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Overdamped Frenkel-Kontorova model with randomness as a dynamical system: Mode locking and derivation of discrete maps

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An overdamped chain of balls connected by ideal springs in a sinusoidal potential is examined and the response to periodic forcing is discussed from the viewpoint of dynamical systems theory. Discrete multivariate mappings are derived from the coupled differential equations that describe the motion, and the behavior of these mappings is discussed.

I. INTRODUCTION

Systems with few nonlinearly interacting degrees of freedom have been intensely studied recently, and many aspects of their behavior are well understood. Studies of discrete mappings have provided much insight into the behavior of physically relevant continuous nonlinear differential equations.[1] However, many nonlinear systems cannot be described using just a few "effective" degrees of freedom. Attempts to study the behavior of many interacting degrees of freedom by examining systems of many coupled nonlinear mappings have revealed a bewildering array of not easily classifiable behaviors.[2] This complexity is a serious obstacle to the elucidation of simple features that may have broad interest.

This paper discusses the dynamic response of a physically realizable nonlinear system with many degrees of freedom, an overdamped chain of balls connected by identical springs of random length in the presence of a strong sinusoidal potential, and a time-dependent forcing. This system has interest with connection to the dynamics of sliding charge-density waves,[3] but here it will be considered as a simple example of a dynamical system with many nonlinearly interacting degrees of freedom. The behavior of this system is simple enough that one can hope to characterize the nature of the asymptotically-long-time orbits, though several features of the dynamics are not easily describable using a mapping with one degree of freedom.

The equation of motion that we consider describes a one-dimensional chain of balls connected by harmonic springs of random length, all with spring constant k, in a sinusoidal potential. Let the unstretched length of the jth spring be a_j, where the a_j's are chosen from a bounded distribution with mean a_0. The energy functional in terms of the position of the jth particle, $\tilde{x}_j$, is

$$U=\frac{k}{2}\sum_{j=1}^{N}(\tilde{x}_{j+1}-\tilde{x}_j-a_j)^2-V\sum_{j=1}^{N}\cos\tilde{x}_j \ . \tag{1.1}$$

If one writes $x_j=\tilde{x}_j+\beta_j$, then the energy is

$$U=\frac{k}{2}\sum_{j=1}^{N}(x_{j+1}-x_j-a_0)^2-V\sum_{j=1}^{N}\cos(x_j-\beta_j) \ , \tag{1.2}$$

if the β's are chosen to satisfy $\beta_{j+1}-\beta_j=a_0-a_j$. Periodic boundary conditions are employed. The dynamics are taken to be purely dissipative, and the system is driven by a spatially uniform but time-dependent force $F(t)$. One can view $F(t)$ as arising from a tilting of the potential for equal-mass balls, or equivalently, one can imagine that the particles have equal charge and a time-dependent electric field is applied to the system. In this paper, $F(t)$ will always be a series of identical well-separated square-wave pulses of magnitude F and duration $t_{\rm on}$. The equation of motion for the jth particle is

$$\dot{x}_j=-V\sin(x_j-\beta_j)+k(x_{j+1}-2x_j+x_{j-1})+F(t) \ . \tag{1.3}$$

This paper will concentrate on the limit of strong pinning $V \gg k$. However, scaling arguments similar to those used by Fukuyama, Lee, and Rice[4] indicate that on long length scales, effectively the pinning is strong, even if the microscopic value of k is large. They show this by considering the elastic energy cost of a distortion on a length scale ξ and comparing it to the gain in pinning energy. In one dimension the elastic energy cost per unit length is $\sim k/\xi^2$, while the $\sqrt{N}$ fluctuations in the pinning lead to an energy gain per unit length $\sim V/\xi^{1/2}$. For fixed k and V, the long-wavelength distortions are effectively strongly pinned since $\xi^{-1/2} \gg \xi^{-2}$ as $\xi \to \infty$. Therefore, understanding the limit of strong pinning provides insight into the weak-pinning limit also, if one considers long enough length scales. One merely considers effective degrees of freedom coupled by effective springs in an effective potential. However, we will not concern ourselves with this question here, and instead ask about the behavior of the model in the strong-pinning regime when a periodic sequence of force pulses is applied.

The considerations in this paper are relevant to three previously published works which stress different aspects of the problem.[5–7] Reference 5 addresses the question of mode locking. If repeated identical square-wave force pulses are applied to the balls (e.g., if the potential is periodically tilted), there is a tendency for the configuration to be displaced by integral or rational numbers of well periods per pulse.[8] Numerical work on finite systems reported in Ref. 5 indicated that mode locking occurs, and that there is no apparent tendency for the locking to diminish as the system size is increased. Here it will be shown that for sufficiently strong pulses and weak springs, the chain moves an integral number of periods for a range of pulse lengths. This result demonstrates that true harmonic mode locking occurs even in the limit of infinitely large systems. The arguments complement those of Ref. 5 because they are valid for all system sizes but do not address the existence of subharmonic mode locking, which the numerical work demonstrated.

