Field-theoretic renormalization group and turbulence

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(Received 16 April 1987)

The recent analysis of the fluctuating Navier-Stokes equations with a random force with a k^{-y} spectrum by Yakhot and Orszag [Phys. Rev. Lett. **57**, 1722 (1986)] is repeated using field-theoretical renormalization techniques. A dimensional expansion around the full equilibrium state in two spatial dimensions is performed to one-loop order. The results differ from those of Yakhot and Orszag in a number of ways. A nontrivial fixed point is found which is the natural extension of that presented by Forster, Stephen, and Nelson [Phys. Rev. A **16**, 732 (1977)], and various correlation functions in crossover form are presented. The application of the model to turbulence, i.e., in cases where the Kolmogorov $\frac{5}{3}$ law is obtained, is analyzed in detail. Unlike the work of Yakhot and Orszag, a choice of the random force correlation exponent ($y = -1.5851$ in three dimensions) is found which gives the Kolmorov $\frac{5}{3}$ law at high wave vectors.

I. INTRODUCTION

Renormalization-group techniques have been successfully applied to a variety of dynamical problems.[1–9] Recently, Yakhot and Orszag[10] have analyzed a variant of model B of Forster *et al.*[3] with an application to fully developed turbulence in mind. (Also see the work of De Dominicis and Martin[11] and of Fournier and Frisch[12] for earlier treatments of the problem.) One major modification to model B was introduced; namely, they allowed the random force correlation to decay with an arbitrary exponent y ($y=-2$ in model A and $y=0$ in model B) and ultimately chose $y=d$ (d is the spatial dimension) in order to obtain the Kolmogorov $\frac{5}{3}$ law for the kinetic energy spectrum in the IR limit. Once this was done, they proceeded to compute other exponents and, remarkably, scaling law amplitudes, all in reasonable agreement with available experiments or simulations. There were however a number of technical difficulties with this approach, stemming from the fact that the critical dimensions of the random force and of the generalized viscosity were different unless $y=-2$ when $d=2$.

Here this problem will be reanalyzed within the context of field-theoretical renormalization techniques,[2,5,6,8,9] in the spirit of Refs. 8 and 9. In particular, a double expansion about $d=-y=2$ will be performed (see, e.g., Ref. 8 for a similar application) and the renormalization will be carried out using the minimal subtraction method.[13] However, unlike what was found in Ref. 8, having the noise exponent different than its equilibrium value leads to nontrivial modifications to other exponents, although they will go continuously over to those found by Forster *et al.*[3] when $y=-2$ for model A.

In the next section, a path integral representation of the Martin-Siggia-Rose generating functional[14,11] is presented.[2] The perturbation expansion associated with this problem is analyzed in Sec. III. As was the case in Ref. 10, the viscosity and the nonlinear coupling constant are found to be marginal when $\epsilon \equiv 4+y-d$ vanishes; however, the noise becomes marginal when $\epsilon+2+y$ vanishes. In order that the theory be *both* IR and UV renormalizable, it is necessary that both quantities vanish simultaneously, i.e., for $d=-y=2$, the equilibrium case discussed in model A.[3] Deviations from this case are discussed within the context of an ϵ expansion in which it is assumed that $2+y \sim O(\epsilon)$. A nontrivial fixed point which differs from that found in Ref. 10, but which is a generalization of that found in Ref. 3, is obtained.

Scaling forms for the energy spectrum, velocity time correlation function, energy dissipation rate, and the average Green's function are presented in Sec. IV. Since the wave-vector dependence of these quantities depends on whether the trivial or nontrivial fixed point is IR or UV stable (this in turn is governed by the sign of ϵ), these quantities are presented in crossover form. In any spatial dimension, there are three choices of y which give the Kolmogorov $\frac{5}{3}$ law, and these are considered in some detail. Finally, Sec. V contains some concluding remarks.

II. PRELIMINARY REMARKS

As is commonly done, the turbulent flow will be modeled by the fluctuating Navier-Stokes equations; i.e.,

$$\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t}+\mathbf{v}(\mathbf{r},t)\cdot\nabla\mathbf{v}(\mathbf{r},t) = -\nabla\left[\frac{p_h(\mathbf{r},t)}{\rho}\right]+\nu\nabla^2\mathbf{v}(\mathbf{r},t)+\mathbf{f}(\mathbf{r},t) \tag{2.1a}$$

and

$$\nabla\cdot\mathbf{v}(\mathbf{r},t)=0. \tag{2.1b}$$

Henceforth, an incompressible fluid with density ρ, kinematic viscosity ν, and hydrostatic pressure p_h, is assumed. The last term on the right hand side of Eq. (2.1a), $\mathbf{f}(\mathbf{r},t)$, is a random forcing function which is assumed to have zero mean and Gaussian statistics. For the moment, the variance of $\mathbf{f}(\mathbf{r},t)$ will not be specified.

Since the flow is assumed to be incompressible, only the transverse part of Eq. (2.1a) is needed; by Fourier transforming Eq. (2.1a) in space and time, this is easily shown to satisfy

$$(i\omega+\nu k^2)v^{\alpha}(\mathbf{k},\omega)-\frac{i}{2}V_{\mathbf{k}}^{\alpha;\beta,\gamma}\int\int\frac{d^dk_1\,d\omega_1}{(2\pi)^{d+1}}v^{\beta}(\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)v^{\gamma}(\mathbf{k}_1,\omega_1)-\mathbf{f}_t(\mathbf{k},\omega)=0, \tag{2.2}$$

where henceforth, Greek superscripts denote Cartesian components, repeated indices are summed, $\mathbf{f}_t$ is the transverse part of the forcing function, the Fourier transforms are defined by

$$\mathbf{v}(\mathbf{k},\omega)\equiv\int d^dr\int dt\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{v}(\mathbf{r},t), \tag{2.3}$$

etc. The vertex function V is defined by

$$V_{\mathbf{k}}^{\alpha;\beta,\gamma}\equiv k^{\beta}\Phi_{\mathbf{k}}^{\alpha,\gamma}+k^{\gamma}\Phi_{\mathbf{k}}^{\alpha,\beta}, \tag{2.4a}$$

where

$$\overleftrightarrow{\Phi}_{\mathbf{k}}\equiv\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}}. \tag{2.4b}$$

The starting point of the field-theoretic renormalization-group calculation is the path integral representation of the Martin-Siggia-Rose[14,11] generating functional, Ξ, which can be obtained from the equations of motion, cf. Eqs. (2.1a)–(2.2), in a variety of ways[2,15]. For example, following Jensen,[2] it follows that

$$\begin{aligned}\Xi(\xi,\zeta)\equiv\int & D[\mathbf{v}(\mathbf{r},t)]D[\tilde{\mathbf{v}}(\mathbf{r},t)]\\ &\times\Big\langle\exp\Big[i\int\int\frac{d^dk\,d\omega}{(2\pi)^{d+1}}\tilde{v}^{\alpha}(-\mathbf{k},-\omega)\cdot\Big\{(i\omega+\nu k^2)v^{\alpha}(\mathbf{k},\omega)-f_t^{\alpha}(\mathbf{k},\omega)\\ &\qquad-\frac{i}{2}V_{\mathbf{k}}^{\alpha;\beta,\gamma}\int\frac{d^dk_1\,d\omega_1}{(2\pi)^{d+1}}\,v^{\beta}(\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)v^{\gamma}(\mathbf{k}_1,\omega_1)\Big\}\\ &\qquad+\xi(-\mathbf{k},-\omega)\cdot\tilde{\mathbf{v}}(\mathbf{k},\omega)+\zeta(-\mathbf{k},-\omega)\cdot\mathbf{v}(\mathbf{k},\omega)\Big]\Big\rangle,\end{aligned} \tag{2.5}$$

where $\langle\ \rangle$ denotes an average over the random forcing function. Note that the domain of the path integral integrations includes only the transverse fields.

