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# Unstable periodic orbits and the dimension of chaotic attractors

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A formulation giving the $q$ dimension $D_q$ of a chaotic attractor in terms of the eigenvalues of unstable periodic orbits is presented and discussed.

A common way of quantitatively characterizing the properties of chaotic attractors, in both experiment and theory, is through their fractal dimensions.[1-5] Here we are concerned with the dimension denoted $D_q$ and introduced for chaotic attractors in Refs. 2 and 3. Typical chaotic attractors are "multifractals;" that is, their dimension $D_q$ varies with the index $q$, thus providing a spectrum of dimension values for the attractor. A Hausdorff-type formulation of the $D_q$ has been given in Ref. 4 and is as follows. We cover the attractor with balls $S_i$ of radius $l_i$, each of which is restricted to be less than $l$, $l_i < l$. The natural probability measure of the attractor in each ball is denoted $p_i$. (That is, the fraction of time a typical chaotic orbit spends in $S_i$ is $p_i$.) A partition function is then defined,

$$\Gamma(q,\tau,\{S_i\},l) = \sum_i p_i^q / l_i^\tau \ , \tag{1}$$

and, for $q > 1$ ($q < 1$), it is maximized (minimized) over all coverings $\{S_i\}$;

$$\Gamma(q,\tau,l) = \begin{cases} \sup(\Gamma(q,\tau,\{S_i\},l)) & \text{for } q > l,\ \tau \geq 0, \\ \inf(\Gamma(q,\tau,\{S_i\},l)) & \text{for } q < 1,\ \tau \leq 0. \end{cases} \tag{2}$$

Next the limit $l \to 0$ is taken,

$$\Gamma(q,\tau) = \lim_{l \to 0}(\Gamma(q,\tau,l)) \ . \tag{3}$$

The function $\tau(q)$ is defined as that value of $\tau$ below which the limit (3) is zero and above which it is infinite;

$$\Gamma(q,\tau) = \begin{cases} \infty & \text{for } \tau > \tau(q), \\ 0 & \text{for } \tau < \tau(q). \end{cases} \tag{4}$$

In terms of $\tau(q)$, the dimension spectrum $D_q$ is then given by

$$(q-1)D_q = \tau(q) \ . \tag{5}$$

The Hausdorff dimension corresponds to $q = 0$, and the information dimension is $D_1 = \lim_{q \to 1} D_q$.

The dependence of $D_q$ on $q$ has been ascribed to the fact that different points on a chaotic attractor may have different singularity scalings,[4] as characterized by their pointwise dimension.[6] [The pointwise dimension[1] at a point $\mathbf{x}$ on the attractor is $D_p(\mathbf{x})$ if $p(l,\mathbf{x})$, the natural probability measure of the attractor in a ball of radius $l$ is centered at $\mathbf{x}$, scales as $l^{D_p}$ for $l \to 0$.]

The reason for different scalings can be understood in terms of the unstable periodic orbits on the attractor. The set of these orbits is dense in the attractor. The importance of unstable periodic orbits has long been recognized in the mathematical literature (e.g., see Bowen[7]) and has recently been emphasized in the present context in Ref. 8. In particular, for any point $\mathbf{x}$ on the unstable manifold of a periodic saddle of period $n$ of a two-dimensional invertible map (the only type of system to be considered in this paper), we show below that

$$D_p(\mathbf{x}) = 1 - \ln\lambda_u / \ln\lambda_s \ , \tag{6}$$

where $\lambda_s < 1$ and $\lambda_u > 1$ are the magnitudes of the stable and unstable eigenvalues of the $n$-times-iterated map at the saddle. Since points on different periodic orbits typically have different $\lambda_u$ and $\lambda_s$, $D_p(\mathbf{x})$ will not be the same for all points on the attractor.[6] To obtain (6), consider a point $\mathbf{x}_0$ on the unstable manifold of a saddle periodic point and two small circles centered at $\mathbf{x}_0$ with radii $l_1$ and $l_2$, where $l_1/l_2 = \lambda_s^{-1}$. We iterate the two circles backward an integral number of periods so that the two circles are now similar ellipses close to the saddle and with their major diameters parallel to the stable manifold of the saddle. We now iterate the $l_2$ ellipse backward one more period. Since it is close to the saddle, its backward iteration by one period is governed by the linearized map at the saddle (i.e., by $\lambda_s$ and $\lambda_u$). Thus, since we chose $l_1/l_2 = \lambda_s^{-1}$, the major diameter of the $l_2$ ellipse is stretched to be precisely the same as that of the $l_1$ ellipse, while its minor diameter is smaller than that for the $l_1$ ellipse by the factor $\lambda_s/\lambda_u$. Due to the smoothness of the attractor along its unstable direction, we have

$$p(l_2,\mathbf{x}_0)/p(l_1,\mathbf{x}_0) = \lambda_s/\lambda_u \ .$$

 

Setting $p(l,\mathbf{x}_0)\sim l^{D_p}$ and $l_2=l_1\lambda_s$, this yields Eq. (6), the desired result.[9]

The main result which we wish to present in this paper is a formulation giving the dimension $D_q$ of a chaotic attractor for an invertible two-dimensional map in terms of the eigenvalues of the dense set of periodic saddles on the attractor [rather than in terms of the measure of coverings of the attractor, i.e., the $p_i$ in Eqs. (1)–(5)]. This formulation is as follows:

$$\tilde{\Gamma}(q,\tau)=\lim_{n\to\infty}\tilde{\Gamma}(q,\tau,n)\ , \tag{7}$$

$$\tilde{\Gamma}(q,\tau,n)=\sum_j{}^{n}\lambda_{sj}^{-\tau+(q-1)}\lambda_{uj}^{-q}\ , \tag{8}$$

where $\sum^n$ denotes the sum over all fixed points of the $n$-times-iterated map which lie in the attractor, $j$ is an index labeling each fixed point, and $\lambda_{sj}$ and $\lambda_{uj}$ are the stable and unstable eigenvalues of the $n$ times iterated map at the $j$th fixed point. The quantity $\tau(q)$ [and hence $D_q$ by Eq. (5)] is determined by Eq. (4) with $\Gamma(q,\tau)$ replaced by $\tilde{\Gamma}(q,\tau)$. Alternatively we can obtain a good approximation to $\tau(q)$ and hence $D_q$ by setting $\tilde{\Gamma}(q,\tau,n)=1$ for some large value of $n$. Since periodic points and their eigenvalues are computationally accessible (e.g., Ref. 8), this provides an alternate way of calculating attractor dimensions. Our claim is that Eq. (8) gives the same value for $D_q$ as Eqs. (1)–(3).

The Hausdorff dimension $f(\alpha)$ of all points on an attractor with pointwise dimension[4] $D_p(\mathbf{x})=\alpha$ can be found by including in Eq. (8) (with $q=0$) only those eigenvalues in a narrow range around $\ln\lambda_{uj}/\ln\lambda_{sj}=1-\alpha$ [cf. Eq. (6)].

Letting $(q-1)$ be small and using (4), the coefficient of the $(q-1)$ term in expansion of (8) in powers of $(q-1)$ yields

$$D_1=1-\lim_{n\to\infty}\left(\sum_j{}^{n}\lambda_{uj}^{-1}\ln\lambda_{uj}\Big/\sum_j{}^{n}\lambda_{uj}^{-1}\ln\lambda_{sj}\right)\ , \tag{9}$$

which is the Kaplan-Yorke formula for the information dimension of the attractor[1,10] in terms of the Lyapunov exponents of typical orbits on the attractor [here we identify the limit of the ratio of the two sums in Eq. (9) with the ratio of the Lyapunov exponents]. Also, Eq. (8) for the case $q=0$ (the Hausdorff dimension) has been previously stated in Ref. 8.

In order to establish the relationship between $D_q$ and periodic orbits [Eq. (8)], in what follows we shall verify it for two examples. [A proof of Eq. (8) for uniformly hyperbolic systems, including higher dimensionality, will be published elsewhere.]

*Example 1. The generalized baker's map.* The generalized "baker's" map was introduced in Ref. 1 as a model for dimension studies which is amenable to analysis yet also has nonconstant stretching and contraction. We divide the square $0\le x,y\le 1$ into a bottom part, $0\le y<\alpha$, and a top part, $\alpha<y\le 1$. We compress the bottom (top) part by a factor $\lambda_a$ ($\lambda_b$) along $x$, and stretch it in $y$ by a factor $\alpha^{-1}$ ($\beta^{-1}$, $\beta=1-\alpha$). We then have two rectangles, both of vertical height unity, one of width $\lambda_a$ and the other of width $\lambda_b$. We place the lower left corner of the $\lambda_a$-width strip at the origin and the lower left corner of the $\lambda_b$-width strip at $x=\frac{1}{2},y=0$. Thus we have a map of the unit square into itself: $x_{n+1}=\lambda(y_n)x_n+\frac{1}{2}u(y_n-\alpha)$; $y_{n+1}=\gamma(y_n)[y_n-\alpha u(y_n-\alpha)]$; where $\lambda(y)=(\lambda_a,\lambda_b)$ for $y\gtrless\alpha$, $\gamma(y)=(\alpha^{-1},\beta^{-1})$ for $y\gtrless\alpha$, and $u(y)$ is the unit step function. Using similarity arguments[1] it can be shown directly from the map that the following transcendental equation determines $D_q$ (cf. Refs. 2 and 3),

