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Geometrical description of Berry's phase

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Berry, Simon, and Aharonov and Anandan have discovered, interpreted, and generalized a geometrical phase factor that occurs for a quantum state evolving around a closed path in the projective Hilbert space of rays. Here this phase is directly given in terms of the holonomies of several natural geometrical structures on the projective space.

Recently, Berry[1] has shown that when a quantum state is adiabatically transported around a circuit as an eigenstate of a Hamiltonian with slowly varying parameters, it acquires a geometrical phase factor in addition to the familiar dynamical phase factor. Simon[2] demonstrated that this geometrical phase factor is the integral of the curvature of a Hermitian line bundle over the parameter space and that it depends only on the circuit of the eigenspace rather than on other aspects of the Hamiltonian. Aharonov and Anandan[3] have found a generalization which removes the restriction to adiabatic evolution. Thus, there is a geometrical factor which occurs for all cyclic evolutions and depends only on the path of the ray in the projective Hilbert space. In this paper this phase is expressed in terms of natural geometrical structures on the space of rays.

Take the Hilbert space to have dimension $n+1$, so that a quantum state is given by $n+1$ complex amplitudes Z^α, where Greek indices (superscripts, not exponents) range from 0 to n:

$$|\psi\rangle = |Z^0, Z^1, \ldots, Z^n\rangle \ . \tag{1}$$

The Hilbert space is, thus, isomorphic to $\mathbb{C}^{n+1}$. Using, henceforth, the Einstein summation convention in which there is an implicit sum over each pair of repeated indices, and using $\delta_{\alpha\beta}$ to lower indices and an overbar to indicate the complex conjugate, a normalized state has

$$\langle\psi|\psi\rangle \equiv \delta_{\alpha\beta}\bar{Z}^\alpha Z^\beta \equiv \bar{Z}_\beta Z^\beta = 1 \ , \tag{2}$$

and thus lies on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. A ray is an equivalence class of states up to overall normalization and phase, so $|\psi\rangle \simeq |\psi'\rangle$ if $|\psi\rangle = c|\psi'\rangle$ or $Z^\alpha = cZ'^\alpha$ for all α with a nonzero complex constant c. Thus, the space of rays is the projective Hilbert space, which is isomorphic to CP^n, complex projective space [also denoted P^n or $P_n(\mathbb{C})$], the set of lines in $\mathbb{C}^{n+1}$ passing through the origin. For $Z^0 \neq 0$, CP^n may be given complex coordinates

$$w^i = Z^i/Z^0 \ , \tag{3}$$

where Latin indices from the middle of the alphabet range from 1 to n.

Using units in which $\hbar = 1$, and denoting time derivatives by an overdot, one may write the evolution of the quantum state by a Hermitian Hamiltonian H as

$$|\dot{\psi}(t)\rangle = -iH(t)|\psi(t)\rangle, \ H(t) = H^\dagger(t) \ , \tag{4}$$

or

$$\dot{Z}^\alpha = -iH^\alpha{}_\beta Z^\beta, \ H^\alpha{}_\beta = \bar{H}^\beta{}_\alpha \ . \tag{5}$$

If the evolution undergoes a circuit in ray space, the original state returns to itself up to a phase factor:

$$|\psi(T)\rangle = e^{i\phi(T)}|\psi(0)\rangle \ . \tag{6}$$

Part of this phase $\phi(T)$ may be identified as a dynamical phase

$$\epsilon(T) = -\int_0^T \frac{\langle\psi(t)|H(t)|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} dt \ , \tag{7}$$

but the remainder,

$$\gamma(T) = \phi(T) - \epsilon(T) \ , \tag{8}$$

is geometrical and depends purely on the closed-path evolution of the ray in the projective Hilbert space,[1-3] i.e., on the path in CP^n.

To obtain an explicit expression for $\gamma(t)$ in terms of the coordinate representation (3) of CP^n (assuming for inessential simplicity that $Z^0 \neq 0$ all along the path), let $\phi(t)$ be the phase of $Z^0(t)/Z^0(0)$ and $\epsilon(t)$ and $\gamma(t)$ be given by Eqs. (7) and (8) with T replaced by t. That is, let

$$\gamma(t) = -\frac{i}{2}\ln\frac{Z^0(t)\bar{Z}^0(0)}{Z^0(0)\bar{Z}^0(t)} + \int_0^t \frac{\bar{Z}^\alpha(t')H_{\alpha\beta}(t')Z^\beta(t')}{\bar{Z}_\gamma(t')Z^\gamma(t')} dt' \ , \tag{9}$$

which obviously reduces to (8) when $t = T$. By using (3), (5), and its complex conjugate, one gets the infinitesimal change of (9) with time as

$$\dot{\gamma}(t)dt = -\frac{i}{2}\left[\frac{dZ^0}{Z^0} - \frac{d\bar{Z}^0}{\bar{Z}^0}\right] + \frac{i}{2}\left[\frac{\bar{Z}_\alpha dZ^\alpha - Z_\alpha d\bar{Z}^\alpha}{\bar{Z}_\gamma Z^\gamma}\right] = \frac{i}{2}\left[\frac{\bar{w}_i dw^i - w_i d\bar{w}^i}{1 + \bar{w}_k w^k}\right] \equiv A \ , \tag{10}$$

where $w_i \equiv \delta_{ij} w^j = w^i$, so

$$\gamma(T) = \oint A \ , \tag{11}$$

the integral of the one-form A around the circuit in CP^n. Thus, A acts as a representative for an Abelian connection for the geometrical phase on the projective Hilbert space.