For Gaussian statistics, the average in Eq. (2.5) is easily carried out, and hence,

$$\Xi(\xi,\zeta)\equiv\int D[\mathbf{v}(\mathbf{r},t)]D[\tilde{\mathbf{v}}(\mathbf{r},t)]\,\exp[iL_0+iL_1+\xi(-\mathbf{k},-\omega)\cdot\tilde{\mathbf{v}}(\mathbf{k},\omega)+\zeta(-\mathbf{k},-\omega)\cdot\mathbf{v}(\mathbf{k},\omega)], \tag{2.6}$$

where

$$iL_0\equiv\int\int\frac{d^dk\,d\omega}{(2\pi)^{d+1}}i\tilde{\mathbf{v}}(-\mathbf{k},-\omega)\cdot(i\omega+\nu k^2)\mathbf{v}(\mathbf{k},\omega)-\tfrac{1}{2}\tilde{\mathbf{v}}(-\mathbf{k},-\omega)\cdot(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\Omega(k,\omega)\cdot\tilde{\mathbf{v}}(\mathbf{k},\omega), \tag{2.7a}$$

$$iL_1\equiv\frac{\lambda}{2}\int\int\int\int\frac{d^dk\,d\omega\,d^dk_1\,d\omega_1}{(2\pi)^{2(d+1)}}\,\tilde{v}^{\alpha}(-\mathbf{k},-\omega)V_{\mathbf{k}}^{\alpha;\beta,\gamma}v^{\beta}(\mathbf{k}-\mathbf{k}_1,\omega-\omega_1)v^{\gamma}(\mathbf{k}_1,\omega_1), \tag{2.7b}$$

and where

$$\langle\mathbf{f}(\mathbf{k},\omega)\mathbf{f}(\mathbf{k}',\omega')\rangle\equiv(2\pi)^{d+1}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\Omega(k,\omega). \tag{2.7c}$$

Following Forster *et al.*[3] a coupling constant, λ, has been included in the definition of the interaction Lagrangian, cf. Eq. (2.7b).

In the absence of persistent forces, the motion of the system is governed by the functional extrema of $\Gamma[\langle\tilde{\mathbf{v}}(\mathbf{r},t)\rangle,\langle\mathbf{v}(\mathbf{r},t)\rangle]$, the generating functional of one particle irreducible (1-IP) diagrams. It is related to Ξ by:

$$\begin{aligned}\Gamma[\langle\tilde{\mathbf{v}}(\mathbf{r},t)\rangle,\langle\mathbf{v}(\mathbf{r},t)\rangle]&\equiv-\ln[\Xi(\xi,\zeta)]+\int d^dr\,dt\;\xi(\mathbf{r},t)\cdot\langle\tilde{\mathbf{v}}(\mathbf{r},t)\rangle\\&\quad+\zeta(\mathbf{r},t)\cdot\langle\mathbf{v}(\mathbf{r},t)\rangle.\end{aligned} \tag{2.8}$$

III. PERTURBATION EXPANSION AND RENORMALIZATION

The presence of L_1 in Eq. (2.6) makes the exact evaluation of Γ impossible, and a perturbative expansion must be used. As is well known,[16] Γ is just the sum of all 1-IP terms in the series.

The one-loop corrections to the low-order vertex functions are represented diagrammatically in Fig. 1. Following Ref. 8, the lines on the diagrams correspond to

$$\langle \mathbf{v}(\mathbf{k},\omega)\mathbf{v}(\mathbf{k}',\omega')\rangle_0 = \frac{2\Omega(k,\omega)k^2\overleftrightarrow{\Phi}_{\mathbf{k}}}{\omega^2+\nu^2k^4}(2\pi)^{d+1}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') \equiv \text{———} , \tag{3.1a}$$

$$\langle \mathbf{v}(\mathbf{k},\omega)\tilde{\mathbf{v}}(\mathbf{k}',\omega')\rangle_0 = \frac{i\overleftrightarrow{\Phi}_{\mathbf{k}}}{i\omega+\nu k^2}(2\pi)^{d+1}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') \equiv \text{———}\sim\sim\sim , \tag{3.1b}$$

and

$$\langle \tilde{\mathbf{v}}(\mathbf{k},\omega)\mathbf{v}(\mathbf{k}',\omega')\rangle_0 = \frac{i\overleftrightarrow{\Phi}_{\mathbf{k}}}{-i\omega+\nu k^2}(2\pi)^{d+1}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') \equiv \sim\sim\sim\text{———} , \tag{3.1c}$$

where the subscript 0 denotes an average in the zeroth-order theory.

At this point, the correlation of the random force must be specified, and, as was done by Yakhot and Orszag,[10] it is assumed that

$$\Omega(k,\omega) \equiv \chi k^{-y}. \tag{3.2}$$

Note that the choices $y=-2$ or 0 correspond to model A (equilibrium) or model B of Ref. 3, respectively. Having y noninteger complicates the renormalization of the theory. As is well known, the starting point is to find the spatial dimension where the primitively divergent (i.e., relevant) quantities are marginal or where they have behaves as the UV cutoff, Λ, to some integer power (typically as Λ^2 which is then eliminated by a mass subtraction[16]). Simple power counting arguments show that the integrals corresponding to Figs. 1(a), and 1(c1)–(c3) are logarithmically UV divergent when

$$\epsilon \equiv 4+y-d=0 \tag{3.3}$$

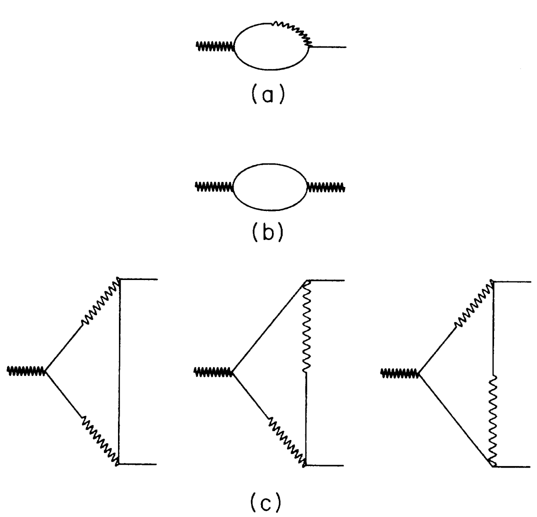

FIG. 1. Primitively divergent one-loop Feynman diagrams.

but that Fig. 1(b) becomes marginal when $\epsilon+2+y$ vanishes. In equilibrium, $y=-2$ and all three classes of integrals become marginal in $d=2$, the upper critical dimension for the problem.[3] Moreover, it is easy to show that these are the only primitively divergent types of vertex functions. On the other hand, Yakhot and Orszag present an expansion around $\epsilon=0$ and argue that $y=d$. If this is the case, then the contributions to the perturbation expansions of the type depicted in Fig. 1(b) will be super-renormalizable UV but nonrenormalizable IR.