References 6 and 7 concerned with a refinement of the mode locking question. If one starts from a "typical" initial configuration, given that the system eventually reaches a state that is invariant under application of a pulse (except for a translation by an integral number of periods), what can one say about the state that is reached? In those papers it is shown that the eventual state is not a typical mode-locked state, but is rather a configuration that yields an anomalous proportion of particles very near maxima in the potential at the instant the pulse is turned off. The ideas were illustrated using heuristically introduced discrete mappings. Here we *derive* discrete mappings by analytically integrating the coupled differential equations and demonstrating that the errors are bounded and can be made as small as desired by adjusting model parameters. The simplest form of the mapping, described by Eq. (3.7) and (3.8), is closely related to the heuristic map of Refs. 6 and 7, and generalizations, such as those presented in Eqs. (3.3), (3.9), and (3.10) are also obtained. The procedure yields insight into the physical limits that can be described adequately by the maps. The mappings can then be used to investigate numerically subharmonic mode locking in very large systems, as discussed in a companion paper.

Thus this paper has two main purposes. The first is to demonstrate using fairly rigorous arguments that even infinitely large systems can display true mode locking. The second is to show that discrete coupled mappings can be derived for this system in a controlled fashion.

The paper is organized as follows. In Sec. II the mode locking behavior of the system is considered. It is shown that in the limit of weak springs and strong forcing, states exist for which each ball moves exactly an integral number of periods of the sinusoidal potential during a force pulse. Sec. III discusses the discrete time mappings that can be shown to describe the behavior of the differential equations and so correspond to an explicit, physically realizable system. Sec. IV is a discussion of the results and of open questions. In a companion paper, we will describe numerical investigations of the mappings that focus on the mode-locking features observed in the large systems and compare them to those found in single-oscillator systems.

As mentioned above, this work is related to previous work relevant to sliding charge-density waves. However, mathematical aspects of the problem that do not bear directly on experiments are stressed here.

II. RESULTS FOR THE COUPLED DIFFERENTIAL EQUATIONS

A. Properties of metastable states

We first consider the metastable states in the absence of a driving force ($F=0$). We show that for those metastable states which have small distortions (where a bound can be put on the ball separations, and hence the spring forces), the balls are all found within a distance of order k/V from a minimum of the potential.[9] The argument is identical to that used in an investigation of the Frenkel-Kontorova model,[10] and proceeds in two steps. First, one uses a necessary (but not sufficient) condition for stability that the second derivative of the energy with respect to small changes in any one ball position must be non-negative:

$$\frac{\partial^2 U}{\partial x_j{}^2} \geq 0 \ , \tag{2.1}$$

or

$$\cos(x_j-\beta_j) \geq \frac{-2k}{V} \ . \tag{2.2}$$

Second, one notes that in order for the force on each particle to be zero, one requires $\sin(x_j-\beta_j) = \frac{k}{V}(x_{j-1}-2x_j+x_{j-1})$, or $|\sin(x_j-\beta_j)| \leq 2kL/V$, if the particle separations are bounded by L. As $kL/V \to 0$ these two bounds can be satisfied simultaneously only if each particle's distance from a well bottom is less than $2kL/V+O((k/V)^2)$. Thus, we have shown that for this type of metastable state, the deviation of each ball position from a well bottom $|\delta x_j| \equiv f_{\text{frac}}((x_j-\beta_j)/(2\pi))| \leq kL/\pi V$, where $f_{\text{frac}}(y)$ denotes the fractional part of y.

Although it is intuitively clear that metastable states such as these exist, we have not proven their existence. However, one can do a perturbative analysis in the parameter k/V which provides fairly convincing evidence that such states do occur. One starts the calculation by setting $k=0$ and choosing ball positions so that all the ball separations are within the range $-(L-4\pi)$ to $(L-4\pi)$. In zeroth order of perturbation theory ($k=0$) each particle is at the bottom of a well, so $(x_j-\beta_j)/2\pi$ is an integer i_j, where $2\pi|i_{j+1}-i_j| \leq L-2\pi$. Now define z_j to be the deviation from the nearest well bottom: $z_j \equiv x_j-\beta_j-2\pi i_j$. The z_j's satisfy

$$0=k(z_{j+1}-2z_j+z_{j-1})-V\sin z_j-kh_j \ , \tag{2.3}$$

where $h_j \equiv [2\pi(i_{j=1}-2i_j+i_{j-1})+\beta_{j+1}-2\beta_j+\beta_{j-1}]$ obeys $h_j \leq 2L$.

