Wiggler-length dependence of the output of a free-electron laser

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A partial differential equation for the probability amplitude in the spherical phase space is derived to describe the nonlinear evolution of a free-electron laser in the Compton regime. A perturbative solution up to an indefinite high order of the electron quantum recoil is obtained. It is used to calculate the laser output, in terms of average photon number, as a function of the wiggler length.

I. INTRODUCTION

We consider a nonlinear quantum model of a free-electron laser (FEL) working in the Compton regime. The FEL is reduced to its very bare essentials; it consists of a single electron propagating along an untapered wiggler at a relativistic speed and interacting with a single mode of radiation propagating along the same direction. On the other hand, we include the nonlinear effects due to the quantum recoil of the electron, treated as a perturbation, up to an indefinite high order.

Quantum-mechanical analysis of an FEL often starts from the Bambini-Renieri Hamiltonian[1] which describes the system in a frame moving along with the electron and the laser radiation at a speed very close to that of light so that (1) the wiggler field appears almost as a plane-wave radiation consisting of quasiphotons (Weizsäcker-William approximation), (2) the frequency of the wiggler field coincides with that of the laser, (3) the electron dynamics becomes nonrelativistic, and (4) the picture of the laser action becomes that of photons scattered back and forth between the wiggler field and the laser field with the electron recoils providing for the conservation of momentum.

The Bambini-Renieri Hamiltonian can be written as[2]

$$\mathcal{H}=\frac{p^2}{2m}+\hbar\omega(a_L^\dagger a_L+a_W^\dagger a_W)+\hbar\Lambda(a_L^\dagger a_W e^{-2ikz}+a_W^\dagger a_L e^{2ikz}) , \qquad (1)$$

where p is the momentum of the electron, $a_L^\dagger$ ($a_W^\dagger$) is the creation operator of the laser (wiggler) field, $k=\omega/c$, and $\Lambda\equiv e^2/2m\omega\epsilon_0 V_W$ is the coupling constant with V_W being the wiggler volume.

This Hamiltonian can be translated into the following difference-differential equation:[3]

$$i\frac{d}{dt}C_n(t)=(-2n\Delta+n^2E)C_n(t)+\Lambda\sqrt{(N-n)(n+1)}C_{n+1}(t)+\Lambda\sqrt{(N-n+1)n}\,C_{n-1}(t) , \qquad (2)$$

where $C_n(t)$ is the probability amplitude that n photons are in the laser field at time t, $\Delta\equiv kp_0/m$ and $E\equiv 2\hbar k^2/m$ are two constants related to the initial momentum and the quantum recoil, respectively, of the electron, and N is the total number of the initial "photons" in the wiggler field.

Equation (2) has been recognized by Bosco *et al.*[3] as one of the various types of generalized Raman-Nath equations (RNE). Hence these authors call it spherical RNE. The original RNE was derived in 1937 to describe light diffraction by ultrasound.[4] The various types of RNE appear in a large number of physical problems, as pointed out by Bosco and Dattoli,[5] and they are all unsolvable because of the existence of the nonlinear term $n^2\epsilon$.

Bosco *et al.*[3] have obtained solutions to Eq. (2) under the simplifying assumption that $\epsilon=0$. Lee[6] has obtained a perturbative solution to the first order of the perturbation parameter ϵ. Most recently, Lee[7] has used the Q representation of atomic coherent states[8] to transform the spherical RNE into a partial differential equation for the probability density function over the spherical phase space and has obtained a perturbative solution up to arbitrarily high order of ϵ. The trouble with this last solution is that the analytic expression is so lengthy that it is very difficult to use it to calculate any observable physical quantities.

We have now realized that it is much easier to deal with the probability amplitude than with the probability density function. It should be pointed out that probability amplitude does not exist for the more famous P representation and the Wigner distribution because they must allow negative values. This is a distinct advantage of the Q representation.

In the following, we will derive the partial differential equation for the probability amplitude, find an almost exact solution, and then use it to calculate the average laser photon number as a function of time.

II. EQUATION FOR THE PROBABILITY AMPLITUDE IN SPHERICAL PHASE SPACE

The density matrix to be constructed from the solution of Eq. (2) is of the following form:

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$$\rho(t)=\sum_{n=0}^{N}\sum_{m=0}^{N}C_m^*(t)C_n(t)\,|\,n\,\rangle\langle\, m\,| \ , \tag{3}$$

where the $|\,n\,\rangle$ are quantum states of definite photon number. The atomic coherent states are defined as

$$|\,\theta,\phi\,\rangle_N\equiv\sum_{n=0}^{N}|\,n\,\rangle\begin{bmatrix}N\\n\end{bmatrix}^{1/2}[\cos(\theta/2)]^{N-n}\times[\sin(\theta/2)]^n e^{-in\phi} \tag{4}$$

and the probability density function over the spherical surface in the Q representation is defined as

$$Q_N(\theta,\phi,t)\equiv\langle\,\theta,\phi\,|\,\rho(t)\,|\,\theta,\phi\,\rangle_N\equiv P_N^*(\theta,\phi,t)P_N(\theta,\phi,t) \ , \tag{5}$$

where

$$P_N(\theta,\phi,t)=\sum_{n=0}^{N}\begin{bmatrix}N\\n\end{bmatrix}^{1/2}C_n(t)[\cos(\theta/2)]^{N-n}\times[\sin(\theta/2)]^n e^{-in\phi} \tag{6}$$

is the probability amplitude. The normalization condition is

$$\frac{N+1}{4\pi}\int_0^{\pi}d\theta\sin\theta\int_0^{2\pi}d\phi\, Q_N(\theta,\phi,t)=1 \ . \tag{7}$$

It will be convenient to introduce the following dimensionless parameters:

$$\delta\equiv\Delta/\Omega,\quad \lambda\equiv\Lambda/\Omega,\quad \epsilon\equiv NE/\Omega,\quad \tau\equiv 2\Omega t \ , \tag{8}$$

where $\Omega\equiv(\Delta^2+\Lambda^2)^{1/2}$.

Using Eqs. (2) and (8), we can obtain the partial differential equation for $P_N(\theta,\phi,\tau)$ as

$$(\hat{D}_0+\epsilon\hat{D}_1)P_N(\theta,\phi,\tau)=0 \ , \tag{9}$$

where

$$\hat{D}_0\equiv i\left\{\frac{\partial}{\partial\tau}+\left[\lambda\cot\left[\frac{\theta}{2}\right]\cos\phi-\delta\right]\frac{\partial}{\partial\phi}-i\lambda e^{i\phi}\frac{\partial}{\partial\theta}\right\} , \tag{9a}$$

$$\hat{D}_1\equiv\frac{1}{2N}\frac{\partial^2}{\partial\phi^2} \ . \tag{9b}$$

III. PERTURBATIVE SOLUTION

We will try to find a perturbative solution to Eq. (9) of the following form:

$$P_N(\theta,\phi,\tau)\equiv P^0(\theta,\phi,\tau)+\epsilon P^1(\theta,\phi,\tau)+\epsilon^2P^2(\theta,\phi,\tau)+\cdots \tag{10}$$

with the initial condition: $C_n(0)=\delta_{n,0}\Longrightarrow P_N(\theta,\phi,0)=P^0(\theta,\phi,0)=[\cos(\theta/2)]^N$ and $P^l(\theta,\phi,0)=0$ for all $l>0$.

Substitution of Eq. (10) into Eq. (9) yields

$$\hat{D}_0P^0(\theta,\phi,\tau)=0 \ , \tag{11a}$$

$$\hat{D}_0P^{l+1}(\theta,\phi,\tau)+\hat{D}_1P^l(\theta,\phi,\tau)=0 \ . \tag{11b}$$

The solution to Eq. (11a) satisfying the initial condition can be written as

$$P^0(\theta,\phi,\tau)=[e^{i\delta\tau/2}F(\theta,\phi,\tau)]^N \ , \tag{12}$$

where

$$F(\theta,\phi,\tau)\equiv\left[\cos\left[\frac{\tau}{2}\right]-i\delta\sin\left[\frac{\tau}{2}\right]\right]\cos\left[\frac{\theta}{2}\right]-i\lambda\sin\left[\frac{\tau}{2}\right]\sin\left[\frac{\theta}{2}\right]e^{i\phi} \ . \tag{13}$$

And, as long as $l\ll N$, the solution to Eq. (11b) can be written in the form

$$P^l(\theta,\phi,\tau)=\begin{bmatrix}N/2\\l\end{bmatrix}[F(\theta,\phi,\tau)]^{N-2l}[G(\theta,\phi,\tau)]^l e^{iN\delta\tau/2} \ , \tag{14}$$

where $G(\theta,\phi,\tau)$ must satisfy the following equation:

$$(\hat{D}_0-\delta)G(\theta,\phi,\tau)=\left[\frac{\partial}{\partial\phi}F(\theta,\phi,\tau)\right]^2=\left[\lambda\sin\left[\frac{\tau}{2}\right]\sin\left[\frac{\theta}{2}\right]e^{i\phi}\right]^2 \ . \tag{15}$$

The solution to Eq. (15) satisfying the initial condition can be written as

$$G(\theta,\phi,\tau)=\left[-i\frac{\lambda^4}{8}h(\tau)\right]\cos^2\left[\frac{\theta}{2}\right]+\left[\frac{\lambda^3}{4}h'(\tau)+i\frac{\lambda^3\delta}{4}h(\tau)\right]\sin\left[\frac{\theta}{2}\right]\cos\left[\frac{\theta}{2}\right]e^{i\phi}+\left[-\frac{\lambda^2\delta}{4}h'(\tau)+i\left[\frac{\lambda^4}{8}h(\tau)+\frac{\lambda^2}{4}h''(\tau)\right]\right]\sin^2\left[\frac{\theta}{2}\right]e^{2i\phi} \ , \tag{16}$$

where

$$h(\tau)=3\sin\tau-\tau[\cos\tau+2] \ , \tag{17}$$

$h'(\tau)$ is the first derivative of $h(\tau)$, and $h''(\tau)$ is the second derivative.