In order to get around this difficulty, a double expansion will be carried out; namely, the perturbation series is expanded about $d=-y=2$ assuming that $\epsilon/(\epsilon+2+y) \sim O(1)$. This is similar to the analysis given at the end of Ref. 8, but as will soon be shown a very different result is obtained.

The theory is renormalized by the minimal subtraction method.[13] Specifically, the integrals are examined slightly below their upper critical dimensions, with the upper cutoff set to infinity, with the external frequencies set to zero, and with all external wave vectors scaled by some wave-vector scale κ. Renormalized viscosity ν_R, noise amplitude χ_R, and coupling constant λ_R are then introduced so as to cancel the poles in ϵ or $(\epsilon+2+y)$. (Thus only the divergent parts of the integrals need to be calculated.)

By using the results for the integrals depicted in Fig. 1 contained in the Appendix, it follows that to one-loop order,

$$\nu = \nu_R\left[1-\frac{A_d u_R}{\epsilon}\right], \tag{3.4a}$$

$$\chi = \chi_R\left[1-\frac{A_d u_R}{\epsilon+2+y}\right], \tag{3.4b}$$

and

$$\lambda = \lambda_R , \tag{3.4c}$$

where

$$A_d \equiv \frac{S_d(d^2-2)}{2d(2\pi)^d(d+2)} \tag{3.5a}$$

and

$$u_R \equiv \frac{\chi_R \lambda_R^2 \kappa^{-\epsilon}}{\nu_R^3} \tag{3.5b}$$

is the dimensionless coupling constant for the problem. In Eq. (3.5a), $S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ is the area of the d-dimensional unit sphere, and hence, $A_2 = 1/(16\pi)$.

Equation (3.4a) was found in Ref. 3 and Eq. (3.4b) is a simple generalization of their result (this was expected, cf. Appendix); moreover, as was argued in Refs. 3 and 4, the fact that $\lambda = \lambda_R$ is due to the symmetry of the Lagrangian under Galilean transformations and will hold to all orders. Finally, for the equilibrium case, the system obeys the fluctuation dissipation theorem;[3,4] this is reflected in the fact that $\nu_R/\chi_R = \nu/\chi$ for $y = -2$.

Once the relationship between the bare and renormalized quantities is known, it is a simple matter to derive a renormalization-group equation by noting that the renormalization procedure has introduced an additional parameter, κ, which is absent from the bare theory. Hence, for any function F

$$\left[\frac{\partial F}{\partial \kappa}\right]_{\nu,\lambda,\chi} = 0, \tag{3.6}$$

or

$$\kappa \left[\frac{\partial F}{\partial \kappa}\right]_{\nu_R,\lambda_R,u_R} + \beta_u \left[\frac{\partial F}{\partial u_R}\right]_{\nu_R,\lambda_R,\kappa} + \beta_\nu \left[\frac{\partial F}{\partial \nu_R}\right]_{u_R,\lambda_R,\kappa} = 0. \tag{3.7}$$

The Wilson functions are given by

$$\beta_u = -u_R \left[\epsilon - u_R A_d \left[3 - \frac{\epsilon}{\epsilon + 2 + y}\right]\right] \tag{3.8a}$$

and

$$\beta_\nu = -A_d u_R \nu_R \tag{3.8b}$$

to one-loop order. Note that the term in $\epsilon/(\epsilon+2+y)$ is absent in the work of Yakhot and Orszag, since the noise renormalization was ignored in computing the coupling dimensionless coupling constant Wilson function.

It is easy to show that the general solution to Eq. (3.7) must have the form

$$F = F[\{\mathbf{k}_i\}_{i=1}^n, \{\omega_i\}_{i=1}^n; U(\kappa), W(\kappa)], \tag{3.9}$$

where, to one-loop order,

$$U(\kappa) \equiv \frac{u_R u^*}{u_R + \kappa^{-\epsilon}(u^* - u_R)}, \tag{3.10a}$$

$$W(\kappa) \equiv \nu_R \left[\frac{u^* \kappa^{-\epsilon}}{u_R + \kappa^{-\epsilon}(u^* - u_R)}\right]^{-\gamma/\epsilon}, \tag{3.10b}$$

$$\gamma \equiv \frac{\epsilon}{3 - \epsilon/(\epsilon + 2 + y)}, \tag{3.11a}$$

and

$$u^* \equiv \frac{\gamma}{A_d}. \tag{3.11b}$$

Note that for $y = -2$, the coupling constant fixed point, u^*, is exactly the same as that in Ref. 3; this is not surprising, but would not be the case if the noise renormalization were omitted.

From Eqs. (3.8) or (3.10) it follows that there are two fixed points: $u_R = u^*$ and $u_R = 0$. The former is stable in the IR limit for $\epsilon > 0$ and the latter for $\epsilon < 0$; the reverse is true in the UV limit. In addition, note that even when it is stable, the mean-field fixed point (i.e., $u_R = 0$) is obtained only for $u_R < u^*$. (Analogous behavior was observed in Ref. 3.)

A further constraint on the functional form of any quantity is imposed by dimensional analysis. For example, if

$$[F] \sim (\text{length})^{d_l} (\text{time})^{d_t}$$

then, cf. Eq. (3.9), it follows that

$$F = R^{d_l}[W(\kappa R) R^{-2}]^{-d_t} F\left[\{R\mathbf{k}_i\}_{i=1}^n, \left\{\frac{\omega_i}{W(\kappa R) R^{-2}}\right\}_{i=1}^n; U(\kappa R)\right] \tag{3.12}$$

for any length scale R. In particular, it must hold for $R = k^{-1}$ (k is one of the external wave vectors), and in this case,

$$F = k^{-d_l - 2d_t}[W(\kappa/k)]^{-d_t} F\left[\frac{\omega}{W(\kappa/k)k^2}, U(\kappa/k)\right], \tag{3.13}$$

where, for the sake of concreteness, a two-point function (which depends only on a single wave vector and frequency) has been assumed.

There are two fixed points, $U = 0$ and $U = u^*$. For $\epsilon > 0$, the former governs the scaling form of any function for $k \gg \kappa$ whereas the latter governs the behavior for $k \ll \kappa$. The reverse is true for $\epsilon < 0$. Analogous behavior was observed in Refs. 3 and 10–12.

