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Absorbing-boundary limit for Brownian motion: Demonstration for a model

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We consider one-dimensional diffusion with an absorbing boundary in the context of a model that retains the essential "missing" boundary condition feature that complicates the solution of the Fokker-Planck equation for this problem. The model solution is obtained and some of its features are briefly discussed but our primary purpose is to demonstrate a limiting process that transforms the solution of a related diffusion problem to that for the absorbing-boundary problem. This approach, which we refer to as the absorbing-boundary limit, consists of obtaining the solution for the infinite-space case with the physical space $-\infty < x < \infty$ characterized by separate friction coefficients for $x \gtrless 0$. The solution obtained will be a function of these two quantities, β_1 for $x>0$ and β_2 for $x<0$, and in the limit $\beta_2 \to 0$ describes a process for which the Brownian particle diffuses to $x=-\infty$ when it crosses the origin and does not return to $x>0$. The solution for $x>0$ is thus identical to that for the case where the origin is an absorbing "boundary." This limiting process provides a new method for obtaining a solution to the Fokker-Planck equation with an absorbing-boundary condition and may lead to a result that is in closed form and more transparent than the eigenfunction expansions recently obtained by other means.

I. INTRODUCTION

The solution of the Fokker-Planck equation (FPE) for one-dimensional Brownian motion in the presence of an absorbing boundary leads to some fairly deep mathematical questions.[1,2] These solutions, only very recently obtained, take the form of complicated eigenfunction expansions that are not particularly transparent relative to much of the information we would hope to extract from them. Experience with the conceptually easier problem of determining the solution of the FPE in the absence of any boundary[3] indicates the possible existence of a simpler, closed form solution for the absorbing-boundary problem. Indeed, for the case of a reflecting boundary it is well known[4] that such a solution follows directly from a consideration of a related problem including an image source, obviating the need to directly confront the boundary-value problem and its inherent complications.

Our purpose here is to demonstrate a new approach to the absorbing boundary problem in which we obtain the solution for a reference system characterized by separate friction coefficients, β_1 for $x>0$ and β_2 for $x<0$, and then take the limit $\beta_2 \to 0$. In this limit the solution describes a process identical to that for the $x>0$ half-space problem with absorbing origin since once the Brownian particle crosses the origin it diffuses to $x=-\infty$ and does not return. The analogy made above with the reflecting-boundary problem is only suggestive since our reference problem presents difficulties of its own. However, these appear to be less formidable than those encountered with the actual absorbing-boundary problem and further consideration of this approach seems warranted.

To demonstrate the limit approach to the absorbing-boundary problem we consider a model for diffusion that retains the essential feature of the FPE in this regard—the lack of boundary data for incoming particles. This model has the additional feature in common with the FPE that in an appropriate limit it contracts to the diffusion equation (DE). Of course, the study of model equations has an intrinsic interest (that sometimes develops into an industry), and the results we obtain also provide the basis for a more detailed study.

II. ONE-DIMENSIONAL DIFFUSION WITH AN ABSORBING BOUNDARY

Consider a system of Brownian particles that are uniformly distributed in the half-space $x>0$ and confined at $x=0$ by an absorbing boundary. In the DE description the particle density $n(x,t)$ satisfies the DE subject to the initial condition $n(x,0)=n_0$ and boundary condition $n(0,t)=0$; the latter is an artifice of the description at the DE level which contains no information about particle velocities and cannot distinguish between incoming and emergent particles at the absorbing boundary.

The solution of the above problem is $n(x,t)=n_0 erf x(4Dt)^{-1/2}$, D the diffusion coefficient. The number of particles absorbed in time t is

$$N(t)=\int_0^t ds\, D\frac{\partial n}{\partial x}(x,s)\,|_{x=0} \propto t^{1/2}, \tag{1}$$

and this growth law holds over the entire range $0 \le t \le \infty$.

The FPE description allows us in principle to include the nonzero density at the absorbing boundary due to incoming particles as a boundary condition, but the absence of the explicit value of this quantity makes this a very difficult problem to solve.[1,2] We will consider a model, similar to the Carleman model of the Boltzmann equation,[5] in which the diffusing particle has velocities $\pm\alpha$ and the distribution functions $u_+(x,+\alpha,t)$,

$u_-(x,-\alpha,t)$ replace $n(x,t)$ in providing the basic level of description. This is only a caricature of the FPE, which contains the full spectrum of particle velocities, but it extends the DE description so that the essential property of nonvanishing density at the absorbing boundary is retained. The u_i satisfy the equations[6]

$$\frac{\partial u_+}{\partial t}+\alpha\frac{\partial u_+}{\partial x}=\beta(u_- - u_+) , \tag{2a}$$

$$\frac{\partial u_-}{\partial t}-\alpha\frac{\partial u_-}{\partial x}=\beta(u_+ - u_-) \tag{2b}$$

where β^{-1} is the mean time between collisions and plays a role similar to the friction coefficient that appears in the FPE.[1-6] For this system $D\propto\beta^{-1}$. We have for the problem described above the initial condition $u_+(x,0)=u_-(x,0)=u_0$ and boundary condition $u_+(0,t)=0$ with no companion boundary condition for $u_-(0,t)$. As we will see, the latter circumstance does not lead to serious problems.