Redefining $\phi(t)$ [say as the phase of $Z^1(t)/Z^1(0)$] for $t\neq 0$ and $t\neq T$ would give Eq. (10) with a gauge-transformed A, but Eq. (11) is invariant, mod2π, under such a gauge transformation.

By Stokes's theorem, one may alternatively write

$$\gamma(T)=\int_S F \ , \tag{12}$$

the integral, over a surface S bounded by the circuit, of the Abelian curvature two-form

$$F\equiv dA=i\frac{\bar{w}_i w_j-(1+\bar{w}_k w^k)\delta_{ij}}{(1+\bar{w}_l w^l)^2}dw^i\wedge d\bar{w}^j \ . \tag{13}$$

Thus, F is an explicit realization of the two-form V given by Eq. (5) of Simon[2] in terms of the coordinates (3) of CP^n.

(Note that my choice of the letters A and F is meant to emphasize the formal similarity with electromagnetism, but of course A and F here live on CP^n and are not to be identified with any actual electromagnetic field.)

Now we may give several ways in which the connection A and its curvature F arises naturally from geometrical structures on CP^n. First, if one takes the S^{2n+1} of normalized states (2) as the Hopf bundle over CP^n, then A and F are (up to a factor of i) the connection and curvature of this U(1) fiber bundle. As a geometrical way of getting this explicitly, let

$$Z^0=\frac{re^{i\phi}}{(1+\bar{w}_k w^k)^{1/2}},\ Z^i=Z^0 w^i \ . \tag{14}$$

Then in terms of $r=\langle\psi|\psi\rangle^{1/2},\phi,w^i$, and $\bar{w}^i$, the standard flat metric on $\mathbb{C}^{n+1}$ becomes

$$\begin{aligned}ds^2(\mathbb{C}^{n+1})&=\delta_{\alpha\beta}dZ^\alpha d\bar{Z}^\beta=dr^2+r^2ds^2(S^{2n+1})\\&=dr^2+r^2[(d\phi-A)^2+ds^2(CP^n)] \ ,\end{aligned} \tag{15}$$

where A is the same as in Eq. (10) and

$$\begin{aligned}ds^2(CP^n)&=2g_{i\bar{j}}dw^i d\bar{w}^j\\&=\frac{(1+\bar{w}_k w^k)\delta_{ij}-\bar{w}_i w_j}{(1+\bar{w}_l w^l)^2}dw^i d\bar{w}^i \ ,\end{aligned} \tag{16}$$

is the Fubini-Study metric[4] on CP^n. An evolution, in which the dynamical phase $\epsilon(t)$ remains zero, corresponds to parallel propagation of the U(1) $e^{i\phi}$ fiber over the CP^n base and, hence, $d\phi-A=0$ along the path. Then $\gamma(T)=\phi(T)=\oint A$ just as Eq. (11) gives.

Second, one may take the Hilbert space $\mathbb{C}^{n+1}$ (with the point 0 blown up to CP^n), to be the natural complex line bundle, with fiber $re^{i\phi}$, over the projective Hilbert space CP^n, as in Chap. XII.3 of Kobayashi and Nomizu[4] and in Example 4.2.2 of Eguchi, Gilkey, and Hanson.[5] Then $F=i\Omega$ in terms of the curvature Ω given on p. 309 of Ref. 4 for $n=1$ and calculated for general n in Example 5.4.1 of Ref. 5. Alternatively, $F/2\pi$ is the first Chern form of this one-dimensional complex vector bundle.[4,5] The Hopf bundle discussed above is then the projection of this line bundle to the circle bundle with $r=1$ and, hence, U(1) fiber $e^{i\phi}$.

Third, one may see that A and F are proportional to the U(1) part of the Riemannian connection and curvature of the Fubini-Study metric (16) on CP^n, which we now calculate. As a complex manifold, CP^n has a complex structure tensor whose components with respect to the complex coordinates w^i and their conjugates $w^{\bar{i}}\equiv\bar{w}^i$ are

$$J^i{}_j=-J^{\bar{i}}{}_{\bar{j}}=-i\delta^i{}_j,\ J^i{}_{\bar{j}}=J^{\bar{i}}{}_j=0 \ . \tag{17}$$

When the raised indices are lowered with the metric (16), one gets the components of the Kähler form

$$J=ig_{i\bar{j}}dw^i\wedge d\bar{w}^j \ . \tag{18}$$

Since CP^n is Kähler, $dJ=0$ (and, in fact, J is covariantly constant), which implies that locally one may write

$$g_{i\bar{j}}=g_{\bar{j}i}=K_{,i\bar{j}},\ g_{ij}=g_{\bar{i}\bar{j}}=0 \ , \tag{19}$$

where the comma denotes partial differentiation.

The Fubini-Study metric (16) may be given by taking the Kähler potential in (19) to be

$$K=\tfrac{1}{2}\ln(1+\bar{w}_k w^k) \ . \tag{20}$$

The Riemannian connection one-forms in the complex coordinate basis are then readily calculable to be

$$\begin{aligned}\omega^i{}_j&\equiv\Gamma^i{}_{jk}dw^k+\Gamma^i{}_{j\bar{k}}d\bar{w}^k=-\frac{\delta^i{}_j\bar{w}_k+\delta^i{}_k\bar{w}_j}{1+\bar{w}_l w^l}dw^k \ ,\\ \omega^{\bar{i}}{}_{\bar{j}}&=\bar{\omega}^i{}_j=-\frac{\delta^i{}_j w_k+\delta^i{}_k w_j}{1+\bar{w}_l w^l}d\bar{w}^k,\ \omega^i{}_{\bar{j}}=\omega^{\bar{i}}{}_j=0 \ ,\end{aligned} \tag{21}$$

and the Riemannian curvature tensor for the Fubini-Study metric may be seen to have components

$$\begin{aligned}R_{abcd}&=g_{ac}g_{bd}-g_{ad}g_{bc}+J_{ac}J_{bd}\\&\quad -J_{ad}J_{bc}+2J_{ab}J_{cd} \ ,\end{aligned} \tag{22}$$

where here and henceforth Latin indices from the beginning of the alphabet range over the n values of the unbarred (holomorphic) coordinates w^i plus the n values of the barred (antiholomorphic) coordinates $w^{\bar{i}}\equiv\bar{w}^i$ (e.g., $x^a=w^i$ for $a=i$ and $x^a=\bar{w}^i$ for $a=i+n$, so a ranges from 1 to $2n$, the real dimension of CP^n).

The connection one-forms Eq. (21) and the curvature two-forms

$$\mathcal{R}^a{}_b\equiv\tfrac{1}{2}R^a{}_{bcd}dx^c\wedge dx^d \ , \tag{23}$$

are (as in any Kähler manifold) purely holomorphic or antiholomorphic in their indices a and b:

$$\begin{aligned}\mathcal{R}^i{}_j&=\bar{\mathcal{R}}^{\bar{i}}{}_{\bar{j}}=2(\delta^i{}_j g_{k\bar{l}}+\delta^i{}_k g_{j\bar{l}})dw^k\wedge d\bar{w}^l \ ,\\ \mathcal{R}^i{}_{\bar{j}}&=\mathcal{R}^{\bar{i}}{}_j=0 \ .\end{aligned} \tag{24}$$

Thus, they rotate the holomorphic and antiholomorphic vectors separately and generate the holonomy group $U(n)=U(1)\times SU(n)$, a subset of the generic holonomy group $SO(2n)$ for a general manifold with real dimension $2n$. The U(1) part of the holonomy is generated by the trace of the holomorphic part of the connection and curvature:

$$\omega \equiv \omega^k{}_k = (n+1)(iA - dK) \ , \tag{25}$$

$$\mathcal{R} \equiv \mathcal{R}^k{}_k = \frac{i}{2} J^a{}_b \mathcal{R}^b{}_a = d\omega = (n+1) iF \ . \tag{26}$$

One may for any Kähler manifold write the Ricci from[4] ρ as

$$\rho = -2i\mathcal{R} = -2iR_{i\bar{j}} dw^i \wedge d\bar{w}^j = 2(\ln \det g_{k\bar{l}})_{,i\bar{j}} dw^i \wedge d\bar{w}^j \ . \tag{27}$$

Then, since $-\rho/4\pi$ is the first Chern form of a Kähler manifold, F is $-2\pi/(n+1)$ times the first Chern form of CP^n and the geometric phase $\gamma(T)$ is $-2\pi/(n+1)$ times the first Chern invariant of the circuit of the ray in the projective Hilbert space.

Because the Fubini-Study metric is Einstein, with

$$R_{bd} \equiv R^a{}_{bad} = 2(n+1) g_{bd} \ , \tag{28}$$

in our normalization, its Ricci form (27) is $-4(n+1)$ times the Kähler form J given in Eq. (18). Hence, one may directly give

$$F = -2J \ . \tag{29}$$

However, the coefficient depends inversely upon the constant of normalization chosen for the metric [e.g., if the coefficient in Eq. (20) were c instead of $\frac{1}{2}$, the coefficient of J in Eq. (29) would be $-1/c$], whereas the coefficient of A (but not of dK) in Eq. (25) and the coefficient of F in Eq. (26) are independent of the scale of the Fubini-Study metric.

Thus, we see explicitly how the geometrical phase factor occurring for the cyclic evolution of a quantum state[1-3] may be generated by the holonomy associated with several natural geometric structures on CP^n, the projective space of rays.

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