IV. SCALING FORMS OF CORRELATION FUNCTIONS

A. The energy spectrum

An important quantity in turbulence is the energy spectrum $E(k)$, ($d_t = -2$, $d_l = 3$). Equations (3.10a), (3.10b), and (3.13) show that

$$E(k)=k[W(\kappa/k)]^2U(\kappa/k)E[U(\kappa/k)], \tag{4.1}$$

where an explicit factor of U has been included to reflect the fact that $E(k)$ must vanish in the absence of the random force. In the appropriate limits, Eq. (4.1) gives

$$E(k)\sim\begin{cases} k^{1-\epsilon}\nu_R^2\left[\dfrac{u^*}{u^*-u_R}\right]^{1-2\gamma/\epsilon} u_R E(0) & \text{if } k^\epsilon\to\infty \\ k^{1-2\gamma}\nu_R^2\left[\dfrac{u^*}{u_R}\right]^{-2\gamma/\epsilon} u^*E(u^*) & \text{if } k^\epsilon\to 0 \end{cases} \tag{4.2}$$

Note that Eq. (4.2) reduces to the result of model A of Ref. 3 when $y\to-2$. In addition, the nonclassical exponent agrees to first order in four dimensions with the results of model B of Ref. 3 when $y=0$ and with the results of Refs. 11 and 12(b) when $\epsilon/(\epsilon+2+y)\ll 3$.

As was noted by Yakhot and Orszag, when $U\to u^*$, the exponent y can be chosen in order to give the Kolmogorov $\frac{5}{3}$ law (e.g., $\gamma=\frac{4}{3}$); this will happen for $\epsilon=\frac{8}{3}$ or for $y=y_\pm$, where

$$y_\pm=\frac{9d-22\pm(100+132d+9d^2)^{1/2}}{12}. \tag{4.3}$$

In three dimensions, $y_+=2.4184$ and $y_-=-1.5851$. Note, however, that as $d\to 2$, only the negative root approaches -2 $[y+2\sim-\frac{2}{5}(2-d)]$ and it is not clear if the choice $y=y_+$ is consistent with the perturbation theory. It is easy to see that ϵ is positive for the choice $y=y_+$ and negative when $d>2$ for $y=y_-$. Hence, in the latter case the nontrivial fixed point governs the high-wave-vector behavior of the energy spectrum. The reverse was concluded by Yakhot and Orszag who found that $y=d$ gave Kolmogorov's $\frac{5}{3}$ law as $k\to 0$ (the difference is again related to the noise renormalization).

The choice $\epsilon=\frac{8}{3}$ also gives the Kolmogorov $\frac{5}{3}$ law at high wave vectors. However, now the energy spectrum will diverge as $k^{-1.068}$ for $k\to 0$ in three dimensions. This is nonintegrable unless cut off at small wave vectors, thereby introducing another parameter. Moreover, as will be shown below, the choice $\epsilon=\frac{8}{3}$, while producing the Kolmogorov static exponent, gives a dynamic exponent which is mean-field and not in agreement with Kolmogorov's analysis.

The scaling form of the energy spectrum can be obtained by expressing the renormalized perturbation expansion in terms of the scaling functions $U(\kappa/k)$ and $W(\kappa/k)$ and ϵ-expanding the explicit κ dependence to the appropriate loop order. To zero-loop order this gives

$$E(k)\sim\frac{S_d(d-1)}{d(2\pi)^d}k[W(\kappa/k)]^2U(\kappa/k)\lambda_R^{-2}. \tag{4.4}$$

Figure 2 shows the behavior of $E(k)$ for various choices of the parameters. As was mentioned before, mean-field behavior is obtained for long wavelengths when $\epsilon<0$; non-mean-field exponents are obtained for high wave vectors, with the crossover occurring at $k\approx\kappa$. Moreover, even for $\epsilon>0$ nontrivial exponents can still be obtained for high wave vectors for the correct choice of y. Hence, in this case, if the application of the model to a turbulent fluid is valid, κ should be identified with the low-wave-vector edge of the inertial subrange.

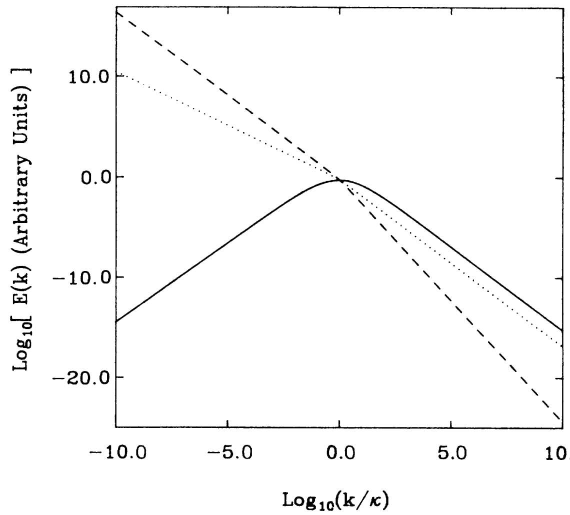

FIG. 2. Examples of the energy spectrum in three spatial dimensions. The exponent y has been chosen to give Kolmogorov's $\frac{5}{3}$ law in some region. The curves correspond to (a) $\epsilon=\frac{8}{3}$ (· · · ·), (b) $y=y_+$ (— — —), and (c) $y=y_-$ (——). In each case, $u_R/u^*=0.5$.

On the other hand, if y is chosen to give some specified exponent (e.g., $\frac{5}{3}$ for $y=y_+$) in the low-wave-vector regime, then κ can be identified with the upper end of the inertial range, although if this is the case, the behavior for the dissipative range will not be what is found experimentally [e.g., in Fig. 2 curve b, $E(k)\sim k^{-2.4184}$ for $k\gg\kappa$]. In addition, this choice will result in an energy spectrum which is nonintegrable at small wave-vectors if $\gamma>1$ (cf. Fig 2, curves a and b). These divergences can be handled elegantly by introducing composite operators [i.e., $\mathbf{v}(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)$] with additive renormalizations[16] into the calculation; to low-order this is tantamount to introducing a lower cutoff.

B. Dynamic velocity correlations

Another quantity of interest is the fluid velocity autocorrelation function, i.e., $\langle\mathbf{v}(\mathbf{r},t)\mathbf{v}(0,0)\rangle$. Denote the space and time Fourier transform of this quantity by $\overleftrightarrow{S}(\mathbf{k},\omega)$. Equation (3.13) implies that

$$\overleftrightarrow{S}(\mathbf{k},\omega)=(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})k^{-d}W(\kappa/k)F\left[\frac{\omega}{W(\kappa/k)k^2},U(\kappa/k)\right]. \tag{4.5}$$

Moreover, by repeating the argument which led to Eq. (4.4) it follows that

$$\overleftrightarrow{S}(\mathbf{k},\omega)\approx(\overleftrightarrow{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})\frac{2U(\kappa/k)W(\kappa/k)k^{-d}}{\lambda_R^2\left[1+\left[\dfrac{\omega}{W(\kappa/k)k^2}\right]^2\right]}. \tag{4.6}$$

Note that Eq. (4.6) reduces to Eq. (4.4) when integrated over frequency and multiplied by $k^{d-1}S_d(d-1)/[d(2\pi)^{d+1}]$.

Equation (4.6) allows dynamic exponents z ($\omega \sim k^z$) to be identified. For $k^\epsilon \to 0$, $z=2-\gamma$, but when $k^\epsilon \to \infty$, $z=2$. In particular, when γ is chosen to make the Kolmogorov $\frac{5}{3}$ law hold, $z=\frac{2}{3}$ for $k^\epsilon \to 0$. As was noted above, $\epsilon = \frac{8}{3}$ also gives a Kolmogorov exponent for the energy spectrum, but in this case, $z=2$ which is inconsistent with Kolmogorov's argument (which gives $z=\frac{2}{3}$).

More generally, an effective wave-vector-dependent viscosity, $\nu_{\text{eff}}(k)$, can be identified from Eq. (4.6); i.e.,

$$\nu_{\text{eff}}(k) \equiv W(\kappa/k) \sim \begin{cases} \nu_R \left[\dfrac{u^*}{u^*-u_R}\right]^{-\gamma/\epsilon} & \text{as } k^\epsilon \to \infty \\ \nu_R \left[\dfrac{u^*}{u_R}\right]^{-\gamma/\epsilon} \left[\dfrac{k}{\kappa}\right]^{-\gamma} & \text{as } k^\epsilon \to 0 \end{cases} . \tag{4.7}$$

The decay of velocity correlations at a given point of the system can be described by $\langle \mathbf{v}(0,t)\mathbf{v}(0,0)\rangle$. From Eq. (4.6) it is easy to show that

$$\langle \mathbf{v}(0,t)\mathbf{v}(0,0)\rangle = \overleftrightarrow{\mathbb{1}} \int_0^\infty dk\, E(k) e^{-W(\kappa/k)k^2|t|} \tag{4.8}$$

to zero-loop order, where $E(k)$ is given by Eq. (4.4).

Of course, $E(k)$ must be IR integrable in order that the right hand side of Eq. (4.8) exist without introducing a lower cutoff. This is the case for the choice $y=y_-$ [cf. Eq. (4.3)] and the resulting correlation function is shown in Fig. 3. For long times,

$$\langle \mathbf{v}(0,t)\mathbf{v}(0,0)\rangle \sim \begin{cases} |t|^{(2+y-d)/2}, & \epsilon<0 \\ |t|^{-2(1-\gamma)/(2-\gamma)}, & \epsilon>0 \end{cases} ; \tag{4.9}$$

in particular, the correlation will decay as $|t|^{-1.293}$ for $y=y_-$ in three dimensions. The long time behavior for $\epsilon<0$ is a generalization of the well known long time tails phenomena.[3,17] Note that the behavior at short times is nonanalytic due to the presence of $t\ln(t)$ terms, although similar terms will arise from the one-loop corrections.

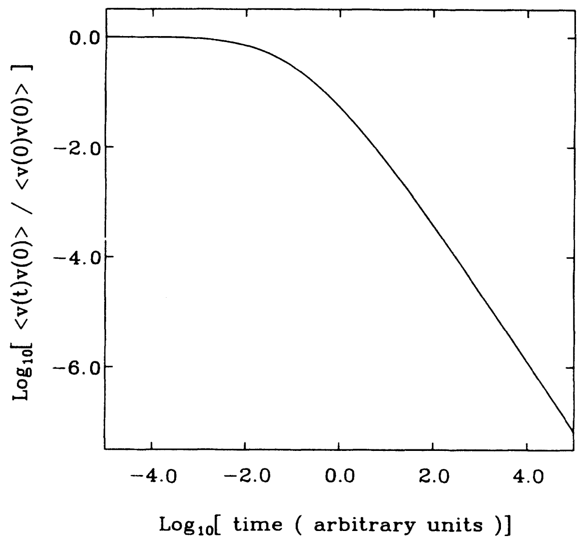

FIG. 3. The local velocity time correlation function for the choice $y=y_-$ and $u_R/u^*=0.5$ in three spatial dimensions. Note, cf. Eq. (4.9), that the asymptotic decay is as $t^{-1.29}$.

C. Energy dissipation

As is well known,[18] the energy dissipation rate per unit mass, $\langle\epsilon\rangle$, is given by

$$\langle\epsilon\rangle = \int \frac{d^dk}{(2\pi)^d} \nu k^2 \langle \mathbf{v}(\mathbf{k})\cdot\mathbf{v}(-\mathbf{k})\rangle, \tag{4.10}$$

where note that all quantities on the right hand side of Eq. (4.10) are not renormalized. Introducing renormalized quantities and putting the result in scaling form gives

$$\langle\epsilon\rangle = \int_0^\infty dk\, d\,\nu_{\text{eff}}(\kappa/k) k^2 E(\kappa/k) \tag{4.11}$$

to zero-loop order. From Eqs. (4.2) and (4.7), it follows that the integral on the right hand side of Eq. (4.11) will be UV integrable only for $\epsilon>4$ or for $\gamma>\frac{4}{3}$ and $\epsilon<0$. If neither condition is met, then an upper cutoff must be introduced (the wave-vector characterizing the dissipative range). Clearly this must be done in equilibrium for $d<6$. In addition, the UV divergence first appears when $\gamma=\frac{4}{3}$, i.e., the choice which produced the Kolmogorov energy spectrum. In addition, the integral is IR divergent for $\epsilon>0$ and $\gamma>\frac{4}{3}$ (this is the situation in Ref. 10) and thus a lower cutoff must be introduced.

D. Linear response and one-loop corrections

The space and time dependence of the average relaxation of small fluctuations is described by the average Green's function. Up to this point, only zero-loop expressions for the correlation functions have been examined; however, using the results of the Appendix [cf. Eq. (A.11)] it follows that the renormalized inverse Green's function, $\Gamma^{(1,1)}(\mathbf{k},\omega)$, satisfies

$$\Gamma^{(1,1)}(\mathbf{k},\omega) \sim i\omega + \nu_R k^2 \left\{ 1 + u_R A_d \left[-\tfrac{1}{2}\ln\left[\frac{8(k^2+i\omega/\nu_R)}{\kappa^2}\right] + H(i\omega/\nu_R k^2)\right]\right\} + O(\epsilon^2), \tag{4.12}$$

where bare quantities were replaced by renormalized ones in the finite parts of the one-loop terms,

$$H(x) \equiv \frac{(1+2x)(1-4x)}{16x} - \frac{2+y}{2\epsilon} + \frac{(1+2x)^2}{(4x)^2}\left[[1+(2x)^2]\ln\left[\frac{1+x}{\frac{1}{2}+x}\right] - \ln(2)\right] + \tfrac{1}{2}\ln(2), \quad (4.13)$$

and where the factor $i\overleftrightarrow{\Phi}_{\mathbf{k}}$ was dropped. The function $H(x)$ has a branch cut for $-1 \le x \le -\frac{1}{2}$, is analytic at $x=0$,

$$H(x) \sim \tfrac{1}{2}\ln(2) + [\ln(2) - \tfrac{37}{48}]x + \cdots ,$$

and decays as

$$[\tfrac{11}{48} - \tfrac{1}{4}\ln(2)]x^{-1} \quad \text{for } |x| \to \infty .$$

In addition, it is finite as $x \to -\frac{1}{2}+$.

Equation (4.12) is the result of the naive perturbation theory and is not consistent with the renormalization-group equation, cf. Eq. (3.7), except when ϵ-expanded. In order to remedy this, ν_R and u_R must be expressed in terms of W and U, and the result brought into a form consistent with Eqs. (3.12) or (3.13). By noting that

$$\nu_R \sim W(\kappa R)[1 - A_d U(\kappa R)\ln(\kappa R)],$$

it is easy to show that

$$\Gamma^{(1,1)}(\mathbf{k},\omega) \sim i\omega + W(\kappa R)k^2\left[1 + \frac{\gamma U(\kappa R)}{u^*}(H[i\omega/W(\kappa R)k^2] - \tfrac{1}{2}\ln\{8R^2[k^2 + i\omega/W(\kappa R)]\})\right] + O(\epsilon^2), \quad (4.14)$$

where γ is given by Eq. (3.11a). Note that the scale parameter, R, has not been specified. Equation (4.14) is consistent with the full perturbation expansion, the renormalization-group equation, and with dimensional analysis. In addition, it is free of any logarithmic singularities as k or $\omega \to 0$.

The space and time Fourier transform of Green's function, $\tilde{G}(\mathbf{k},\omega)$ is obtained by inverting the right hand side of Eq. (4.14); i.e.,

$$\tilde{G}(\mathbf{k},\omega) = \frac{1}{W(\kappa/k)k^2(is + 1 + \gamma U(\kappa/k)\{-\frac{1}{2}\ln[8(1+is)] + H(is)\}/u^*)}, \quad (4.15)$$

where $s \equiv \omega/[W(\kappa/k)k^2]$ and where, as in the previous section, the scale parameter has been set to k^{-1}. Simple dynamic scaling laws will be obtained when $U(\kappa/k)$ attains its fixed point values, cf. Sec. IV B. Nonetheless, the time dependence associated with Eq. (4.15) is nontrivial. This behavior is shown in Fig. 4, where the inverse Laplace transform of the right hand side of Eq. (4.15) was taken, with time scaled by $k^2W(\kappa/k)$.

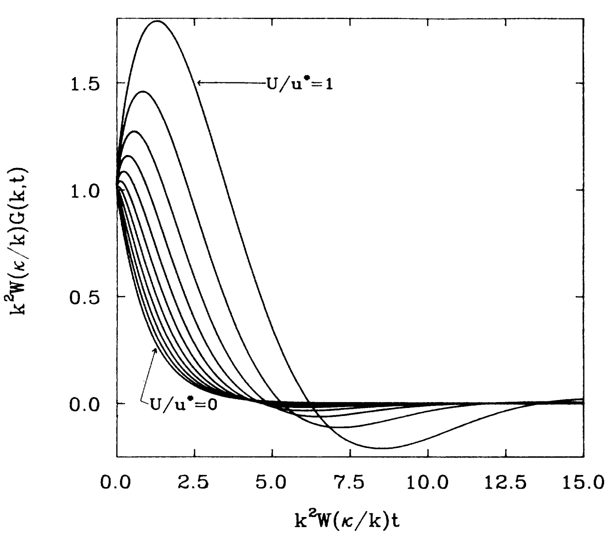

FIG. 4. The time representation of the scaled Green's function for various values of $U(\kappa/k)/u^*$ for the choice $y = y_-$ in three dimensions. The curves were obtained by numerically inverting the Laplace transform. The figure shows the differences between the classical ($U/u^* = 0$, exponential decay) and nonclassical ($U/u^* = 1$) behaviors. The latter is clearly nonexponential and will be observed for high wave vectors. (The curves are evenly spaced in increments of $U/u^* = 0.1$.)

Note that choosing $R = k^{-1}$ introduces a $k^2\ln(k)$ nonanalyticity at $k=0$ in the denominator of the right hand side of Eq. (4.15). This nonanalyticity was not present in the renormalized perturbation theory; it reflects the fact that linking the scale parameter to k when the latter is zero does not make too much sense, and an alternate match condition should be considered [e.g., $\omega R^2/W(\kappa R) \equiv 1$].

Similar problems occur in equilibrium critical phenomena; e.g., in magnetic systems, the match condition is chosen differently depending on whether the external magnetic field is zero (away from the critical temperature) or nonzero (at the critical temperature). Finally, note that a uniform approximation has been proposed by Rudnick and Nelson.[19] An analogous approximation would be to choose the scale R such that

$$(kR)^2 + \left|\frac{\omega R^2}{W(\kappa R)}\right|^2 = 1, \quad (4.16)$$

although in the nonclassical limit, R must be determined numerically.

The coefficient of the explicit k^2 factor on the right hand side of Eq. (4.14) defines a generalized frequency and wave-vector-dependent viscosity, which, when $k=0$, becomes

$$\nu_{\text{eff}}(p) = W(\kappa R)\left[1 - \frac{\gamma U(\kappa R)\ln[8pR^2/W(\kappa R)]}{2u^*}\right], \quad (4.17)$$

where $p \equiv i\omega$ is a Laplace transform variable. Now a natural choice for the match condition is $8pR^2/W(\kappa R) \equiv 1$ which implies that $\nu_{\text{eff}}(z) = W[\kappa R(p)] = 8R^2(p)p$. Repeating the analysis which led to Eq. (4.7) shows that $\nu_{\text{eff}} \sim p^{-1/(2-\gamma)}$ in the nonclassical regime; in particular, for the Kolmogorov choices of y, $\nu_{\text{eff}} \sim p^{-3/2}$.

VI. DISCUSSION

In this work a generalization of model A of Forster *et al.*[3] was analyzed using field-theoretical renormalization techniques. In the equilibrium limit (i.e., $y=-2$) the results reduce to those found in Ref. 3. In addition, they agree to $O(\epsilon)$ with those of model B for constant low-wave-vector forcing. Thus the renormalization scheme introduced here continuously interpolates between these two very different limits.

This calculation was motivated by the recent application of the fluctuating Navier-Stokes equations to turbulence by Yakhot and Orszag.[10] While many of the results found here are similar to those found in Ref. 10, there are still numerous differences, all arising from the way in which the noise renormalization was handled. Most important is the wave-vector regions in which nontrivial exponents are obtained for the energy spectrum (cf. Sec. IV A). This paper is concluded with some general comments and some open questions.