Rather than solve for the u_i directly we consider $w(x,t)\equiv u_+(x,t)+u_-(x,t)$ and $v(x,t)\equiv u_+(x,t) - u_-(x,t)$, which for this model also represent the density and (α^{-1} times) the particle current, respectively. Apparently this particular feature allows us to consider the distribution function(s) and its (their) moments on the same footing and provides the simplification that allows us to directly solve the absorbing boundary problem. Introducing $W(x,t)\equiv w(x,t)-2u_0$, we have

$$\frac{\partial W}{\partial t}+\alpha\frac{\partial v}{\partial x}=0 , \tag{3a}$$

$$\frac{\partial v}{\partial t}+\alpha\frac{\partial W}{\partial x}=-2\beta v , \tag{3b}$$

with $W(x,0)=v(x,0)=0$, $W(0,t)\equiv W_0(t)=-[v(0,t)+2u_0]$. Solving for W we find

$$\overline{W}(x,s)=\overline{W}_0(0)e^{-qx} , \tag{4a}$$

$$\overline{v}(x,s)=\overline{W}_0(0)\frac{S^{1/2}}{(S+2\beta)^{1/2}}e^{-qx} , \tag{4b}$$

where the overbar indicates a Laplace transform with s the transform variable and $q\equiv[s(s+2\beta)]^{1/2}$.

III. THE BOUNDARY SOLUTION

The boundary solution follows directly from Eqs. (4); the explicit solution for $x>0$ will not be required here (or later) and so we relegate this to an appendix. Setting $x=0$ in Eqs. (4) and making use of the relationship between W and v at that point [see immediately below Eq. (3)] we have from Eq. (4b)

$$W_0(0)=\frac{2u_0}{s}(s+2\beta)^{1/2}[s^{1/2}+(s+2\beta)^{1/2}]^{-1}$$

or

$$\overline{w}_0(0)=2u_0 s^{-1/2}[s^{1/2}+(s+2\beta)^{1/2}]^{-1} , \tag{5}$$

which provides the "missing" boundary condition. Note that at $x=0$ Eq. (4a) reduces to an identity.

In the remainder of this section we digress from our primary goal and briefly examine some of the properties of the boundary solution. To invert w_0 we take note of the branch points at the origin and at -2β and apply the Laplace inversion theorem;[7] the sole contribution for $\beta>0$ comes from the branch cut between these two points and we find

$$w_0(t)=\frac{2u}{\pi}\int_0^1 dz\,(1-z)^{1/2}z^{-1/2}e^{-2\beta tz} , \tag{6}$$

which is an integral representation for the confluent hypergeometric function $\Phi(\frac{1}{2};2;-2\beta t)$.[8] For small values of the time we have $w_0(t)\approx 1-|b|t$, whereas at large values of t, $w_0(t)\approx|b'|t^{-1/2}$; of course $n(0,t)$ is identically zero in the DE description. The number of particles absorbed in time t is found by integrating Eq. (6) over time. From the above results it immediately follows that at short times this quantity is proportional to t and at long times to $t^{1/2}$. This last result confirms our expectations that at long time the DE description prevails except where there is an *a priori* conflict as in the preceding example.

IV. ABSORBING BOUNDARY LIMIT

Our primary purpose is to use the known solution, Eqs. (4), *to demonstrate* a limiting process that can be used in conjunction with a more tractable problem to provide the solution to the absorbing boundary problem. Our hope is that in the case of the FPE this will lead to a simpler form of solution than those obtained so far.[1,2]

Consider the identical problem as studied above except, for $x>0$ we set $\beta=\beta_1$ and the particles can enter the region $x<0$ in which $\beta=\beta_z$; the initial condition remains the same. This is the reference problem, and in this context the "missing" boundary condition is no longer a complication. The solutions for $x\lessgtr 0$ are completed by matching at the boundary, which then eliminates any need to further consider the boundary value in the solution process. At this point we then take the limit in which the diffusion coefficient for $x<0$ becomes infinite, here $\beta_2=0$ (friction coefficient $\xi_2=0$ in the case of the FPE). In this limit the particles "diffuse" to $x=-\infty$ when they cross from $x>0$ to $x<0$ and they do not return, i.e., they are absorbed.

