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Eigenenergies of the $r^2+\lambda r^2/(1+gr^2)$ potential obtained by the shifted $1/N$ expansion

Y. P. Varshni

Ottawa-Carleton Institute for Physics, University of Ottawa, Ottawa, Canada K1N 6N5

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Energy eigenvalues are obtained for the three-dimensional potential $V(r)=\frac{1}{2}[r^2+\lambda r^2/(1+gr^2)]$, where λ and g are parameters, using the shifted-$1/N$-expansion method. Results are obtained for nine sets of n_r and l values corresponding to $n=0$, 1, 2, 3, and 4. It is found that certain levels which are degenerate in the limit $\lambda=0$ do not remain so as λ increases. This splitting is studied as a function of g and of λ. It is also shown that with a negative λ, this potential gives a sequence for energy levels which is identical with that which occurs in the shell model of the nucleus.

I. INTRODUCTION

There have been several investigations[1-14] on the energy levels and wave functions for the one-dimensional potential

$$V(x)=x^2+\lambda x^2/(1+gx^2)\ . \tag{1}$$

As summarized by Mitra,[1] this type of interaction occurs in several areas of physics. In particular, this type of potential occurs when considering models in laser theory,[15,16] and in quantum field theory with a nonlinear Lagrangian.[17,18] The Schrödinger equation for potential (1) is not solvable analytically for arbitrary values of λ and g. A variety of methods have been used to determine the eigenvalues and wave functions for the potential (1). Numerical values for the eigenenergies have been given by several authors[1-5,9,13] for various combinations of λ and g. The existence of a class of exact solutions, when certain algebraic relations between λ and g hold, has also been demonstrated.[6-9]

In the present paper we investigate the eigenvalues of the three-dimensional analogue of Eq. (1),

$$V(r)=\frac{1}{2}\left[r^2+\frac{\lambda r^2}{1+gr^2}\right], \tag{2}$$

using a variant of the $1/N$-expansion method. We include the factor $\frac{1}{2}$ in $V(r)$ so that in the limit $\lambda\to 0$, the eigenvalues will have the familiar sequence $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$.

In recent years, the large-N-expansion method, where N is the number of spatial dimensions, has been applied to calculate the eigenvalues for a number of potentials.[19-22] In this method $1/N$ is used as a perturbation expansion parameter. For spherically symmetric potentials, a natural choice for the expansion parameter is $1/k$ where $k=N+2l$, $l(l+N-2)\hbar^2$ being the eigenvalue of the square of the N-dimensional orbital angular momentum.[19] The convergence of this method is, however, rather slow. Recently Sukhatme and Imbo[23] have proposed a modification of this method, called the shifted $1/N$ expansion, which considerably improves the analytic simplicity and convergence of the perturbation series for the energy eigenvalues. The modification simply consists of using $1/\bar{k}$ as an expansion parameter, where $\bar{k}=k-a=N+2l-a$, and a is a suitable shift. The shifted-$1/N$-expansion method has been discussed and applied to a number of potentials by Imbo *et al.*[24] Several applications of the shifted-$1/N$-expansion method have been made.[25-28]

In Sec. II we summarize the expressions for calculating the energy eigenvalues. Calculations are carried out for various combinations of n_r and l corresponding to $n=0$, 1, 2, 3, and 4. The results are discussed in Sec. III. A possible application of the potential (2) with negative λ to shell model of the nucleus is also pointed out.

II. ANALYTIC EXPRESSION FOR THE EIGENVALUES

The procedure for determining the energy eigenvalues in the shifted-$1/N$-expansion formalism is given in the paper of Imbo *et al.*[24] Hence, for sake of brevity, we omit the intermediate steps and give here only the final expressions. We shall use atomic units ($\hbar=m=e=1$) in obtaining the formulas.

