDER QUASICLASSICAL SYMMETRY TRANSFORMATIONS

Our next step is to analyze in a more general manner the quasiclassical equivalence between the linear-plus-Coulomb Hamiltonian

$$\tilde{\epsilon}=\tilde{\epsilon}(x)=\frac{\tilde{d}_0^{\,2}}{x^2}-\frac{\alpha}{x}+\kappa x \tag{3.1}$$

and the quartic anharmonic oscillator

$$\epsilon=\epsilon(x)=\frac{d_0^2}{x^2}+\mu x^2+\lambda' x^4 \tag{3.2}$$

discussed previously.[8] The l dependence concerns $\tilde{d}_0^{\,2}$ and d_0^2, as shown in Sec. II. Recall that this equivalence comes from the symmetry transformation

$$p^2\rightarrow\frac{1}{\rho^2}x^{2+2\rho}\left[p^2-\frac{1}{4x^2}(1-\rho^2)\right], \tag{3.3}$$

in which ρ is a free parameter. This transformation produces the energy eigenvalues corresponding to the transformed Hamiltonian forms in terms of suitable ρ values. So far the quasiclassical equivalence between Eqs. (3.1) and (3.2) has been analyzed, in particular, by keeping unchanged the basic d_0^2 evaluation. This is, of course, a noncovariant description, since keeping fixed d_0^2, the transformation properties concern exclusively the energy levels and the couplings. Now we have the opportunity to combine Eq. (3.3) with the fixing condition (2.18). Then the generalization emphasized above can be readily accomplished. More definitely, Eq. (2.18) allows us to establish the covariance behavior of d_0 under Eq. (3.3).

First, Eq. (2.18) shows that the self-consistent fixing conditions characterizing Eqs. (3.1) and (3.2) are given by[13]

$$\tilde{k}(\tilde{k}-2l-1)=2(1+2n_r)[\tfrac{1}{2}(\alpha\tilde{x}_0+3\kappa\tilde{x}_0^{\,3})]^{1/2} \tag{3.4}$$

and

$$k(k-2l-1)=4x_0^2(1+2n_r)(\mu+3\lambda' x_0^2)^{1/2} \tag{3.5}$$

to first order, where

$$\tilde{d}_0^{\,2}=\frac{\tilde{k}^2}{4}=\tfrac{1}{2}\alpha\tilde{x}_0+\tfrac{1}{2}\kappa\tilde{x}_0^{\,3} \tag{3.6}$$

and

$$d_0^2=\frac{k^2}{4}=\mu x_0^4+2\lambda' x_0^6, \tag{3.7}$$

respectively. Accordingly,

$$\tilde{\epsilon}(\tilde{x}_0)=\tfrac{3}{2}\kappa\tilde{x}_0-\frac{\alpha}{2\tilde{x}_0} \tag{3.8}$$

and

$$\epsilon(x_0)=2\mu x_0^2+3\lambda' x_0^4, \tag{3.9}$$

so that

$$\tilde{x}_0=\frac{1}{3\kappa}\{\tilde{\epsilon}(\tilde{x}_0)+[\tilde{\epsilon}^{\,2}(\tilde{x}_0)+3\alpha\kappa]^{1/2}\} \tag{3.10}$$

and

$$x_0^2=\epsilon(x_0)\{\mu+[\mu^2+3\lambda'\epsilon(x_0)]^{1/2}\}^{-1}. \tag{3.11}$$

Thus Eqs. (3.4) and (3.5) become

$$\tilde{k}(\tilde{k}-2l-1)=2\tilde{x}_0(1+2n_r)[\tilde{\epsilon}^{\,2}(\tilde{x}_0)+3\alpha\kappa]^{1/4} \tag{3.12}$$

and

$$k(k-2l-1)=4x_0^2(1+2n_r)[\mu^2+3\lambda'\epsilon(x_0)]^{1/4}, \tag{3.13}$$

respectively. On the other hand, the symmetry transforms of Eqs. (3.1) and (3.2) read

$$\beta^*\frac{\tilde{d}_0^{\,2}}{x^2}-\beta^*\tilde{\epsilon}x^2+\beta^*\kappa x^4=\beta^*\alpha \tag{3.14}$$

and

$$\beta\frac{d_0^2}{x^2}-\beta\frac{\epsilon}{x}+\beta\lambda' x=-\beta\mu \ , \tag{3.15}$$

which are defined, in general, up to the scale factors β^* and β. Then Eq. (3.14) reproduces Eq. (3.2) via $\epsilon=\beta^*\alpha$, $d_0^2=\beta^*\tilde{d}_0^{\,2}$, $\mu=-\beta^*\bar{\epsilon}$, and $\lambda'=\beta^*\kappa$, whereas the inverse transformation is characterized by $\bar{\epsilon}=-\beta\mu$, $\tilde{d}_0^{\,2}=\beta d_0^2$, $\alpha=\beta\epsilon$, and $\kappa=\beta\lambda'$. Combining these equivalence results with Eqs. (3.12) and (3.13), one finds immediately the covariance criterion