If one writes $z_j=\sum_{n=0}^{\infty} z_j^{(n)}$, where $z_j^{(n)}$ is the contribution of the nth order of perturbation theory, then one wishes to show that the high-order terms are small enough so that the sum converges. Suppose that it has been shown that $\delta_j^{(n)}=\sum_{m=0}^{n-1} z_j^{(m)}$ obeys

$$\frac{k}{V}(\delta_{j+1}^{(n)}-2\delta_j^{(n)}+\delta_{j-1}^{(n)})-\sin\delta_j-\frac{k}{V}h_j=A_j^{(n)}\left[\frac{8k}{V}\right]^n , \tag{2.4}$$

where each $|A_j^{(n)}| \le A_n$, that $|z_j^{(n-1)}| \le B_{n-1}(8k/V)^{n-1}$, and that $\cos\delta_j^{(n)} > \frac{1}{2}$, where A_{n-1} and B_{n-1} are constants of order unity. It will now be shown that a consistent solution is obtained with $|z_j^{(n)}| \le B_n(8k/V)^n$, and B_n and $A_n \le A_{n-1}$.

The nth order of perturbation theory yields an equation for $z_j^{(n)}$ for $n>1$:

$$\frac{k}{V}(z_{j+1}^{(n)} - 2z_j^{(n)} + z_{j-1}^{(n)}) - \sin(\delta_j^{(n)} + z_j^{(n)}) = 0 \ . \tag{2.5}$$

Since it is assumed that $|z_j^{(n)}| \ll 1$, this equation can be expanded to yield

$$\frac{k}{V}(z_{j+1}^{(n)} - 2z_j^{(n)} + z_{j-1}^{(n)}) - z_j^{(n)}\cos\delta_j^{(n)} = A_j^{(n)}(8k/V)^n + O((z_j^{(n)})^2) \ . \tag{2.6}$$

The consistency of this expansion is assured if $z_j^{(n)}$ obeys the bound $|z_j^{(n)}| \le B_n(8k/V)^n$, with B_n of order unity. If one chooses $z_j^{(n)} = -A_j^{(n)}(8k/V)^n/\cos\delta_j^{(n)}$, then direct calculation yields $B_{n+1} \le A_n/4$ and $A_{n+1} \le A_n$. The first-order term in the expansion $z_j^{(1)} = h_j k/V$ obeys the conditions required by the induction argument and it is easily seen that $\delta_j^{(\infty)}$ is of order k/V, so that the condition $\cos\delta_j^{(n)} > \frac{1}{2}$ is obeyed for all n. Therefore, the induction argument shows that the coefficient of each term in the expansion is bounded by a constant independent of n. For $8k/V \ll 1$, the perturbation sum converges, so the expansion is well-behaved. Therefore, it is consistent that metastable states with bounded ball separations do indeed exist.

A slight modification of this argument can be used to show that the number of metastable states grows at least exponentially with the number of particles N when $F=0$ and $V \gg k$. This result is important because it means that the number of states that are invariant under a well-defined time evolution and stable to small perturbations is extremely large, unlike the situation for most one-particle equations of motion.

To show the result, one again starts with a metastable state for which all the spring distortions are less than some bound L. Another metastable state is constructed by moving the jth ball by one potential period, keeping the other balls basically fixed. Because $k \ll V$, the potential barriers are high enough so that this process is possible, and there will be another metastable state with that one particle moved forward by 2π (up to corrections that tend to zero with k/V). Again, one can imagine doing the perturbation theory in k/V for this new assortment of wells to verify that the new configuration does correspond to a metastable state. This procedure can be done with each particle independently, so long as one requires that each degree of freedom remains within $\sim 2\pi$ of its position in the original configuration. This process can be viewed as forming a bit sequence where each degree of freedom is in one of two states (either the original position, or displaced forward by 2π), so the number of possibilities in this restricted situation is $2^N - 1$, where N is the number of balls in the chain. (The state with each particle displaced by one well is equivalent to the original configuration.) Clearly, this number is a lower bound on the number of metastable states.

In other words, one can start with $k=0$ and construct explicitly an exponentially large set of configurations, each of which has ball separations bounded by L. Given a configuration in this set, the perturbation theory in k/V outlined above can be used to find a metastable state where each ball is in the same potential minimum as when $k=0$.

B. Existence of harmonic mode locking

Here it is shown that harmonic mode locking occurs even in the infinite size limit in the limit of very strong pinning ($V \gg k$) and very strong driving pulses ($F \gg V$). By this it is meant that application of a square-wave force pulse of a given amplitude leads to motion of the system by an integral number of potential wells for a range of pulse lengths.