The energy eigenvalues in the shifted $1/N$ expansion are given in terms of r_0, which is determined from the position of the minimum of the effective potential

$$V_{\text{eff}}(r)=\frac{\hbar^2}{8mr^2}+\frac{V(r)}{\bar{k}^2} . \tag{3}$$

For the potential (2), with $N=3$,

$$\bar{k}^2=\frac{4r_0^4(1+2gr_0^2+g^2r_0^4+\lambda)}{(1+gr_0^2)^2} \tag{4}$$

and the equation for determining r_0 is found to be

$$(2l+1)+(2n_r+1)\left[\frac{4(1+gr_0^2+3g^2r_0^4+g^3r_0^6+\lambda)}{(1+2gr_0^2+g^2r_0^4+\lambda)(1+gr_0^2)}\right]^{1/2}=\frac{2r_0^2(1+2gr_0^2+g^2r_0^4+\lambda)^{1/2}}{(1+gr_0^2)} , \tag{5}$$

where n_r is the radial quantum number. The oscillator quantum number $n=2n_r+l$.

The final expression for the eigenvalues for the potential (2) is as follows:

$$E=\frac{\bar{k}^2}{r_0^2}\left[\frac{1}{8}+\frac{1}{8}\frac{(1+gr_0^2)(1+gr_0^2+\lambda)}{(1+2gr_0^2+g^2r_0^4+\lambda)}+\frac{\gamma^{(1)}}{\bar{k}^2}+\frac{\gamma^{(2)}}{\bar{k}^3}+O\left[\frac{1}{\bar{k}^4}\right]\right] . \tag{6}$$

The quantities $\gamma^{(1)}$ and $\gamma^{(2)}$ appearing in the corrections to the leading order of the energy expansion are

$$\gamma^{(1)}=\tfrac{1}{8}(1-a)(3-a)+(1+2n_r)\tilde{\varepsilon}_2+3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4-\frac{1}{\omega}[\tilde{\varepsilon}_1^2+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3+(11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2] , \tag{7}$$

$$\begin{aligned}\gamma^{(2)}=&(1+2n_r)\tilde{\delta}_2+3(1+2n_r+2n_r^2)\tilde{\delta}_4+5(3+8n_r+6n_r^2+4n_r^3)\tilde{\delta}_6\\&-\omega^{-1}[(1+2n_r)\tilde{\varepsilon}_2^2+12(1+2n_r+2n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4\\&\quad+2(21+59n_r+51n_r^2+34n_r^3)\tilde{\varepsilon}_4^2+2\tilde{\varepsilon}_1\tilde{\delta}_1+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\delta}_3+30(1+2n_r+2n_r^2)\tilde{\varepsilon}_1\tilde{\delta}_5+6(1+2n_r)\tilde{\varepsilon}_3\tilde{\delta}_1\\&\quad+2(11+30n_r+30n_r^2)\tilde{\varepsilon}_3\tilde{\delta}_3+10(13+40n_r+42n_r^2+28n_r^3)\tilde{\varepsilon}_3\tilde{\delta}_5]\\&+\omega^{-2}[4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2+36(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3+8(11+30n_r+30n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2\\&\quad+24(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4+8(31+78n_r+78n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4+12(57+189n_r+225n_r^2+150n_r^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4]\\&-\omega^{-3}[8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3+108(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2+48(11+30n_r+30n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3+30(31+109n_r+141n_r^2+94n_r^3)\tilde{\varepsilon}_3^4] ,\end{aligned} \tag{8}$$

in which

$$\tilde{\varepsilon}_j=\varepsilon_j/(2\omega)^{j/2},\quad \tilde{\delta}_j=\delta_j/(2\omega)^{j/2} ,$$

$$\omega=\frac{1}{2}\left[\frac{4(1+3gr_0^2+3g^2r_0^4+g^3r_0^6+\lambda)}{(1+2gr_0^2+g^2r_0^4+\lambda)(1+gr_0^2)}\right]^{1/2} ,$$

$$a=2-2(2n_r+1)\omega ,$$

$$\delta_1=-\tfrac{2}{3}\delta_2=-(1-a)(3-a)/4 ,$$

$$\delta_3=-\tfrac{4}{5}\delta_4=2\varepsilon_1=-\tfrac{4}{3}\varepsilon_2=2-a ,$$

$$\varepsilon_3=-\frac{1}{2}+\frac{r_0^2(-1+gr_0^2)g\lambda}{2(1+gr_0^2)^2(1+2gr_0^2+g^2r_0^4+\lambda)} ,$$

$$\varepsilon_4=\frac{5}{8}-\frac{r_0^2(-10gr_0^2+5g^2r_0^4+1)g\lambda}{8(1+gr_0^2)^3(1+2gr_0^2+g^2r_0^4+\lambda)} ,$$

$$\delta_5=-\frac{3}{4}+\frac{r_0^4(3gr_0^2-1)(gr_0^2-3)g^2\lambda}{4(1+gr_0^2)^4(1+2gr_0^2+g^2r_0^4+\lambda)} ,$$

$$\delta_6=\frac{7}{8}-\frac{r_0^4(-35g^2r_0^4+7g^3r_0^6+21gr_0^2-1)g^2\lambda}{8(1+gr_0^2)^5(1+2gr_0^2+g^2r_0^4+\lambda)} .$$