$$\frac{\tilde{k}(\tilde{k}-2l-1)}{k(k-2l-1)}=\frac{\sqrt{\beta}}{2}=\frac{1}{2\sqrt{\beta^*}} \ , \tag{3.16}$$

which shows that $\beta^*\beta=1$, as one might expect. Further, $\tilde{k}=\sqrt{\beta}k$, so that

$$\sqrt{\beta}=\frac{\tilde{k}}{2\tilde{k}-1-2l}=\frac{k+1+2l}{2k} \ , \tag{3.17}$$

which determines the scale factor in terms of k or $\tilde{k}$. Next let us assume the additional parametrizations

$$k=1+2l+f_0(n_r)(2d_0^{(0)}-1) \tag{3.18}$$

and

$$\tilde{k}=1+2l+f_0(n_r)(2\tilde{d}_0^{\,(0)}-1) \ , \tag{3.19}$$

in which $f_0(0)=1$, whereas $d_0=d_0^{(0)}$ and $d_0=\tilde{d}_0^{\,(0)}$ are responsible for the GSE's.[14] Then the covariance criterion exhibits the simple linear form

$$\tilde{d}_0^{\,(0)}=\tfrac{1}{2}d_0^{(0)}+\tfrac{1}{4} \ , \tag{3.20}$$

which favors, in turn, the relevance of Eqs. (3.18) and (3.19). One sees immediately that if $\kappa=\lambda'=0$, Eq. (3.20) is synonymous with Eq. (3.17) of Ref. 8, this time in terms of $\tilde{d}_0^{\,(0)}(\kappa=0)=d_0(n=1)=1$ and $d_0^{(0)}(\lambda'=0)=d_0(n=-2)=\frac{3}{2}$.

Particular fixing conditions relying on Eq. (2.27) can also be done. Assuming, e.g., the dominance of the Coulomb term in Eq. (3.1) and setting correspondingly $\tilde{k}=k_1(1)$ yields

$$\tilde{d}_0^{\,2}=-\frac{k_1^2}{4}+\frac{k_1}{2}(1+2l)+(1+2n_r)\left[\frac{\alpha\tilde{x}_0+3\kappa\tilde{x}_0^{\,3}}{2}\right]^{1/2} \ , \tag{3.21}$$

if $\alpha>0$, where $k_1=k_1(1)$ and

$$k_1^2=2\alpha\tilde{x}_0+2\kappa\tilde{x}_0^{\,3} \ . \tag{3.22}$$

It is obvious that $k_1(1)$ comes from the familiar first-order evaluation[2]

$$k=k_1(n)=1+2l+(1+2n_r)(2-n)^{1/2} \tag{3.23}$$

implied by the attractive power potential $V(x)=V_n(x)$, now for $n=1$. Then Eq. (3.21) gives

$$\tilde{d}_0=\tilde{d}_1(\kappa)=\left[\frac{\alpha\tilde{x}_0+3\kappa\tilde{x}_0^{\,3}}{2}\right]^{1/4} \ , \tag{3.24}$$

for $l=n_r=0$. The self-consistent solution reads $\tilde{d}_0=\tilde{d}_0^{\,(1)}(\kappa)$, such that

$$\tilde{d}_0^{\,2}=2\tilde{d}_0+4(2\alpha\tilde{x}_0+6\kappa\tilde{x}_0^{\,3})^{1/2} \ , \tag{3.25}$$

for the GSE, which works via Eq. (3.6). The comparison of $\tilde{d}_0^{\,(1)}(\kappa)$ and $\tilde{d}_1(\kappa)$ with the exact numerical result $\tilde{d}_0(\kappa)$ (Ref. 15) is presented in Table II. The present data refer to $\alpha=2$. We see that $\tilde{d}_0(\kappa)$ is reproduced more precisely by $\tilde{d}_1(\kappa)$, except for the large-κ region ($\kappa>2$), in which $\tilde{d}_0^{\,(1)}(\kappa)$ becomes more accurate.[16] In addition, the particular fixing $\tilde{k}=k_1(-1)$ can also be used satisfactorily for large $\kappa>0$ values. Then the GSE is given by

$$\tilde{d}_0=\tilde{d}_1^{\,(-1)}(\kappa)=\left[-\frac{1}{2}+\left[\frac{\alpha\tilde{x}_0+3\kappa\tilde{x}_0^{\,3}}{2}\right]^{1/2}\right]^{1/2} \ , \tag{3.26}$$

where

$$\alpha\tilde{x}_0+\kappa\tilde{x}_0^{\,3}=\tfrac{1}{2}(1+\sqrt{3})^2 \ . \tag{3.27}$$

However, this latter approach breaks down for negative κ values, as shown in Table II. Moreover, for relatively small-κ values like $0<\kappa<2$, both $\tilde{d}_0^{\,(1)}(\kappa)$ and $\tilde{d}_1(\kappa)$ are better than $\tilde{d}_1^{\,(-1)}(\kappa)$. Similar descriptions can be done for the anharmonic oscillator and, in general, for any spherically symmetrical potential.