We consider the effects of very short but intense pulses, so that $F \gg 1$ but $Ft_{\rm on}$ is of order unity. Here we show that this model displays harmonic mode locking in that states exist in which every particle in the chain moves exactly an integral number of potential periods for a nonzero range of pulse durations $t_{\rm on}$. By hypothesis, the starting metastable configuration is chosen to be one where the particle separations are bounded by L. When the pulse is turned on, the system is in a metastable state with each particle within a distance $2kL/V$ from the bottom of a well, as shown above in Sec. II A. The force is applied, and the motion is governed by Eq. (1.3). In Sec. III A it is shown that for $F \gg 1$ and $Ft_{\rm on} = O(1)$ the ball separations remain bounded during the entire pulse by L', where $L' \sim L$. This result is intuitively reasonable because $t_{\rm on}$ is so short the configuration does not have time to change much. If $F \gg V \gg k$, then the applied force, which does not change the ball separations, always dominates.[11] Since by assumption $F \gg V \gg k$, after time $t_{\rm on}$ in the presence of the force, the jth particle's position is bounded by $x_j^{\max}(t_{\rm on}) = x_j(t=0) + Ft_{\rm on} + (V+2kL')t_{\rm on}$ and below by $x_j^{\min}(t_{\rm on}) = x_j(t=0) + Ft_{\rm on} - (V+2kL')t_{\rm on}$. Since $x_j(t=0)$ is within $2kL/V$ of a well bottom, given a $\delta > 4kL'/V$, if $Ft_{\rm on}$ is chosen in the range $\pi+\epsilon$ to $3\pi-\epsilon$, where $\epsilon \ge (V+2kL')t_{\rm on} + 2kL/V + \delta$, then $x_j(t_{\rm on}) - x_j(t=0)$ is between $\pi+\delta$ and $3\pi-\delta$ for all j. This condition clearly can be achieved for small enough $t_{\rm on}$ and k/V.

Now consider the motion of the jth particle after the force is turned off. Eventually the system will reach a new metastable state, and if the ball separations are still bounded by L' each particle will end up within a distance $\sim 2kL'/V$ from a well bottom. Just as the pulse is turned off each particle is not too close to any well top, and it is intuitively reasonable that each particle merely falls down towards the nearest minimum.

To demonstrate that this is so, suppose that a particle jumps over a well top during the relaxation process. In order to do this, it must first pass a point where $|\sin(x_j - \beta_j)| \ge 4kL'/V$. Consider the first ball to reach such a point, and denote its position x_l. Once again, x_l obeys (1.3), with $F=0$. Since x_l is the position of the first particle to reach a potential zero, the spring force is still bounded by $2kL'$, so the force on the lth particle must have the same sign as $-V\sin(x_l-\beta_l)$. Therefore, the particle moves in the same direction as in the absence of the springs, and it remains in the well it was in just as the pulse was turned off. Thus, each particle moves the same number of potential periods, and the system exhibits harmonic mode locking.

Thus, we have shown that in the limit where $F \gg V \gg k$, metastable states exist for which application of a force pulse causes each particle to move by exactly an integral number of wells, no matter how many particles are in the system. Although the argument applies for only a limited range of parameters, it answers in the affirmative the question of principle of whether harmonic mode locking can ever occur in an infinite system with randomness.

III. CONSTRUCTION OF THE MAPPING

A. Integration of the equations of motion

In this section the differential equation (1.3) is integrated to derive discrete time-coupled maps that can be used to facilitate investigation of the system. The integration over the time variable can be performed whenever the force F is very large and Ft_{on} is of order unity, but the map simplifies considerably when one also considers the limit $V \gg k$. The errors introduced in the procedure are of order k/F and k/V, so they can be made small by setting the model parameters appropriately. This step is extremely useful if one wishes to study the behavior of large systems numerically.

The construction relies on the fact that when $F \gg k$ and Ft_{on} is of order unity, then the changes in the spring forces during any given pulse are small, even though many pulses can induce large excursions and cause large changes in the spring forces. Thus, one can integrate the equations of motion assuming that the spring forces are constant during a given pulse and make errors that are proportional to k/F.

First assume the configuration $\{x_j(n)\}$ at the start of the nth pulse is known, and define $t=0$ as the start of the pulse. Now consider the equation of motion during the period that the pulse is on:

$$\dot{x}_j = k(x_{j+1}-2x_j+x_{j-1}) - V\sin(x_i-\beta_j)+F \ , \tag{3.1}$$

or

$$\dot{u}_j = k(u_{j+1}-2u_j+u_{j-1}) - V\sin u_j - kd_j + F \ , \tag{3.2}$$

where $u_j = x_j-\beta_j$ and $d_j = -(\beta_{j+1}-2\beta_j+\beta_{j-1})$. If k were zero, then the equations would decouple and each one could be integrated analytically [using the substitution $z_j=\tan(u_j/2)$]. In fact, if one writes $c_j = u_{j+1}-2u_j+u_{j-1} = c_{j0}(n)+\delta c_j(t)$, where $c_{j0}(n)$ is the value of c_j at $t=0$, then if $|\delta c_j(t)|$ is less than a bound $\delta c_{\max}$, the equation of motion can be integrated to yield