For any given choice of n, l, λ, and g, Eq. (5) determines r_0. The energy eigenvalue can then be calculated from Eq. (6). For the limit $\lambda\to 0$, $r_0=(2n_r+l+\frac{3}{2})^{1/2}$, $\bar{k}=2r_0^2$, and from Eq. (6) we recover the result $E=(2n_r+l+\frac{3}{2})$ for the three-dimensional harmonic oscillator.[29]

III. RESULTS AND DISCUSSION

Before coming to the results for potential (2), it is useful to consider the results for the three-dimensional harmonic oscillator. The levels having n_r and l values which satisfy $n=2n_r+l$ are degenerate. The n_r and l values for $n=0$–4 are shown in Table I.

TABLE I. n_r and l values for various levels.

n	n_r	l	Spectroscopic designation
0	0	0	1s
1	0	1	1p
2	1	0	2s
	0	2	1d
3	1	1	2p
	0	3	1f
4	2	0	3s
	1	2	2d
	0	4	1g

The three-dimensional harmonic oscillator is often used in discussions of the nuclear shell model[30] to furnish a simple reference set of levels. In nuclear physics literature, in spectroscopic notation, the levels are usually labeled by (n_r+1) and l values. Thus $n_r=1$, $l=1$ is a $2p$ level, and so on; we shall also use the same notation.

Calculations of eigenenergies for the potential (2) were carried out for 225 sets of (λ,g) values (15 values each of λ and g). A fraction of these results is shown in Table II. In each case nine values are shown corresponding to the various n_r and l combinations for $n=0-4$. The subsequent discussion and figures are based on all the results obtained and not merely on Table II.

There are no other eigenvalue results available by any other method, consequently it is not possible to infer the accuracy of the present results by a direct comparison.

TABLE II. The first nine energy levels for different values of λ and g in increasing order of excitation.

			g				
λ	n_r	l	0.1	1	10	100	1000
0.1	0	0	1.5600	1.5260	1.5044	1.5005	1.5000
	0	1	2.5932	2.5328	2.5047	2.5005	2.5000
	1	0	3.6156	3.5313	3.5046	3.5005	3.5000
	0	2	3.6220	3.5369	3.5048	3.5005	3.5000
	1	1	4.6383	4.5354	4.5048	4.5005	4.5000
	0	3	4.6472	4.5395	4.5049	4.5005	4.5000
	2	0	5.6546	5.5339	5.5047	5.5005	5.5000
	1	2	5.6590	5.5382	5.5048	5.5005	5.5000
	0	4	5.6694	5.5412	5.5049	5.5005	5.5000
1	0	0	2.0399	1.7561	1.5438	1.5049	1.5005
	0	1	3.3521	2.8261	2.5471	2.5050	2.5005
	1	0	4.5835	3.8126	3.5459	3.5050	3.5005
	0	2	4.6309	3.8674	3.5481	3.5050	3.5005
	1	1	5.8112	4.8528	4.5475	4.5050	4.5005
	0	3	5.8803	4.8938	4.5486	4.5050	4.5005
	2	0	6.9850	5.8379	5.5471	5.5050	5.5005
	1	2	7.0191	5.8815	5.5483	5.5050	5.5005
	0	4	7.1037	5.9118	5.5489	5.5050	5.5005
10	0	0	4.8095	3.7028	1.9366	1.5492	1.5050
	0	1	7.9069	5.5357	2.9704	2.5497	2.5050
	1	0	10.795	6.7480	3.9590	3.5495	3.5050
	0	2	10.918	7.0450	3.9811	3.5498	3.5050
	1	1	13.647	7.9813	4.9752	4.5497	4.5050
	0	3	13.844	8.3600	4.9862	4.5499	4.5050
	2	0	16.305	8.8512	5.9707	5.5497	5.5050
	1	2	16.419	9.2492	5.9827	5.5498	5.5050
	0	4	16.686	9.5688	5.9892	5.5499	5.5050
100	0	0	14.891	13.353	5.8056	1.9922	1.5499
	0	1	24.695	21.118	7.1819	2.9968	2.5500
	1	0	34.256	26.910	8.0514	3.9955	3.5500
	0	2	34.401	27.988	8.3055	3.9980	3.5500
	1	1	43.770	32.401	9.2414	4.9973	4.5500
	0	3	44.009	33.980	9.3601	4.9986	4.5500
	2	0	53.045	36.359	10.195	5.9969	5.5500
	1	2	53.187	37.211	10.323	5.9982	5.5500
	0	4	53.519	39.119	10.391	5.9989	5.5500
1000	0	0	47.271	45.628	32.370	6.4183	1.9992
	0	1	78.660	74.828	44.520	7.4675	2.9997
	1	0	109.80	101.68	46.680	8.4540	3.9995
	0	2	109.95	103.05	50.363	8.4802	3.9998
	1	1	140.89	128.08	50.554	9.4733	4.9997
	0	3	141.14	130.31	52.748	9.4858	4.9999
	2	0	171.74	152.26	49.720	10.468	5.9997
	1	2	171.89	153.56	53.015	10.482	5.9998
	0	4	172.23	156.58	54.264	10.489	5.9999