(1) This calculation resulted in three possible choices for the noise exponent which yield the Kolmogorov $\frac{5}{3}$ law in some region; however, only one of them connects continuously with equilibrium. Moreover, only this one gives the Kolmogorov spectrum for high wave vectors and the others require IR cutoffs.

(2) The choice $y=y_+$ is closest in spirit to what was found in Ref. 10; specifically, the noise disappears at high wave vectors and has its maximum effect at small ones. Not too surprisingly, many of the results presented in Ref. 10 can be reproduced with no or only slight modifications. On the other hand, the choice $y=y_-$ results in a more equilibriumlike noise in that it grows as the wave vector is increased. The growth, however, is as $k^{1.585}$, which is less than that found in equilibrium. The most important difference between these two fixed points is in whether they are IR or UV stable; the former gives the Kolmogorov $\frac{5}{3}$ law as $k \to 0$, whereas the latter gives it as $k \to \infty$. One exception to this observation occurs when $u_R = u^*$, in which case $U(\kappa/k)$ does not flow with k; however, there is no obvious reason why $u_R = u^*$ in general.

(3) If the model for $y=y_+$ (or that given by Yakhot and Orszag) is to be applied to turbulence, then additional IR cutoffs must be introduced into the theory for phenomena on scales longer than those characterizing the inertial range. Moreover, the nature of the energy spectrum in the dissipative range is probably not correct. The choice $y=y_-$ gives a Kolmogorov spectrum in the UV limit and thus does not require an explicit IR cutoff, although the form of the energy spectrum at small wave vectors is universal and is probably too simple to be correct in general.

(4) Strictly speaking, away from walls, the random force should be interpreted as the divergence of a random stress tensor. In equilibrium, the random stress is δ-correlated in space (hence, the choice $y=-2$ in Ref. 3). The choice $y=y_-$ corresponds to a $k^{-0.585}$ decay in the random stress correlation function in three dimensions.

(5) It is still not clear why making $\gamma=\frac{4}{3}$ is relevant to turbulence. Nonetheless, it is interesting to note (cf. Sec. IV C) that this is the value which makes the total kinetic energy dissipation rate marginal, and in particular, implies that the dependence of the dissipation rate on the details of processes in the far dissipative range (for $y=y_-$) or in the sub-inertial range (for $y=y_+$) will only be logarithmic (this is just Kolmogorov's scaling hypothesis).

(6) The renormalized parameters, u_R and χ_R are related to the bare ones by Eqs. (3.4a) and (3.4b) to one-loop order. For $\epsilon/(\epsilon+2+y) \sim O(1)$, it is easy to show that the general term in these series must have the form

$$(u_R/\epsilon)^n c_n[\epsilon/(\epsilon+2+y)],$$

where the fluctuation dissipation theorem implies that the coefficients $c_n(1)$ must be equal in the χ and ν renormalizations. Of course, it remains to be shown whether or not the perturbation expansions are useful in regions of interest (in particular for ϵ large). Moreover, the relationship between the bare and renormalized parameters must be known before amplitudes appearing in the scaling functions can be computed and the radius of convergence or asymptotic nature of the series given in Eqs. (3.4a) and (3.4b) is known. Nonetheless, even if the series relating the bare and renormalized parameters cannot be summed explicitly it is assumed that the renormalized quantities are finite and that the *functional relationships* derived from the renormalization group remain valid (the two amplitudes relating u_R and ν_R to the bare quantities would then be fit to experiment).

(7) Finally, it has been possible to derive crossover forms for the various correlation functions to zero- or one-loop order. Except in a full ϵ expansion, these have depended on the choice of the scale parameter $R=k^{-1}$. It is well known that other match conditions [e.g., $\omega = W(\kappa/R)R^{-2}$] will give alternate resummations of the perturbation series which may be more appropriate for certain situations. These will be investigated in the future.

ACKNOWLEDGMENTS

Many discussions with Victor Yakhot are gratefully acknowledged. Portions of this work were supported by the Alfred P. Sloan Research Foundation, and by the Camille and Henry Dreyfus Foundation.

APPENDIX

Here the evaluation of the one-loop integrals to $O(1)$ will be presented. As was mentioned in the text the singular parts of the integrals are easy to obtain once the nature of the singularity is known. Moreover, it is easy to show that the one-loop viscosity correction, cf. Fig. 2(a), diverges as ϵ^{-1} as $\epsilon \rightarrow 0$, and that the noise correction, cf. Fig. 1(b), diverges as $(\epsilon+2+y)^{-1}$. Since the coefficients of singular parts of the integrals for the case $y=-2$ are given by Forster *et al.*[3], Eqs. (3.4a) and (3.4b) are trivial modifications of their results.

The finite parts of the integrals must be known if scaling forms valid to one-loop order are desired. For example, consider the one-loop correction to the viscosity shown in Fig. 2(a). This diagram corresponds to the integral

$$\Delta \overleftrightarrow{v} k^2 \equiv \frac{i\chi V_{\mathbf{k}}^{\alpha;\beta,\gamma}}{(2\pi)^d \nu} \int d\mathbf{k}_1 \frac{V_{\mathbf{k}_1}^{\beta;\beta',\gamma'} \Phi_{\mathbf{k}-\mathbf{k}_1}^{\gamma,\gamma'}}{|\mathbf{k}-\mathbf{k}_1|^{2+y}[i\omega+\nu(k_1^2+|\mathbf{k}-\mathbf{k}_1|^2)]}, \tag{A.1}$$

where the intermediate frequency integration has been performed. Next, the explicit form of V [cf. Eq. (2.4a)] and transversality are used, and the change of variable $\mathbf{k}_1 \rightarrow \mathbf{k}_1 + \frac{1}{2}\mathbf{k}$ is made; Eq. (A.1) thus becomes

$$\Delta \overleftrightarrow{v} k^2 = \frac{i\chi\lambda V_{\mathbf{k}}^{\alpha;\beta,\gamma}}{(2\pi)^d \nu} \int d\mathbf{k}_1 \frac{(k_1^{\beta'} \Phi_{\mathbf{k}_1+\frac{1}{2}\mathbf{k}}^{\beta,\gamma'} + k^{\gamma'} \Phi_{\mathbf{k}_1+\frac{1}{2}\mathbf{k}}^{\beta,\beta'}) \Phi_{\mathbf{k}_1-\frac{1}{2}\mathbf{k}}^{\gamma,\gamma'}}{|\mathbf{k}-\frac{1}{2}\mathbf{k}_1|^{2+y}[i\omega+2\nu(k_1^2+\frac{1}{4}k^2)]}, \tag{A.2a}$$

$$\equiv A+B, \tag{A.2b}$$

where A corresponds to the integral resulting from the first term in the brackets in the numerator of Eq. (A.2a) and B to the other. In addition, since $\Delta\overleftrightarrow{v}$ is a transverse rank-2 tensor function of $\mathbf{k}$ in an isotropic system, it must be proportional to $\overleftrightarrow{\Phi}_{\mathbf{k}}$. Moreover, the proportionality constant is obtained by taking the trace of the right hand side of Eq. (A.2a) and dividing the result by $(d-1)$.