IV. CONCLUSIONS

Proof is given that $1/N$ energy expansions can be interpreted adequately in terms of quasiclassical minimizations of corresponding Hamiltonian forms. This means that the concavity conditions characterizing both methods should be simultaneously valid, as discussed in Sec. II. In view of this latter requirement one obtains the fixing condition (2.15), which acts to sth order ($s\geq 1$). Having chosen a such that Eq. (2.15) is satisfied, one finds, order by order, the energy levels with the help of Eqs. (2.16) and (2.17). Thus Eq. (2.15) leads to self-consistent $d_0=k/2$ solutions acting correspondingly to each order. We emphasize that accuracy can then be enhanced merely by using such improved k parameters, at least up to certain limits. Generalizations towards potentials which are not spherically symmetrical[17] can be performed in a similar manner. Next we would like to note that general qualitative and symmetry properties of the $1/N$ method seem to be described adequately in terms of the first-order approximation. Moreover, exclusive higher-order calculations are not automatically responsible for a better accuracy, owing to inherent limitations of their validity degrees.

In general, one has trouble with Eq. (2.18) if the dominating potential energy term is not "stable," i.e., if it does not subject itself to a GSE. We have to realize that in such cases the energy level approaches a critical value, like the well-known zero-energy limit. Then we are faced with limitations on accuracy characterizing, in general, both variational and perturbative methods. It is obvious that within such regions the higher-order counterparts of Eq. (2.18) proposed above should be invoked. However, such improvements may concern progressively decreasing domains of underlying couplings, so that accuracy cannot be extended beyond certain limits. Such

TABLE II. Comparison of $\bar{d}_0^{(1)}(\kappa)$, $\bar{d}_1(\kappa)$, and $\bar{d}_1^{(-1)}(\kappa)$ with the numerical $\bar{d}_0(\kappa)$ evaluation ($\alpha=2$).

κ	$\bar{d}_0(\kappa)$	$\bar{d}_0^{(1)}(\kappa)$	$r\%$	$\bar{d}_1(\kappa)$	$r\%$	$\bar{d}_1^{(-1)}(\kappa)$	$r\%$
−0.08	0.976 651	0.978 812	0.22	0.976 298	−0.03	0.562 779	−42.37
−0.06	0.983 273	0.984 329	0.11	0.983 034	−0.02	0.785 902	−20.07
−0.04	0.989 279	0.989 701	0.04	0.989 172	−0.01	0.858 261	−13.24
0	1	1	0	1	0	0.930 605	−6.94
0.1	1.021 662	1.023 149	0.14	1.021 179	−0.05	1.009 930	−1.15
0.4	1.063 851	1.073 786	0.93	1.059 971	−0.36	1.093 467	2.78
0.6	1.082 438	1.096 747	1.32	1.076 291	−0.57	1.119 401	3.41
1.2	1.118 732	1.140 112	1.91	1.107 255	−1.02	1.161 454	3.82
1.6	1.134 655	1.158 134	2.07	1.120 562	−1.24	1.177 596	3.78
2	1.147 059	1.171 695	2.15	1.130 843	−1.41	1.189 493	3.70
125	1.309 325	1.317 197	0.60	1.262 216	−3.60	1.318 614	0.71
1000	1.342 278	1.341 809	−0.03	1.288 759	−3.99	1.342 162	−0.01

limitations refer, e.g., to the $B \gg 2$ region of the Hellmann potential.[18] Furthermore, one realizes that the accuracy of Eq. (2.27) will be enhanced if the dominant constituent of the potential, for example, $V_{(1)}(x)$, gives additionally the most singular contribution. Then the corresponding k_1 parametrization works reasonably well even outside the dominance region. This has been confirmed by the reasonable accuracy of the $\tilde{k}=k_1(1)$ choice $[V_{(1)}(x)=-\alpha/x]$ in the large-$\kappa>0$ region, as displayed in Table II. Similarities with other perturbation approaches can also be noticed. So the requirement that the sum of corrections to the zeroth-order term vanishes, order by order, has also been used before in order to construct a convergent Rayleigh-Schrödinger perturbation theory for the one-dimensional anharmonic oscillator.[19] However, in general, neither the starting zeroth-order input nor the corresponding unperturbed Hamiltonian are uniquely involved.[20]

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[11]This concerns the term $(n+2)(n-1)(1+6n_r+6n_r^2)/144k$ displayed on the right-hand side of Eq. (19) of Ref. 2.

[12]See Table III of Ref. 8. Recall that the input GSE data for $d_0(n)$ have been given to four digits only. The present results have been slightly improved for $n=-1$ and -4. Concerning the $0<n<2$ potentials, see also M. C. Dumont-Lepage *et al.*, J. Phys. A **13**, 1243 (1980).

[13]This time the kinetic energy reads p^2/m_0 instead of $p^2/2m_0$, so that the f and g functions should be rescaled as $f\to f/2$ and $g\to g/2$.

[14]Equation (A4) of Ref. 8 has been used, now for $\lambda=0$.

[15]The GSE data have been taken over from Table I of F. M. Fernández *et al.*, Phys. Lett. **111A**, 104 (1985), as well as from Table IX of F. M. Fernández *et al.*, J. Phys. A **18**, 1389 (1985).

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