$$u_j(t_{\text{on}}) = 2\left[\frac{V}{F'_j} + \tan^{-1}\left\{\alpha_j \tan\left[\frac{F'_j t_{\text{on}}}{2}\alpha_j + \tan^{-1}\left[\frac{1}{\alpha_j}\tan[\tfrac{1}{2}u_j(t=0)] - V/F'_j\right]\right]\right\}\right] + 2\pi f_{\text{int}}\left[\frac{F'_j\alpha_j}{2\pi}t_{\text{on}}\right] + O(kt_{\text{on}}\delta c_{\max}) \ , \tag{3.3}$$

where

$F'_j = F + k[c_{j0}(n)-d_j]$, $\alpha_j = (1-V^2/F'^2_j)^{1/2}$, $f_{\text{int}}(y)$ denotes the largest integer not greater than y, and the $\tan^{-1}$ is defined to be in the first or second quadrant.

Now we show that a $\delta c_{\max}$ of order unity can be found. Suppressing the pulse label n, we examine $\delta c_j(t)$, which obeys the equation of motion

$$\begin{aligned}\frac{d}{dt}\delta c_j(t) &= k[c_{j+1}(t)-2c_j(t)+c_{j-1}(t)] \\ &\quad -k(d_{j+1}-2d_j+d_{j-1}) \\ &\quad -V(\sin u_{j+1} - 2\sin u_j + \sin u_{j-1}) \ .\end{aligned} \tag{3.4}$$

Since $\delta c_{\max}(t)$ is the largest $|\delta c_j(t)|$ and each d_j is less than 8π, one has

$$\frac{d}{dt}|c_{\max}(t)| \le 4\{V + k[c_{\max}(t)+8\pi]\} \ . \tag{3.5}$$

Therefore,

$$\delta c_{\max}(t) \le \left[\frac{V}{k} + 8\pi + c_{\max}(t=0)\right](e^{4kt_{\text{on}}}-1) \ . \tag{3.6}$$

Since it has been assumed that $Ft_{\text{on}} = M = O(1)$ and $k \ll F$, $kt_{\text{on}} \ll M$. Thus, $\exp(4kt_{\text{on}})-1 = 4M(k/F) + O(k/F)^2$. Also by assumption, at $t=0$ the system is in a metastable state, so $c_{\max}(t=0) \le V/k$ (since $|\sin x| \le 1$). Since $V \lesssim F$, these bounds imply that $\delta c_{\max}(t) \le 8M + O(k/F)$. Therefore $\delta c_{\max}$ is of order unity and the discrete form (3.3) is accurate up to corrections of order k/F.

If one considers the limit $F \gg V$, then (3.3) can be simplified considerably. To lowest order in V/F, one finds

$$\begin{aligned}u_j(t_{\text{on}}) &= u_j(t=0) + F'_j t_{\text{on}} \\ &\quad + \frac{V}{F'_j}\{\cos[F'_j t_{\text{on}} + u_j(t=0)] - \cos[u_j(t=0)]\} \ ,\end{aligned} \tag{3.7}$$

where, once again, $F'_j = F + k[u_{j+1}(t=0) - 2u_j(t=0) + u_{j-1}(t=0) - d_j]$. Strictly speaking, these approximations are all valid simultaneously only in the limit $V^2/F^2 \ll k/V$, so that it is consistent to include the spring terms and to neglect the corrections that are second order in V.

Now that $\{u_j(t_{\rm on})\}$, the configuration just as the pulse ends, is known, one needs to find $\{u_j(t\to\infty)\}$, the metastable state reached when the field is turned off and the configuration relaxes. This state can always be found by searching numerically for zeros of the force equations, but substantial simplification occurs if $V \gg k$. Once again the u_j's obey (3.2), but now with $F=0$. If the $u_{j+1}-2u_j+u_{j-1}$ can be bounded by $2L$, then as $k/V\to 0$ the potential term dominates the spring term except within a region of width $\lesssim 2kL/V$ near the tops and bottoms of the wells. Therefore, for most possible ball locations, the spring forces can be neglected. It is clear physically that each particle just rolls down to the nearest well bottom, and arguments similar to those in Sec. II A can be applied to show this as long as no particles are too close to a well top. Since the ball separations are still bounded after the pulse, after relaxation each particle will be within $2kL/V$ of a well bottom. Therefore, it is reasonable to take

$$u_j(n=1)\equiv u_j(t\to\infty)=2\pi f_{\rm int}[u_j(t=t_{\rm on})/2\pi+1/2]\ . \tag{3.8}$$