$$1=A+B\ , \tag{10}$$

where $A=\lambda_a^{-(q-1)(D_q-1)}\alpha^q$, $B=\lambda_b^{-(q-1)(D_q-1)}\beta^q$. We can specify an orbit for the generalized baker's transformation by its symbolic itinerary which specifies whether the orbit's location on successive iterates is in the top (symbolized by a 1) or in the bottom (symbolized by a 0). Thus a periodic orbit of period $n$ which spends $k\le n$ of its $n$ iterates in $y>\alpha$ is represented by a string of $n$ symbols with $k$ ones and $n-k$ zeros. The eigenvalues associated with such an orbit are $\lambda_s=\lambda_a^{n-k}\lambda_b^{k}$, $\lambda_u=\alpha^{-(n-k)}\beta^{-k}$. Equation (8) yields

$$\tilde{\Gamma}(q,\tau,n)=\sum_{k=0}^{n}N_{nk}A^{(n-k)}B^{k}\ , \tag{11}$$

where $N_{nk}$ is the number of fixed points of the $n$ times iterated map which belong to periodic orbits which spend $k$ iterates in the top ($y>\alpha$). Thus $N_{nk}$ is the number of ways of arranging $k$ zeros and $n-k$ ones,

$$N_{nk}=\begin{pmatrix}n\\k\end{pmatrix}\ . \tag{12}$$

Hence Eq. (11) is just a binomial expansion, $\tilde{\Gamma}=(A+B)^n$. Letting $n\to\infty$, Eq. (4) thus yields Eq. (10).

*Example 2.* The previous example has the property that setting $\tilde{\Gamma}(q,\tau,n)=1$ gives precisely the desired result, Eq. (10), for all $n$, rather than only in the $n\to\infty$ limit. The generalized baker's map is exceptional in this regard. A more typical example, which is still analytically treatable, is illustrated in Figs. 1(a) and 1(b). Again, we divide the unit square into top ($y>\alpha$) and bottom ($y<\alpha$) parts. We again horizontally compress the two parts by $\lambda_a$ and $\lambda_b$. The bottom part is vertically stretched by $\alpha^{-1}$, as before. The difference is that we now vertically compress the top part by $\alpha/\beta$. The parts are then reassembled in the square as shown in the figure. Again, we can specify orbits by a string of ones (tops) and zeros (bottoms). In this

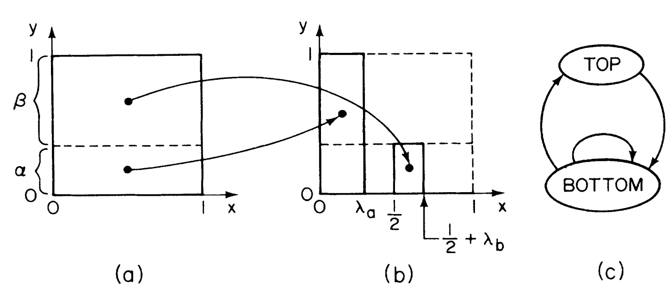


FIG. 1. (a) and (b) Schematic illustration of the map. (c) Orbit can go from the bottom to either the bottom or the top, but can only go from the top to the bottom.

case, however, an orbit point in the top is always mapped to the bottom [Fig. 1(c)]. Thus a one is *always* followed by a zero. Replacing $B$ by $\tilde{B}=(\beta/\alpha)^q\lambda_b^{-(q-1)(D_q-1)}$ (to account for the compression by $\alpha/\beta$ as opposed to the stretching by $1/\beta$ in example 1), we see that Eq. (11) still applies. Equation (12) for $N_{nk}$, however, does not. To find $N_{nk}$ we first note that the number of fixed points of the $n$-times-iterated map is the number of possible sequences of length $n$ which contain $k$ ones and $n-k$ zeros, subject to the constraint that a zero follows a one (except when the last symbol is a one). We consider two cases: (a) the last symbol is a zero, and (b) the last symbol is a one. In case (a), to find the contribution to $N_{nk}$ from such sequences, we regard the sequence (1,0) as a single symbol denoted by a 2. Thus a period-$n$ orbit which is located in the top $k$ times is represented by a string of $n-k$ symbols of which $k$ are two's and $n-2k$ are zeros (clearly $k\le n/2$). There are $\binom{n-k}{k}$ such symbol sequences. Sequences ending in the top [case (b)], on the other hand, end in a one. Since the sequence represents a periodic orbit it must also start with a zero. All the rest of the symbols can be thought of as zeros and twos. For this case the zero-two sequence has $n-k-1$ symbols of which $k-1$ are two's. There are $\binom{n-k-1}{k-l}$ sequences of this type. Thus we have

$$N_{nk}=\begin{pmatrix}n-k\\k\end{pmatrix}+\begin{pmatrix}n-k-1\\k-1\end{pmatrix}, \tag{13}$$

and $N_{nk}=0$ for $k>n/2$. Using Stirling's approximation to expand $Z(\kappa)\equiv 1/n\ln(N_{nk}A^{(n-k)}\tilde{B}^k)$ for large $n$, we have

$$Z(\kappa)\cong(1-\kappa)\ln(1-\kappa)-\kappa\ln\kappa-(1-2\kappa)\ln(1-2\kappa)+\kappa\ln\tilde{B}+(1-\kappa)\ln A\ ,$$

where $\kappa=k/n$ and $\frac{1}{2}>\kappa>0$. The quantity $Z$ is concave down ($d^2Z/d\kappa^2<0$) and has one maximum in $\frac{1}{2}>\kappa>0$. The location of this maximum is given by $\kappa_0(1-\kappa_0)=(1-2\kappa_0)^2\tilde{B}/A$. Since the summand in Eq. (11) is $\exp[nZ(\kappa)]$, if $n\to\infty$ and $Z(\kappa_0)<0$, then $\tilde{\Gamma}(q,\tau)\to 0$. On the other hand, if $Z(\kappa_0)>0$, then $\tilde{\Gamma}(q,\tau)\to\infty$. Thus from (4) we have the condition $Z(\kappa_0)=0$. This gives a transcendental equation for $D_q$,

$$1=A+A\tilde{B}\ . \tag{14}$$

This is the result of applying Eq. (8). The question is, is it correct? To show that it is, we now obtain Eq. (14) by a rigorous independent method: the similarity technique[1-5] [used, for example, in Refs. 2 and 3 to obtain Eq. (10)]. We write the sum in the partition function, $\sum p_i^q/l_i^\tau$, as a sum over the top region plus a sum over the bottom region, $\Gamma(l)=\Gamma_T(l)+\Gamma_B(l)$. Similarly we write $\Gamma_B$ as a sum over the bottom left ($x<\frac{1}{2}$) region plus a sum over the bottom right region,

$$\Gamma_B(l)=\Gamma_{BL}(l)+\Gamma_{BR}(l)\ . \tag{15}$$

Applying the map to one of the $S_i$ coverings in the bottom, we see that it is compressed by $\lambda_a$ and elongated by $1/\alpha$. Thus this $S_i$ covering can be covered by $(\alpha\lambda_a)^{-1}$ coverings of radius $(l_i\lambda_a)$. Each of these new coverings has a probability $(p_i\alpha\lambda_a)$. Inserting this information in the partition function we have that

$$\Gamma_T(l\lambda_a)=\frac{\beta}{\alpha\lambda_a}\left[\frac{(\alpha\lambda_a)^q}{\lambda_a^\tau}\Gamma_B(l)\right]=\frac{\beta}{\alpha}\frac{\alpha^q}{\lambda_a^{(q-1)(D_q-1)}}\Gamma_B(l)\ .$$

Thus,

$$\Gamma_T(l)=\frac{\beta}{\alpha}\frac{\alpha^q}{\lambda_a^{(q-1)(D_q-1)}}\Gamma_B(l/\lambda_a)\ .$$

Similarly,

$$\Gamma_{BR}(l)=\frac{(\beta/\alpha)^{q-1}}{\lambda_b^{(q-1)(D_q-1)}}\Gamma_T(l/\lambda_b)\ .$$

Also, $\Gamma_{BL}=(\alpha/\beta)\Gamma_T$. Combining these in (15) and letting $l\to 0$ [Eqs. (3) and (4)], we obtain the equation for $D_q$,

$$1=\frac{\alpha^q}{\lambda_a^{(q-1)(D_q-1)}}+\frac{\beta^q}{(\lambda_a\lambda_b)^{(q-1)(D_q-1)}}\ , \tag{16}$$

which is (14).[11]

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[10] J. Kaplan and J. A. Yorke, in *Functional Differential Equations and the Approximation of Fixed Points,* Lecture Notes in Math, Vol. 730, edited by A. Dold and B. Eckmann (Springer, Berlin, 1978), p. 228.

[11] It is also interesting to calculate the topological entropy from the periodic orbits (Ref. 7), $S_{top}=\lim_{n\to\infty}[(1/n)\sum_k N_{nk}]$. For example 1, this gives $S_{top}=\ln 2$, while for example 2 we obtain $S_{top}=\ln G$, where $G$ is the golden mean. The latter result is also confirmed by using the transition matrix corresponding to Fig. 1(c), $\begin{pmatrix}0&1\\1&1\end{pmatrix}$, whose largest eigenvalue is $G$.