It is a simple matter to show that in the case of the DE the limit described above reproduces the known result. We now show that our earlier result, Eq. (5), also follows from this limiting process. For the reference problem Eq. (4) will provide the solution for $x>0$, with β replaced by $\beta 1$, and the solution for $x<0$ follows as

$$\overline{w}_2(x,s)=\overline{w}(0,s)e^{-q_2|x|} , \tag{7a}$$

$$\overline{v}_z(x,s)=-\overline{w}(0,s)s^{1/2}(s+2\beta_2)^{-1/2}e^{-q_2|x|} , \tag{7b}$$

where $q_i=q(\beta_i)$. Matching the solutions for v_i at $x=0$ we find

$$\overline{w}(0,s)=(s+2\beta_2)^{1/2}[(s+2\beta_2)^{1/2}+(S+2\beta_1)^{1/2}]^{-1}\frac{2u_0}{s} \tag{8}$$

so that for $\beta_2 \to 0$ we recover Eq. (5). Although the reflecting boundary is not of special interest we note that the limit corresponding to that boundary condition is $\beta_2 \to \infty$ for which Eq. (8) reduces to the initial uniform value.

The absorbing boundary limit approach appears to offer a relatively simple methodology for attacking the absorbing boundary problem. We have absorbed much of the literature related to such problems in radiative and neutron transport theory as well as the recent work cited earlier and did not enounter any mention of a similar method.

V. CONCLUDING REMARKS

We have studied a model for particle diffusion which offers insights into boundary value problems not properly described by the DE. An exact solution for the model has been obtained, and we have confirmed that only possibly at long times can agreement with the DE be expected. Our main interest has been to use the model to demonstrate a limiting process that we believe will provide a new approach to absorbing boundary problems and possibly simplified forms for their solution.

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APPENDIX

For completeness we include the solutions of Eqs. (4) that follow directly by standard methods[9] after some preliminary manipulations. For $t > x/\alpha$

$$W(x,t) = W_0(t - x/\alpha)e^{-\beta x/\alpha} + (\beta x/\alpha)\int_{x/\alpha}^{t} d\tau\, W_0(t-\tau)e^{-\beta\tau}\frac{I_1(\beta z^{1/2})}{z^{1/2}} ,$$

$$v(x,t) = W_0(t - x/\alpha)e^{-\beta x/\alpha} - \beta\int_{x/\alpha}^{t} d\tau\, W_0(t-\tau)e^{-\beta\tau}\left[I_0(\beta z^{1/2}) - \frac{\tau I_1(\beta z^{1/2})}{z^{1/2}}\right] ,$$

and for $t < x/\alpha$, $W(x,t) = v(x,t) = 0$. Here I_n is the standard modified Bessel function and $z \equiv \tau^2 - x^2/\alpha^2$. Note that the boundary effect propagates with a finite speed in contradistinction to the infinite signal speed that is a characteristic of the DE.

[1]R. Beals and V. Protopopescu, J. Stat. Phys. **32**, 565 (1983). See also the references cited here for earlier work.

[2]T. Marshall and E. Watson, J. Phys. A **18**, 3531 (1985).

[3]S. Harris and J. Monroe, J. Stat. Phys. **17**, 377 (1977).

[4]M. Bartlett, *An Introduction to Stochastic Processes* (Cambridge University Press, Cambridge, United Kingdom, 1960), Sec. 5.21.

[5]S. Harris, *An Introduction to the Theory of the Boltzmann Equation* (Holt, Rinehart, and Winston, New York, 1971), Sec. 8.3.

[6]Z. Schuss, *Theory and Applications of Stochastic Differential Equations* (Wiley, New York, 1980), Chap. 6. The equations considered here are the backward equations. As is customary, we consider the forward equations.

[7]Professor J. Tasi provided the inversion proof for the result that follows and I gratefully acknowledge his contribution.

[8]See, e.g., I. Gradshtyn and I. Ryzhik, *Tables of Integrals, Series, and Products,* edited by A. Jeffery (Academic, New York, 1965), Sec. 3.383.

[9]These equations appear as a special case in the theory of electrical transmission lines. See H. Carslaw and J. Jaeger, *Operational Methods in Applied Mathematics* (Oxford University Press, Oxford, United Kingdom, 1948), Chap. IX.