We have assumed that not too many particles are very near well tops just as the pulse ends at $t=t_{\rm on}$. However, as shown in Refs. 6 and 7, the process of applying repeated pulses to the system tends to cause an accumulation of particles at well tops at the end of the pulse, so it is desirable to improve the approximation by explicitly accounting for the behavior near the well tops. Once again, we use the fact that the changes in the spring forces in the course of one pulse are very small, so up to $O(k/F)$ corrections, the jth particle rolls forward at the end of the pulse if $\tilde f_j=k[u_{j+1}(t_{\rm on})-2u_j(t_{\rm on})+u_{j-1}(t_{\rm on})]-V\sin u_j(t_{\rm on})-kd_j>0$. Since the springs only play a role very near the well tops and bottoms, the sine can be expanded to yield a mapping of the form

$$\begin{aligned}u_j(n+1)\equiv u_j(t\to\infty)=2\pi f_{\rm int}&\left[\frac{u_j(t_{\rm on})}{2\pi}+\Theta(f_j)\right]\\&+\frac{k}{V}[u_{j+1}(t_{\rm on})-2u_j(t_{\rm on})\\&+u_{j-1}(t_{\rm on})-d_j]\ ,\end{aligned} \tag{3.9}$$

where

$$f_j=k[u_{j+1}(t_{\rm on})-2u_j(t_{\rm on})+u_{j-1}(t_{\rm on})]+V\left[f_{\rm frac}\left[\frac{u_j(t_{\rm on})}{2\pi}\right]-\tfrac{1}{2}\right]-kd_j\ ,$$

and $\Theta(x)=1$ if $x>0$ and 0 otherwise.

This mapping is valid even for particles that are very near the well tops unless the f_j's are so small that the difference in spring forces on a particle between $t=t_{\rm on}$ and $t\to\infty$ [which is bounded by $4\pi k/V$ rather than $2\pi kL/V$ for the integer function (3.8)] is sufficient to change the well into which the ball rolls. A metastable state with bounded ball separations that has not been specially selected will have almost every ball in place where (3.9) describes the ensuing metastable state. Whether the atypical states reached by applying repeated pulses to a given configuration are adequately described by (3.9) is an issue that has been investigated numerically, with results that are described in the companion paper. Certainly the subharmonic mode-locking behavior of (3.9) corresponds more closely to that of the coupled differential equations (1.3) than that of (3.8).

When numerical work is performed, it is useful to have a mapping with a quadratic maximum for which numerical errors do not propagate between steps. A simple way to achieve this is to allow for quadratic maxima but not quadratic minima (e.g., one can imagine a potential with cusps at the well bottoms but not the tops), so that each $u_j(t\to\infty)$ is still an integer:

$$u_j(n+1)\equiv u_j(t\to\infty)=2\pi f_{\rm int}(u_j(t_{\rm on})/2\pi)+\Theta(f_j)\ . \tag{3.10}$$

Since a truncation is performed at each step, the numerical errors do not propagate. The mode-locking behavior of this type of mapping will also be discussed in the companion paper.

In order for the derivation of these mappings to be valid, one must place bounds on the ball separations (so the eventual metastable state has each ball very near a well bottom) no matter how many pulses are applied. We have shown that if the ball separations are bounded at order unity when a given pulse starts they remain bounded at the end of that pulse, but we have not excluded the possibility that many pulses could cause large distortions to build up. Here we show that for our system with random spring lengths, if one starts from a uniform configuration, the particle separations do remain bounded at order unity.[12]

The argument uses the fact that if k were 0, then the equation of motion for every particle would be the same. Since k/V and k/F are small, the kd_j are all small and each particle is within $2kL/V$ of a well bottom at the start of the pulse. The randomness causes the particles to move at different velocities, so that distortions build up. Since the distortions cause the velocity to become more uniform, if the d_j are bounded we expect the distortions to be bounded also.

For simplicity, consider the limit $V\ll F$, so that (3.7) applies. (The argument for $V\sim F$ is much more complicated, but the physical reasoning is identical.) Note that, up to order $(V/F)^2$ and $(k/V)^2$ corrections, the quantity $\delta F_j=k(u_{j+1}-2u_j+u_{j-1}-d_j)$ obeys

$$\begin{aligned}\delta F_j(t_{\rm on})=\delta F_j(t=0)+kt_{\rm on}[&\delta F_{j+1}(t=0)-2\delta F_j(t=0)\\&+\delta F_{j-1}(t=0)]\ .\end{aligned} \tag{3.11}$$

Let δF_l be the δF_j with the largest absolute value. If $\delta F_l>0$, then by hypothesis, $\delta F_{l+1}-2\delta F_l+\delta F_{l-1}<0$, so δF_l must decrease during the pulse. Similarly, if $\delta F_l<0$, then it must increase between $t=0$ and $t_{\rm on}$. Therefore, δF_j must remain bounded and of order k, implying the ball separations stay of order unity.