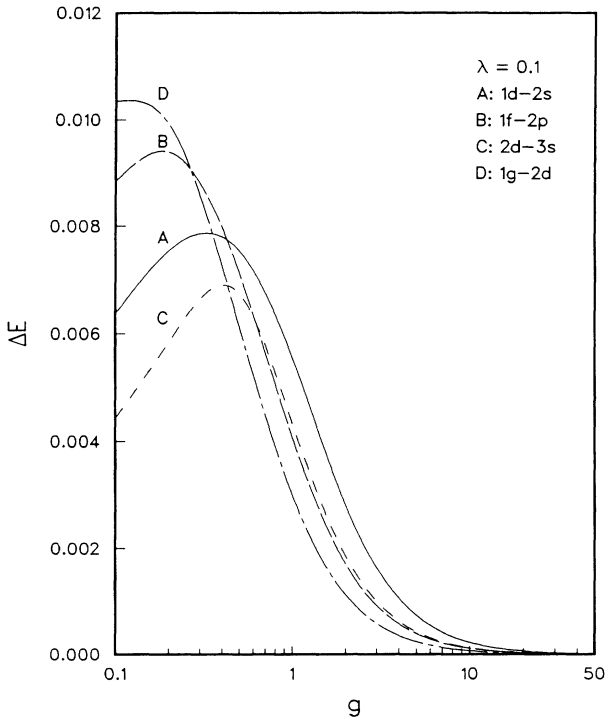

FIG. 1. ΔE for the four splittings vs g for $\lambda = 0.1$.

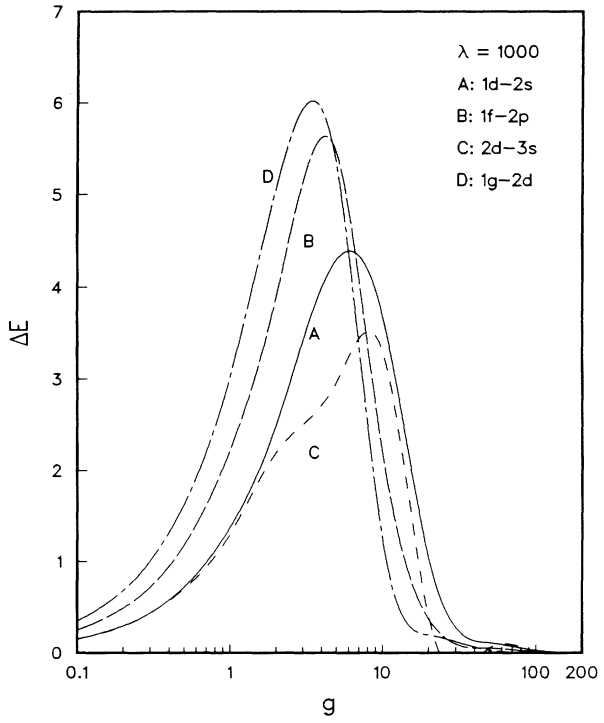

FIG. 3. ΔE vs g for $\lambda = 1000$.