We can now put the components of the solution together. Substitution of Eqs. (12) and (14) into Eq. (10) gives

$$P_N(\theta,\phi,\tau)=\{[F(\theta,\phi,\tau)]^2+\epsilon G(\theta,\phi,\tau)\}^{N/2}e^{iN\delta\tau/2}$$
$$=\left[A(\tau)\cos^2\left[\frac{\theta}{2}\right]+B(\tau)\sin\left[\frac{\theta}{2}\right]\cos\left[\frac{\theta}{2}\right]e^{i\phi}+C(\tau)\sin^2\left[\frac{\theta}{2}\right]e^{2i\phi}\right]^{N/2}e^{iN\delta\tau/2}, \quad (18)$$

where the explicit expressions for $A(\tau)$, $B(\tau)$, and $C(\tau)$ can be obtained by using Eqs. (13) and (16) as follows:

$$A(\tau)=[(1+\delta^2)\cos\tau+\lambda^2]/2 -i[\delta\sin\tau+(\epsilon\lambda^4/8)h(\tau)], \quad (19a)$$

$$B(\tau)=[\lambda\delta(\cos\tau-1)+(\epsilon\lambda^3/4)h'(\tau)] +i[-\lambda\sin\tau+(\epsilon\lambda^3\delta/4)h(\tau)], \quad (19b)$$

$$C(\tau)=[\lambda^2(\cos\tau-1)-(\epsilon\lambda^2\delta/4)h'(\tau)] +i[(\epsilon\lambda^4/8)h(\tau)+(\epsilon\lambda^2/4)h''(\tau)]. \quad (19c)$$

IV. AVERAGE PHOTON NUMBER

Now we can use the solution to calculate the average photon number which represents the laser output. The rigorous way to carry out this calculation is very tedious. Fortunately, we can take a short cut as follows: Using Eq. (18), we obtain the probability density function $Q_N(\theta,\phi,\tau)\equiv|P_N(\theta,\phi,\tau)|^2$ as some expression raised to the power of $N/2$. When $N\gg1$, $Q_N(\theta,\phi,\tau)$ is a very sharply peaked distribution almost like a Dirac δ function in the spherical phase space, with τ as a parameter; then we should be able to approximate the expectation value of an expression by its value at the peak of $Q_N(\theta,\phi,\tau)$ located at (θ_M,ϕ_M).

Let n be the photon number; then the normalized laser output can be represented by

$$\frac{\langle n\rangle}{N}\equiv\langle(1-\cos\theta)/2\rangle\approx(1-\cos\theta_M)/2 . \quad (20)$$

And from Eq. (18) we can derive the following relation:

$$\cos\theta_M=(|A|-|C|)(|A|^2+|C|^2+\tfrac{1}{2}|B|^2+\tfrac{1}{2}|B^2-4AC|)^{-1/2} . \quad (21)$$

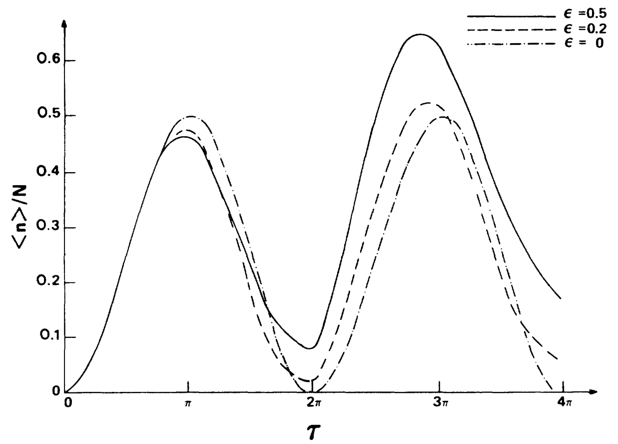

FIG. 1. FEL output vs wiggler length. The values of the parameters are $\lambda=\delta=1/\sqrt{2}$ and $\epsilon=0$, 0.2, and 0.5.

Using Eq. (17) in Eqs. (19), then Eqs. (19) in Eq. (21), and then Eq. (21) in Eq. (20), we can obtain an explicit analytic expression for the laser output as a function of time or wiggler length, with λ and ϵ as two independent parameters. (δ is not an independent parameter because of the relation $\delta^2+\lambda^2=1$.)

In Fig. 1 we present some examples of the numerical evaluation of $\langle n\rangle/N$ with $\lambda=\delta=1/\sqrt{2}$ and $\epsilon=0$, 0.2, and 0.5. It indicates that, for $\epsilon=0$, the output is a perfect periodic function of τ, as is known from the exact solution for this special case,[3] and that the perturbation causes the first peak to be slightly lower but the second peak to be substantially higher, especially for higher values of ϵ. This implies that, if we have a wiggler long enough to reach the second peak of the output, we can enhance the efficiency of the FEL.

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