Now consider the period after the force is turned off and the particles fall into the nearest minimum. Although $f_{\rm int}(x)-f_{\rm int}(y)$ can be greater than $x-y$ (e.g., if $x=0.9$ and $y=1.1$), here we know that at the start of the pulse, each x_j is within $O(k/V)$ of being an integer, and that each particle moves the same distance, up to $O(k/F)$ corrections, during the pulse. These facts enable us to show that $|\delta F_j(n+1)| \le |\delta F_j(n)|$.

Since the d_j's are bounded, problems can arise only if the quantity $|u_{j+1}-2u_j+u_{j-1}|$ becomes large, so consider that limit. For simplicity consider the simplest relaxation (3.8). Say $\delta F_j \gg 1$, so that while the pulse is on δF_j decreases. Since

$$f_{\rm int}(x_{j+1})-2f_{\rm int}(x_j)+f_{\rm int}(x_{j-1})=f_{\rm int}(x_{j+1}-2x_j+x_{j-1})-f_{\rm int}(f_{\rm frac}(x_{j+1})-2f_{\rm frac}(x_j)+f_{\rm frac}(x_{j-1}))$$

and since

$$x_{j+1}(t_{\rm on})-2x_j(t_{\rm on})+x_{j-1}(t_{\rm on})<x_{j+1}(0)-2x_j(0)+x_{j-1}(0)$$

(if they are equal than the particles have translated rigidly with respect to each other and the fractional parts must be equal), problems can arise only if

$$f_{\rm frac}(x_{j+1}(t_{\rm on}))-2f_{\rm frac}(x_j(t_{\rm on}))+f_{\rm frac}(x_{j-1}(t_{\rm on}))<-1\ . \tag{3.12}$$

This could happen for, e.g., $x_{j+1}=x_{j-1}=0.1$ and $x_j=0.9$. However, $f_{\rm frac}(x_j)=0$ at $t=0$, and since δF_j is decreasing, x_j increases faster than $\frac{1}{2}(x_{j-1}+x_{j+1})$. This fact implies that (3.12) holds only if $x_{k+1}-x_k$ is large for at least one of the two cases $k=j$ and $k=j-1$. Direct examination of (3.7) shows that the change in $x_{j+1}-x_j$ during any pulse is of order k/F, so (3.12) can never hold. Thus, the ball separations remain bounded at order unity even if infinitely many pulses are applied.

Generalizing these arguments to the case (3.9) is more complicated, but the physical principles are the same. Thus, if one starts the iteration process with a metastable state with bounded ball separations, the ball separations remain bounded no matter how many pulses are applied. This step completes the demonstration that the coupled differential equations (1.3) can be integrated to yield discrete coupled maps, with controllable errors.

B. Modifications of the model leading to other mappings

So far only the model described by (1.3) has been considered. An obvious question is whether the mapping constructed above is robust in the sense that the construction can be performed even if the model is generalized. For instance, one could allow the spring constant k, the potential strength V, or applied force F to depend on j, so that the equation of motion is now

$$\dot{x}_j=k_j(x_{j+1}-x_j)-k_{j-1}(x_j-x_{j-1})-V_j\sin(x_j-\beta_j)+F_j\ .$$

Physically, these variations would correspond to a distribution of spring constants and of ball masses and/or charges. These charges do not affect the underlying symmetry of the metastable states under rigid translation by 2π. We assume here that the variations in, say, $V(\delta V)$ are bounded and of order V. In this section it is shown that the time integration can be performed for many variations of the original equations of motion. However, the metastable state reached after the configuration is allowed to relax sometimes cannot be described by either (3.8) or (3.9), because the spring forces become comparable to V after many pulses.

The key to integrating the equations of motion to obtain the first step of the mapping is that the change in the spring forces during a given pulse must be small compared to the total force acting on the system. As long as $t_{\rm on}$ is kept small and F correspondingly large, while it is on, the applied force will always be by far the largest force acting on each particle, and neglecting the changes in the spring forces is valid, if k/F is small. The arguments presented above that yield (3.3) can be generalized easily to handle spatial variations in k, V, and F in the limit of small k.

However, in order for the second step of the mapping to be describable using (3.8) or (3.9), the ball separations must remain bounded no matter how many pulses are applied. Otherwise, the spring forces can grow to the point that the metastable state no longer has every particle very close to the bottom of a well.