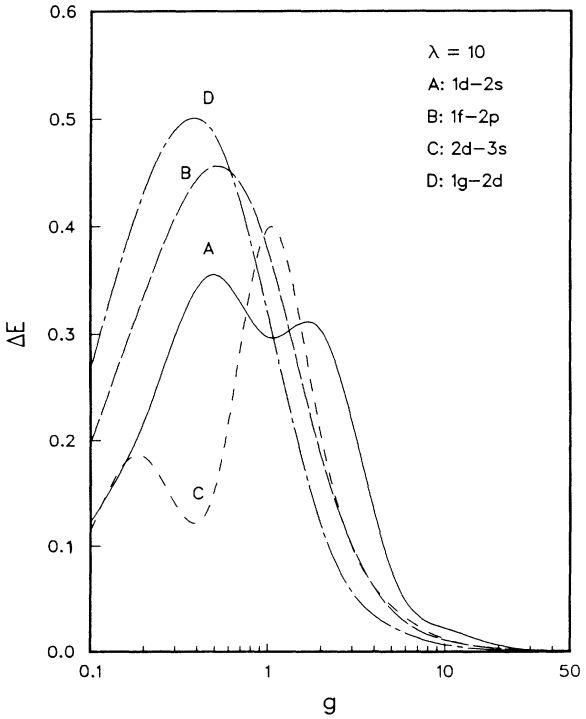

FIG. 2. ΔE vs g for $\lambda = 10$.

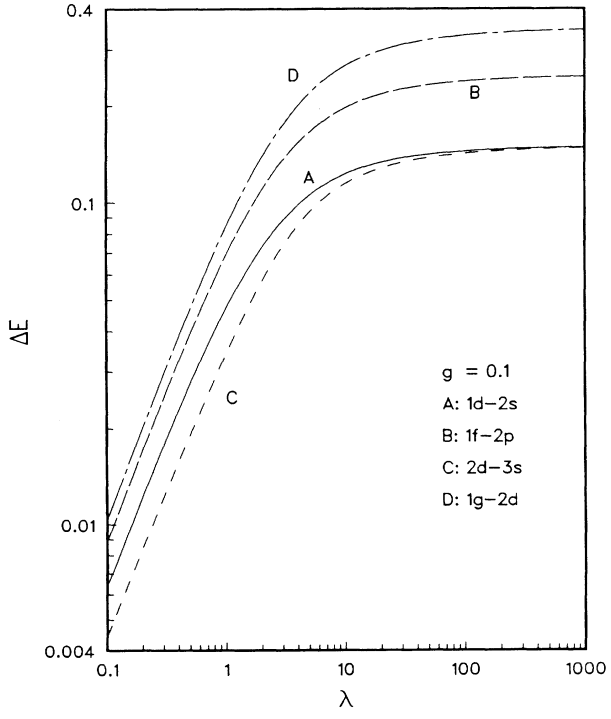

FIG. 4. ΔE vs λ for $g = 0.1$.

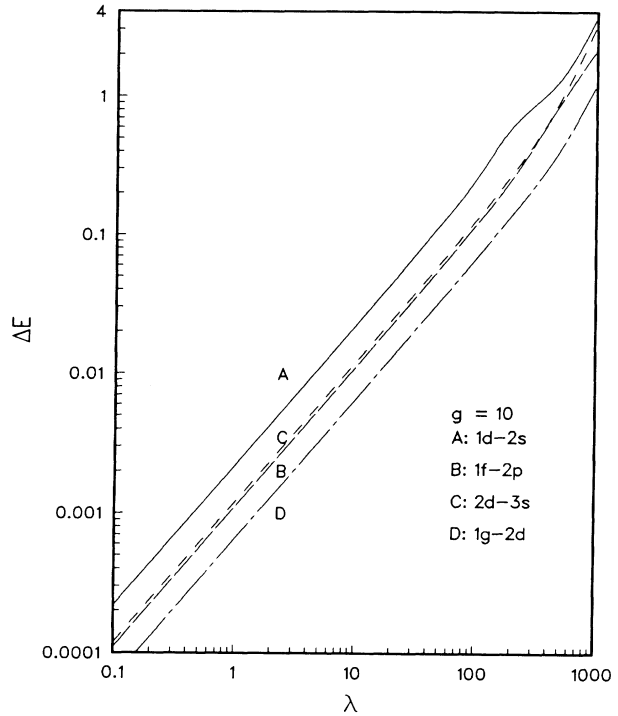

FIG. 5. ΔE vs λ for $g=10$.