First consider A. Divide the integral in half, let $k_1 \rightarrow -\mathbf{k}_1$ in one part, and carry out all the indicated contractions; this gives

$$A = -\frac{i\lambda^2\chi\overleftrightarrow{\Phi}_{\mathbf{k}}}{(2\pi)^d \nu(d-1)} \int d\mathbf{k}_1 \frac{\mathbf{k}\cdot\mathbf{k}_1 k_1^2 (k_1^2-\frac{1}{4}k^2)[1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}_1)^2]}{[(k_1^2+\frac{1}{4}k^2)^2-(\mathbf{k}\cdot\mathbf{k}_1)^2][i\omega+2\nu(k_1^2+\frac{1}{4}k^2)]} \left[\frac{1}{|\mathbf{k}_1-\frac{1}{2}\mathbf{k}|^{2+y}} - \frac{1}{|\mathbf{k}_1+\frac{1}{2}\mathbf{k}|^{2+y}} \right], \tag{A.3}$$

where, in addition, terms which are odd under $\mathbf{k}_1 \rightarrow -\mathbf{k}_1$ have been dropped. Equation (A.3) can be rewritten in d-dimensional polar coordinates as

$$A = -\frac{i\lambda^2\chi\overleftrightarrow{\Phi}_{\mathbf{k}} S_{d-1} k^{2-\epsilon}}{(2\pi)^d \nu^2 (d-1)} \int_0^\infty dk_1 \frac{k_1^{d+2}(k_1^2-\frac{1}{4})}{i\omega^*+2(k_1^2+\frac{1}{4})}$$
$$\times \int_0^\pi d\theta \frac{\sin^d(\theta)\cos(\theta)}{(k_1^2+\frac{1}{4})^2-[k_1 cos(\theta)]^2}$$
$$\times \left[\frac{1}{[k_1^2+\frac{1}{4}-k_1\cos(\theta)]^{(2+y)/2}} - \frac{1}{[k_1^2+\frac{1}{4}+k_1\cos(\theta)]^{(2+y)/2}} \right], \tag{A.4}$$

where all lengths have been scaled by k^{-1} and $\omega^* \equiv \omega/(\nu k^2)$.

It is easy to see that this integral is marginal when $\epsilon=0$ and that it vanishes when $y=-2$; hence, for the renormalization scheme discussed in the text, only the leading order dependence is needed and it follows that

$$A \sim -\frac{iu\nu A_d(2+y)k^2\overleftrightarrow{\Phi}_{\mathbf{k}}}{2\epsilon} + O(\epsilon). \tag{A.5}$$

The same steps can be repeated for B which becomes

$$B = \frac{iu\nu S_{d-1}\overleftrightarrow{\Phi}_{\mathbf{k}} k^{2-\epsilon}}{(2\pi)^d (d-1)} \int_0^\infty dk_1 \int_0^\pi d\theta \frac{k_1^{d+1}\sin^d(\theta)}{[k_1^2+\frac{1}{4}-k_1\cos(\theta)]^{2+y/2}[i\omega^*+2(k_1^2+\frac{1}{4})]} \left[d-2+\frac{k_1\cos(\theta)[2k_1\cos(\theta)+1]}{k_1^2+\frac{1}{4}-k_1\cos(\theta)} \right], \tag{A.6}$$

where the symmetrization under $\mathbf{k}_1 \rightarrow -\mathbf{k}_1$ was not performed. The singular part of this integral is contained in

$$B' \equiv \frac{iu\nu S_{d-1}\overleftrightarrow{\Phi}_{\mathbf{k}} k^{2-\epsilon}}{(2\pi)^d (d-1)} \int_0^\infty dk \int_0^\pi d\theta \frac{k_1^{d+1}\sin^d(\theta)}{(k_1^2+\frac{1}{4})^{2+y/2}[i\omega^*+2(k_1^2+\frac{1}{4})]} \left[d-2+\frac{2k_1^2\cos^2(\theta)}{k_1^2+\frac{1}{4}} \right]. \tag{A.7}$$

This integral can be performed and gives

$$B'=\frac{iu\nu S_d k^2(2k/\kappa)^{-\epsilon}\overleftrightarrow{\Phi}_{\mathbf{k}}}{4d(2\pi)^d}\left[(d-2)B\left[\frac{d}{2}+1,\frac{\epsilon}{2}\right]F\left[1,\frac{\epsilon}{2};3+\frac{y}{3};-2i\omega^*\right]+\frac{2}{d+2}B\left[\frac{d}{2}+2,\frac{\epsilon}{2}\right]F\left[1,\frac{\epsilon}{2};4+\frac{y}{2};-2i\omega^*\right]\right],$$

where $B(x,y)$ is the beta function and $F(a,b;c;z)$ is the hypergeometric function. In ϵ-expanded form this last result becomes

$$B'\sim\frac{i\nu u A_d\overleftrightarrow{\Phi}_{\mathbf{k}}}{\epsilon}\left[1-\epsilon\ln(2k/\kappa)-\frac{\epsilon}{2}\left[\frac{(1+2i\omega^*)^2}{(2i\omega^*)^2}\ln(1+2i\omega^*)-\frac{1}{2i\omega^*}\right]\right]+O(\epsilon), \quad (A.8)$$

where recall that $d-2\sim O(\epsilon)$. The remaining part of B is finite and may be evaluated at $d=-y=2$; i.e.,

$$B-B'\sim\frac{i2u\nu S_{d-1}k^2}{(2\pi)^d(d-1)}\int_0^\infty dk_1\int_0^\pi d\theta\,\frac{k_1^7\cos^4(\theta)\sin^2(\theta)}{(k_1^2+\frac{1}{4})^2[i\omega^*+2(k_1^2+\frac{1}{4})]\{(k_1^2+\frac{1}{4})^2-[k_1\cos(\theta)]^2\}}. \quad (A.9)$$

This integral can be expressed in terms of elementary functions as

$$iuA_dk^2\nu\overleftrightarrow{\Phi}_{\mathbf{k}}\left[-\tfrac{1}{8}-\frac{i\omega^*}{2}+i\omega^*(1+i\omega^*)\ln\left[\frac{2(1+i\omega^*)}{(1+2i\omega^*)}\right]-\frac{3}{16i\omega^*}+\frac{1}{4i\omega^*}\left[1+\frac{1}{4i\omega^*}\right]\ln[(1+i\omega^*)(1+2i\omega^*)]\right]. \quad (A.10)$$

Finally, combining Eqs. (A.5), (A.8) and (A.10) gives

$$\Delta\overleftrightarrow{\nu}\sim i\nu uA_d\overleftrightarrow{\Phi}_{\mathbf{k}}\left[\frac{1}{\epsilon}-\tfrac{1}{2}\ln\left[\frac{8(k^2+i\omega/\nu)}{\kappa^2}\right]+H(i\omega^*)\right]+O(\epsilon), \quad (A.11)$$

where $H(i\omega^*)$ is given by Eq. (4.13).

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