No matter how large V/k is, even if there is no randomness, one can always find a configuration in which the springs are stretched so much that the spring force is of the same order of magnitude as the force from the potential. However, if one were to start with a uniform configuration and apply pulses to it, for some types of randomness the distortions grow up to a bound of order unity, and for some models the distortions grow up to bounds of order V/k. In the latter instance, the simple integer function mapping (3.8) and the quadratic expansion (3.9) are not valid.

If all the randomness is in the springs, then the smallness of k ensures that arguments analogous to those in Sec. III A can be used to show the ball separations remain bounded at order unity. However, when one attempts to generalize these arguments to the cases where either F or V have random components, then they break down. Although the analogously defined δF_j can be shown to decrease from pulse to pulse, the random component analogous to kd_j is no longer of order k so that the distortions are not bounded at order unity.

A simple example of the process by which the ball separations can grow to $O(V/k)$ is the case where β_j (or d_j) is zero for all j, $V_j = V_1$ for odd j, and $V_j = V_2$ for even j. If one starts from a uniform configuration, then by symmetry $x_j(n) = x_1(n)$ for all odd j and $x_j(n) = x_2(n)$ for all even j. If $V_1 > V_2$ and $V_j/F \ll 1$, then using (3.7) and (3.8), it is straightforward to show that the fixed point configurations have $x_2 - x_1$ of order V/k for ranges of Ft_{on}.

Thus, when one attempts to generalize the model, one finds that the balls are often relatively far from the well bottoms in a metastable state, so that the second step of the map (3.9) may not be valid. Nonetheless, even if the map can no longer be derived from the differential equations, it still embodies the essential physical features of metastability and feedback.

C. Relation to previous heuristic mappings

A phenomenological mapping

$$y_j(n) = x_j(n) + t_{\text{on}}\{k[x_{j+1}(n) - 2x_j(n) + x_{j-1}(n)] + F - d_j\}\ , \tag{3.13}$$

and

$$x_j(n+1) = f_{\text{int}}(y_j(n) + 1/2) \tag{3.14}$$

is used in Refs. 5 and 7. (In Ref. 7, d_j is taken to be 0 for all j). This map clearly is closely related to the map defined by (3.7) and (3.8), the major difference being that there is no V/F term in (3.13). The physical reasoning behind this simplification is that the potential wells induce an oscillation in each particle's velocity but do not cause systematic changes in the configuration, whereas the changes in the spring forces from pulse to pulse are vital to mode locking, since they cause the strongly pinned regions of the chain to be pulled forward and the weakly pinned regions to be held back. In other words, the spring forces provide systematic feedback that causes the velocity of the chain to become more uniform, a vital feature of the physics. The potential wells are necessary to obtain many metastable states (indeed, they are the only source of nonlinearity), but it is assumed that they play a vital role only when the pulses are off.

The mapping used in Ref. 6 also differs from those derived here because the random forces d_j were allowed to be of order unity rather than of order k. As discussed above, this difference means that the integer truncation (3.14) can no longer be derived from the differential equations.

IV. DISCUSSION

This paper has examined the response of an overdamped chain of particles connected by springs in a sinusoidal potential to square-wave force pulses. It was shown that mode-locking behavior is observed in the system even in the limit of infinite size. In addition, it was shown that discrete time multivariate mappings can be derived in a controlled fashion from the differential equations describing the motion. Numerical investigation of mode-locking behavior of very large systems is made possible by use of the map.

A major advantage of this work is that the dynamical properties of the chain can be examined using controlled approximations. Both the demonstration of mode locking and the construction of the discrete dynamical systems are put on a firm footing, since the errors are bounded and can be made as small as desired by adjusting the model parameters. However, one may hope that the work has more general applicability, which would be the case if the system considered here has properties that are typical of large classes of nonlinear systems with many degrees of freedom.

Aside from the results already obtained on this system that are discussed in Refs. 5–7, many promising avenues remain to be explored. The mode-locking behavior of the different maps derived here is discussed in a companion paper. In addition, a very appealing feature of the mappings is that there is some control on the amount of complexity. For the parameter ranges of the mapping considered in this paper, as well as for the continuous time equations of motion (1.3), only fixed points and periodic orbits are observed for any initial configuration. However, if the "feedback" parameter k is increased, then more complex behavior (including apparent chaos) is observed. This issue will be discussed in greater detail in the companion paper outlining the numerical results.

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[8]Harmonic mode locking corresponds to a displacement of each ball by an integral number of well periods per pulse, while subharmonic mode locking corresponds to a rational number (e.g., a translation by $2\pi m$ every n pulses).

[9]There also exists an extremely large number of metastable states whose distortions are extremely large (of order V/k), but we will defer considering them until Sec. III C.

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[11]The existence of mode locking serves to show that $L' \leq L + 2\pi$.

[12]However, in Sec. III B we shall see that this condition does not hold for some related sets of differential equations.