However, some evidence on this point can be obtained from the magnitude of the $\gamma^{(1)}$ and $\gamma^{(2)}$ terms. For most values of λ and g the contributions of $\gamma^{(1)}$ and $\gamma^{(2)}$ terms are quite small and it would be reasonable to conclude that the calculated energies are satisfactory. However, there is a domain of λ,g values where this is not true for some levels, especially the $2s$ level. This domain is very roughly defined by $\lambda^{1/2}/g \approx 3-30$. In this region the contributions of $\gamma^{(1)}$ and $\gamma^{(2)}$ together sometimes becomes as high as a few percent of the leading term for the $2s$ level. Three other levels—$1s$, $1p$, and $1d$—also show a somewhat similar behavior but of a lesser magnitude. In these cases, the series in (6) appears to be not strongly convergent and the calculated energies may be less accurate than the others. When λ is very large and g is small, the magnitudes of $\gamma^{(1)}$ and $\gamma^{(2)}$ terms become rather large for the $3s$ level and the calculated value may have a substantial error. An example of such a situation occurs in Table II for $\lambda=1000$ and $g=10$. When both λ and g are large, the eigenenergies tend to the values 2, 3, 4, 5, 6, . . . which correspond to an "elevated" harmonic oscillator.

A perusal of Table II shows that for $\lambda \neq 0$, the degeneracy between certain levels for $n=2$, 3, and 4 (see Table I) is removed. We have studied this splitting as a function of λ and g. There are four splittings in our data, $E(1d)-E(2s)$, $E(1f)-E(2p)$, $E(2d)-E(3s)$, and $E(1g)-E(2d)$. Figures 1–3 show these splittings as a function of g for three values of λ, namely, $\lambda=0.1$, 10, and 1000. It will be noticed from Figs. 1–3 that in all cases the splittings tend to disappear when $g \to \infty$.

TABLE III. Energy eigenvalues for two negative values of λ.

Level	E ($\lambda=-0.5$, $g=0.1$)	E ($\lambda=-1$, $g=0.1$)
$1s$	1.1721	0.7806
$1p$	2.0000	1.4307
$1d$	2.8544	2.1407
$2s$	2.8962	2.2549
$1f$	3.7291	2.8941
$2p$	3.7832	3.0256
$1g$	4.6198	3.6806
$2d$	4.6800	3.8182
$3s$	4.7114	3.9082

Figures 4 and 5 show these splittings as a function of λ for $g=0.1$ and 10; for $g=1000$, the splittings are too small. In this case it will be noticed that ΔE values increase continuously with λ. In Fig. 4 each curve appears to be approaching a constant value. But in Fig. 5 this tendency is not to be seen, the splittings continue to increase with increase in λ. These five figures provide a sampling of the types of behavior of the splittings that may be exhibited by the potential (2).

In the shell model of the nucleus, the potential is believed to be intermediate between that of a three-dimensional harmonic oscillator and a square well.[30] More accurately, the nuclear potential is such that the degeneracy of levels in the harmonic oscillator potential is removed such that the level with the highest l value lies deepest. Thus the ordering of levels in the shell model is $1s$, $1p$, $1d$, $2d$, $1f$, $2p$, $1g$, $2d$, $3s$,. . . . Exactly such a result is obtained when we allow λ in the potential (2) to be negative. Two typical sets of results are shown in Table III. These results and those for other negative λ values suggest that with a suitable choice of λ and g values, Eq. (2) may closely approximate the nuclear potential. This possibility is being pursued.

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[1]A. K. Mitra, J. Math. Phys. **19**, 2018 (1978).
[2]S. K. Kaushal, J. Phys. A **12**, L253 (1979).
[3]S. Galicia and J. Killingbeck, Phys. Lett. **71A**, 17 (1979).
[4]N. Bessis and G. Bessis, J. