Integral equation for the laser field in a long-pulse free-electron laser

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The free-electron laser is described by the self-consistent solution of the coupled system of the Lorentz force equation for the electrons and the Maxwell equation for the laser field. We show that, near saturation and at both low- and high-gain regime, the evolution of each longitudinal mode in the optical cavity can be written as an integral equation. We present numerical solutions of the integral equations for a free-electron-laser oscillator.

I. INTRODUCTION

The free-electron-laser (FEL) problem is highly nonlinear and its solution requires intensive numerical computations which are very costly. In this work we study an approximate analytical approach for an FEL oscillator. There are several reasons for obtaining approximate but reliable analytic or semianalytical expressions for the laser field. First, the analytical results, valid in a subset of values of the parameters, can be used as benchmarks for more realistic and accurate simulations covering wider range of parameters. Second, the CPU time needed to assess the gain-depression effect produced by real electron beams (involving energy spread, emittance, betatron oscillation, etc.) can be drastically reduced by averaging the analytical results with the distributions; we could even use experimentally determined distribution functions.

We present an integral equation for the laser field near saturation (threshold) and at both low- and high-gain regime of a long-electron-pulse FEL. The theory is one dimensional, i.e., the diffraction effects of the radiation field are neglected. Some of the transverse effects of the laser can be taken into account by means of the inclusion of the phase $(m+\frac{1}{2})\arctan\phi(\tau)$ in the simpler one-dimensional formulation. In Sec. II we review the well-known single-mode Colson formulation[1] of a free-electron laser. In Sec. III we discuss the multimode case; integral equations for each of the modes are presented and their numerical solution for an FEL oscillator are shown for parameters corresponding to the University of California at Santa Barbara (UCSB) FEL oscillator. Finally, in Sec. IV we discuss results and conclusions.

II. SINGLE-MODE THEORY

We self-consistently solve the system of Lorentz-Maxwell differential equations, i.e., the reduced wave equation after the slow wave approximation, for the laser field $\alpha(\tau)$ is

$$\dot{a}(\tau)=-j\langle e^{-i[\zeta_{i0}+\nu_{i0}\tau+\delta\zeta_i(\tau)]}\rangle_{\zeta_0,v_0}\,, \tag{1}$$

and

$$\delta\zeta_i(\tau)=\int_0^\tau d\tau'(\tau-\tau')\mathrm{Re}[a(\tau')e^{i[\zeta_{i0}+\nu_{i0}\tau'+\delta\zeta_i(\tau')]}] \tag{2}$$

is the Lorentz equation for each electron; $\{i=1,\ldots,N_e\}$. $\zeta_i(\tau)=\zeta_{i0}+\nu_{i0}\tau+\delta\zeta_i(\tau)$ is the electron relative phase with respect to the radiation field, j is the current parameter, and $\langle\ \rangle$ represents an average over the electrons initial phases ζ_{i0} and initial phase velocities ν_{i0}. We refer the reader to our previous publication[2] for the notation and definitions of the symbols used in this article.

A reasonable ansatz for weak fields near the onset of saturation is to neglect $\delta\zeta(\tau')$ in the exponential of Eq. (2) but keep it in Eq. (1) (we drop the electron index in the discussion that follows). We now examine different regimes defined according to values of $|a(1)|$ and j.

(a) $|a(1)|<1$ and $j\ll 1$ correspond to the weak-field and small-gain regime. In this case, we expand $e^{-i\delta\zeta(\tau)}\approx 1-i\delta\zeta(\tau)$ in Eq. (1) and keep the first-order term. The integral equation for the laser field can be exactly solved[1,3] and we obtain

$$a(\tau)=a(0)+\tfrac{1}{2}jia(0)\times\left\langle\int_0^\tau d\tau'\int_0^{\tau'}d\tau''(\tau'-\tau'')e^{-i\nu_0(\tau'-\tau'')}\right\rangle_{\nu_0}\,, \tag{3}$$

where the average over the electrons initial phases has been explicitly done using

$$\langle e^{-i\zeta_0}\rangle=\int_0^{2\pi}\frac{d\zeta_0}{2\pi}e^{-i\zeta_0}=0$$

since the electrons are uniformly distributed in phase. Notice that we have succeeded in removing the explicit presence of the electron variable ζ_0 in Eq. (3). Let us consider now as a simple example a high-quality electron beam of initial velocity ν_0. This is described by a distribution function $f(\nu_{i0})=\delta(\nu_{i0}-\nu_0)$. The single-pass power gain is given by

$$G=\frac{|a(1)|^2-|a(0)|^2}{|a(0)|^2}\approx -j\,\mathrm{Im}[g(\nu_0)]$$
$$=j\frac{2-2\cos\nu_0-\nu_0\sin\nu_0}{\nu_0^3}\ , \tag{4}$$

where the gain spectrum is

$$g(\nu_0)=\left\langle \int_0^1 d\tau' \int_0^{\tau'} d\tau''(\tau'-\tau'')e^{-i\nu_0(\tau'-\tau'')}\right\rangle_{\nu_0} .$$

The phase of the laser field $a(\tau)=|a(\tau)|e^{i\phi(\tau)}$ is

$$\phi(\tau)=j\frac{2\sin(\nu_0\tau)-\nu_0\tau[1+\cos(\nu_0\tau)]}{2\nu_0^3} . \tag{5}$$

The maximum of the low current and weak-field gain is $G_{\max}=0.135j$ at $\nu_0=2.6$ and the homogeneous gain bandwidth is $\delta\nu_0\approx 2$. For the UCSB FEL there are 150 longitudinal modes.

(b) $|a(1)|<1$ and $j\gg 1$ correspond to the weak field and high-gain regime. Expanding the exponential in Eq. (1) and keeping the first order term, we obtain

$$\dot a(\tau)=\tfrac{1}{2}ji\left\langle \int_0^{\tau} d\tau'(\tau-\tau')e^{-i\nu_0(\tau-\tau')}a(\tau')\right\rangle_{\nu_0} . \tag{6}$$

Defining $b(\tau)=a(\tau)e^{i\nu_0\tau}$ and taking the successive derivatives we can write

$$\frac{d^3b(\tau)}{d\tau^3}-i\nu_0\frac{d^2b(\tau)}{d\tau^2}-\tfrac{1}{2}ijb(\tau)=0\ ,$$

which leads to the well-known cubic equation for the exponential gain coefficient $\beta^3-i\nu_0\beta^2-\frac{1}{2}ij=0$ if we set $b(\tau)=\sum_{i=1}^{3}A_ie^{\beta_i\tau}$. The single-pass power gain is in this case,

$$G=\frac{(1+\frac{2}{3}\nu_0+\frac{4}{9}\nu_0^2)}{9}\exp\left[\sqrt{3}\tau\left[\frac{j}{2}\right]^{1/3}-\frac{\nu_0^2}{3\sqrt{3}}\left[\frac{j}{2}\right]^{-1/3}\right]$$

with a maximum $G_{\max}=\frac{1}{9}\exp[\sqrt{3}(j/2)^{1/3}]$ at $\nu_0=0$ and a characteristic width $\delta\nu_0\approx 4.22j^{1/6}$. This case was discussed in Ref. 3 where the degradation of the gain in a single-pass FEL for different electron beam energy and angular spread were investigated.

(c) $|a(1)|\geq 1$ near the onset of saturation and corresponding to both low- and high-gain regimes. We keep the exponential in Eq. (1) and using[4] the integral definition of the Bessel functions

$$J_n(z)=i^{-n}\frac{1}{2\pi}\int_0^{2\pi}d\zeta_0 e^{in(\zeta_0+\psi)}e^{iz\cos(\zeta_0+\psi)}\ ,$$

we explicitly perform the average over the initial electron phases ζ_0. The integral equation for the laser field is

$$a(\tau)=a(0)+ji\int_0^{\tau}d\tau'\langle e^{-i\nu_0\tau'}J_1[\rho(\tau')]e^{i\psi(\tau')}\rangle_{\nu_0}\ , \tag{7}$$

where we have defined

$$A(\tau)=\int_0^{\tau}d\tau'(\tau-\tau')a(\tau')e^{i\nu_0\tau'}=\rho(\tau)e^{i\psi(\tau)} .$$

In the limit of small $\rho(\tau)<1$, Eq. (7) reproduces Eq. (6).

Equation (7) is our most important result; the weak saturation properties of the free-electron laser are described by the factor $J_1(\rho)/\rho$. In Fig. 1 we show a comparison of the exact solution of the Maxwell-Lorentz equations, our model, and the linear theory. Equation (7) reproduces very well the initial linear growth of the field to the onset of saturation, but the amplitude reaches a plateau faster than the exact solution. The steady amplitude of the laser field is 20% larger than the value predicted by this model. The approximation becomes less reliable as we increase the current parameter j and the resonator and out-coupling losses.

The numerical analysis is performed using a fourth-order integration subroutine to calculate $A(\tau)$ at each point of a time mesh consisting of typically 100 points. The field $a(\tau)$ is obtained by a simple Runge-Kutta method. As we have pointed out in the Introduction, this formalism permits us to easily carry out the average over the inhomogeneities of the electron beam.

III. MULTIMODE THEORY

Up to this point the formalism we have developed has only included a single longitudinal mode of frequency ω. However, understanding the start-up of a nonideal FEL oscillator and its ultimate coherence capabilities can only be tackled by a theory with a spectrum of longitudinal modes.

Two main formalisms have been used to describe a multiparticle-multimode theory of a free-electron laser: the Hamiltonian and the space-time approach. In the Hamiltonian formalism,[2] the Hamiltonian of the system is written in the rest frame of the cold-electron beam and explicit evolution equations for the laser modes are obtained. This approach has been applied to describe both short- and long-pulse problems.[5] A totally equivalent road is to apply the space-time formalism which we will follow in this work.

To generalize our single-mode description to a many-mode one, we follow the arguments of Refs. 1 and 2 and

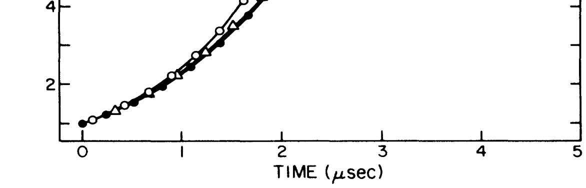

FIG. 1. Comparison of the laser field time evolution. ●, exact theory simultaneous solution of Eqs. (1,2); △, this model, solution of Eq. (7); and ○, linear theory, solution of Eq. (6).

define the laser field $a(z,\tau)$ and the electron phase $\zeta(z,\tau)$ with a spatial dependence z. Assuming that the slowly varying amplitude and phase approximation remain valid and after performing the change of variables $\tau=t(c/L)$ and $\bar{z}=z-ct$ the Maxwell wave equation reads,

$$a(\bar{z},\tau)=a(\bar{z},0)-j(\bar{z})\int_0^\tau d\tau'\langle e^{-i\zeta(\bar{z},\tau')}\rangle_{\zeta_0,\nu_0} . \quad (8)$$

The electron phase $\zeta(\bar{z},\tau)$ is governed by the Lorentz equation,

$$\left[\frac{\partial}{\partial\tau}-s\frac{\partial}{\partial\bar{z}}\right]\nu(\bar{z},\tau)=\mathrm{Re}[a(\bar{z},\tau)e^{i\zeta(\bar{z},\tau)}] ,$$

$$\left[\frac{\partial}{\partial\tau}-s\frac{\partial}{\partial\bar{z}}\right]\zeta(\bar{z},\tau)=\nu(\bar{z},\tau)$$

with formal solution

$$\zeta(\bar{z},\tau)=\zeta_0+\nu_0\tau+\int_0^\tau d\tau'(\tau-\tau')\mathrm{Re}\{a[\bar{z}+s(\tau-\tau'),\tau'] \times e^{i\zeta[\bar{z}+s(\tau-\tau'),\tau']}\} . \quad (9)$$

In the equations above we have left out a factor $[1-\nu(\bar{z},\tau)/2\pi N]$ which is close to unity unless the electrons lose an appreciable amount of energy during the interaction. The "slippage distance" $s=L(1-\beta)\approx N\lambda$ is the characteristic length in the longitudinal direction and represents the distance the radiation phase front moves ahead of a set of electrons, initially coincident, after traversing the undulator.

For our purposes it is convenient to write the Maxwell-Lorentz self-consistent system of equations for a free-electron laser as

$$a(\bar{z},\tau)=a(\bar{z},0)+ij(\bar{z})\int_0^\tau d\tau'\left\langle e^{-i\nu_0\tau'}J_1[\rho(\bar{z}+s\tau',\tau')]\frac{A(\bar{z}+s\tau',\tau')}{\rho(\bar{z}+s\tau',\tau')}\right\rangle_{\nu_0} . \quad (10)$$

$A(\bar{z},\tau)$ is defined as

$$A(\bar{z},\tau)=\int_0^\tau d\tau'(\tau-\tau')a(\bar{z}-s\tau',\tau')e^{i\nu_0\tau'} \quad\text{and}\quad \rho(\bar{z},\tau)^2=|A(\bar{z},\tau)|^2 .$$

Assuming the radiation field is a periodic function of $\bar{z}$ with period L the length of the resonator, we can expand $a(\bar{z},\tau)$ in Fourier series obtaining $(q=\pi/L)$

$$\begin{aligned}A(\bar{z}+s\tau,\tau)&=\int_0^\tau d\tau'(\tau-\tau')e^{i\nu_0\tau'}\sum_{n=-\infty}^{\infty}a_n(\tau')e^{inq[\bar{z}+s(\tau-\tau')-L\tau']}\\&=A_0(\tau)+\sum_{n(\neq 0)}A_n(\tau)e^{inq(\bar{z}+s\tau)} ,\end{aligned} \quad (11)$$

where we have defined

$$A_n(\tau)=\int_0^\tau d\tau'(\tau-\tau')a_n(\tau')e^{i\tau'(\nu_0-nqs-n\pi)} .$$

The argument of the Bessel function in Eq. (10) after making use of the assumption that the fundamental "strong" mode, $A_0(\tau)$ is greater than the sidebands "weak" mode, i.e., $|A_0|\gg|A_m|$ $(m\neq 0)$ becomes

$$\rho(\bar{z}+s\tau,\tau)=|A_0|\left[1+\frac{1}{|A_0|^2}\mathrm{Re}\left[A_0(\tau)\sum_{m(\neq 0)}A_m^*(\tau)e^{-imq\bar{z}}\right]\right] .$$

This assumption is drawn from the experimental indication that the UCSB FEL (Ref. 6) oscillates mainly in a single mode. Replacing this last expression of $\rho(\bar{z},\tau)$ in $J_1[\rho(\bar{z},\tau)]$ we obtain

$$J_1(\rho(\bar{z}+s\tau,\tau))\approx J_1(|a_0|)+\frac{J_1'(|A_0|)}{|A_0|}\mathrm{Re}\left[A_0(\tau)\sum_{m(\neq 0)}A_m^*(\tau)e^{-imq\bar{z}}\right] ,$$

where $J_1'(z)=dJ_1(z)/dz$ is the derivative of the Bessel function. Introducing this approximate expression for the Bessel function in Eq. (10) and keeping terms of the same order, we obtain the following integral equation for the laser field (we drop the bar over $z\equiv\bar{z}$):

$$a(z,\tau)=a(z,0)+ij(z)\int_0^\tau d\tau'\left\langle e^{-i\nu_0\tau'}\frac{A(z+s\tau,\tau')}{|A_0(\tau')|}\left[J_1[|A_0(\tau')|]+\left[\frac{J_1'[|A_0(\tau')|]}{|A_0(\tau')|}-\frac{J_1[|A_0(\tau')|]}{|A_0(\tau')|^2}\right]\times\mathrm{Re}\left[A_0(\tau')\sum_{m(\neq 0)}A_m^*(\tau')e^{-imqz}\right]\right]\right\rangle_{\nu_0} . \quad (12)$$

In an oscillator configuration we solve Eq. (12) supplemented with the boundary condition $a^{(n)}(z=0,\tau=0)=e^{(-1/2)Q}a^{(n-1)}(z=L,\tau=1)$, where Q represents the losses at both mirrors of the resonator and $a^{(n)}$ is the field amplitude at the end of the nth pass. Introducing the change of variable, $a(z,\tau)e^{(-1/2)Q\tau}=\bar{a}(z,\tau)$, the boundary condition for the field reads

$\bar{a}^{(n)}(0,0)=\bar{a}^{(n-1)}(L,1)$.

Expanding both sides of Eq. (12) in Fourier series we derive a system of integral equations for each one of the laser field modes; for the fundamental strong mode [we drop the bar over the field $a(z,\tau)$],

$$a_0(\tau)=a_0(0)+ij_0\int_0^\tau d\tau' e^{-i\nu_0\tau'}\left[\frac{A_0(\tau')}{|A_0(\tau')|}J_1[|A_0(\tau')|]+\left[\frac{J_1'[|A_0(\tau')|]}{|A_0(\tau')|^2}-\frac{J_1[|A_0(\tau')|]}{|A_0(\tau')|^3}\right]\times\left[\tfrac{1}{2}A_0(\tau')\sum_{m(\neq 0)}A_m^*(\tau')A_m(\tau')+\tfrac{1}{2}A_0^*(\tau')\sum_{m(\neq 0)}A_{-m}(\tau')A_m(\tau')\right]\right] \quad (13a)$$

and for the sidebands ($l\neq 0$),

$$a_l(\tau)a_l(0)+ij_0\int_0^\tau d\tau' e^{-i\tau'(\nu_0-qsl)}\left[\frac{A_l(\tau')}{|A_0(\tau')|}J_1[|A_0(\tau')|]+\left[\frac{J_1'[|A_0(\tau')|]}{|A_0(\tau')|^2}-\frac{J_1[|A_0(\tau')|]}{|A_0(\tau')|^3}\right]\times\left[\tfrac{1}{2}A_0(\tau')\sum_{m(\neq 0)}A_m^*(\tau')A_{m+l}(\tau')+\tfrac{1}{2}A_0^*(\tau')\sum_{m(\neq 0)}A_{l-m}(\tau')A_m(\tau')\right]\right] . \quad (13b)$$

In writing Eqs. (11) we have left out the average over the initial resonance parameter ν_0 for simplicity and assumed that the current is a constant function of z, i.e., $j(z)=j_0$ which is valid for the long electron pulse at the UCSB free-electron laser.

We point out that the saturation mechanism of the laser is explicitly shown in Eq. (13) and Eq. (7) by the presence of the Bessel functions; also, these integral equations for the Fourier component of the laser field do not display any explicit dependence on the electron variables.

IV. RESULTS AND CONCLUSIONS

A basic question in FEL theory and in conventional lasers in general is: can side (weak) modes build up from spontaneous emission levels in the presence of a single quasi-cw fundamental (strong) mode, i.e., is single-mode operation stable? A closely related question refers to the intrinsic linewidth of a very long electron pulse FEL.[7] To explore these questions and to provide an explanation of the observed experimental facts, we have solved numerically the set of integral equations (13).

Time domain measurements of the laser power for pulses shorter than 5 μsec (≈ 100 passes) strongly suggest that it operates in a stable single-cavity longitudinal mode having an extremely small bandwidth. Equation (13) gives the time evolution of the amplitude and phase of the laser field in an oscillator configuration. The multipass formalism contains four parameters: the dimensionless current j which controls the gain per pass, the loss coefficient Q, the mode separation $\Delta\equiv qs$, and finally, the electron beam energy decrease rate r. In the UCSB FEL the electron beam energy decreases monotonically at a constant rate $r(V/\mu\text{sec})$ determined by the electron recovery efficiency.

We typically use 50 time steps along the undulator length and the Simpson's rule to calculate the time integral in our evolution equations. The functions

$$A_n(\tau)=\int_0^\tau d\tau'(\tau-\tau')a_n(\tau')e^{i\tau'(\nu_0-nqs-n\pi)}$$

are computed using a modified Filon scheme[8] to take into account the fast variations of the exponential part. The $A_n(\tau)$ are finite Fourier integrals and any straightforward integration method for large values of the exponent would require a large number of points to obtain a reasonable accuracy. Filon suggested using a quadratic approximation for the relatively slowly varying part of the integrand $(\tau-\tau')a_n(\tau')$ instead of the entire integrand. We have tested the integral subroutine with known functions, obtaining better than 0.02% deviations from the exact values.

Our equations do not describe correctly the start-up of the laser as the $a_0(\tau)$ mode is singled out and assumed to be larger than all other modes $a_l(\tau)$ present in the problem. Furthermore, the formalism also entails that a certain degree of coherence in the laser field has already been achieved. In the first few passes in the cavity, the laser amplitude is small and consequently, all longitudinal modes evolve independently of each other. During this linear regime, mode competition between the longitudinal modes lying inside of the gain bandwidth produces a line narrowing of the laser spectrum. It is at this point in the laser evolution that our formalism has validity and describes fairly accurately the physical situation.

Typically we take as an initial laser field in our simula-

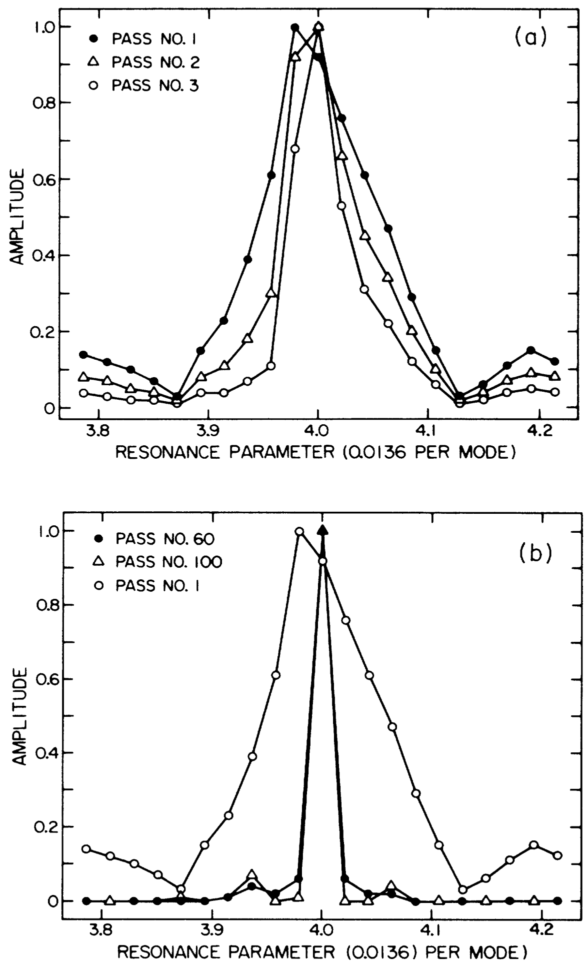

FIG. 2. Plot of the normalized dimensionless field amplitude of 21 consecutive longitudinal modes at different round trip passes. Parameters used are $j=10.0$, $Q=0.2$, $\triangle=0.0136$, $r=0.0$, and initial resonance parameter $\nu_0=4.0$. (a) ●, pass No. 1; △, pass No. 2; and ○, pass No. 3. (b) ○, pass No. 1; ●, pass No. 60; and △, pass No. 100.

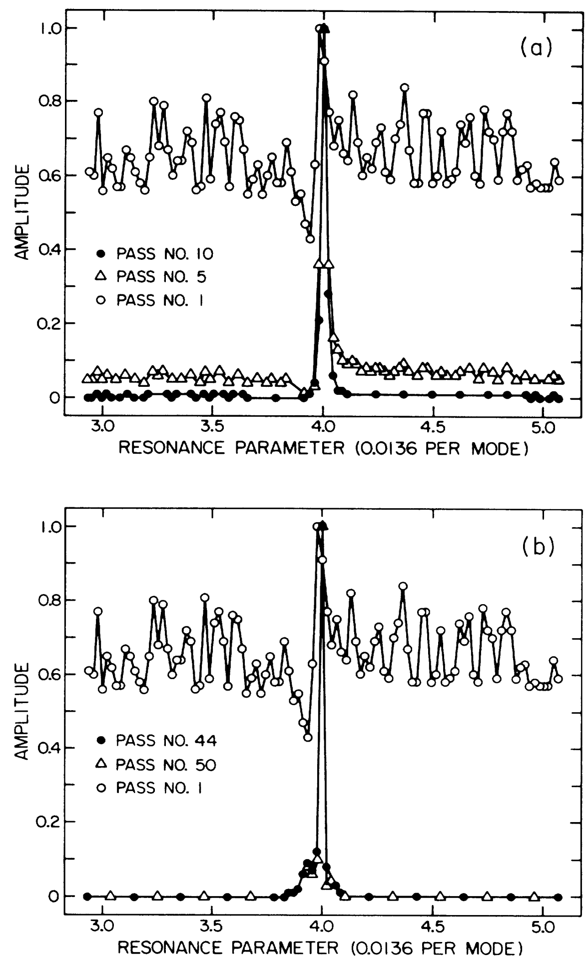

FIG. 3. Same as Fig. 2. Parameters are $j=10.0$, $Q=0.2$, $\Delta=0.27$, $r=3000$, and initial resonance parameter $\nu_0=2.6$. (a) ○, pass No. 1; △, pass No. 5; and ●, pass No. 10. (b) ○, pass No. 1; ●, pass No. 44; and △, pass No. 50.

tion the amplitude $|a_0(0)| \approx 0.001 > |a_l(0)|$ and a random phase $\phi_l(0)$. We present graphics of the spectrum of the laser field at different round trip passes in the cavity. Figure 2(a) plots the spectrum for the first three passes; we observe the beginning of the mode selection and narrowing of the spectrum. In Fig. 2(b) we compare the spectrum at the pass $n=1$, 60, 100; clearly the signal becomes single mode. Figure 3 is the same as Fig. 2 with a broader input spectrum. These results agree with the experimental evidence of stable single longitudinal mode operation of a very long electron beam FEL.

In conclusion, we have shown that the self-consistent system of coupled Lorentz-Maxwell equations describing an FEL can be written as a system of integral equations for the Fourier components of the laser field in the cavity. This formalism is well suited for the analysis of a multimode FEL.

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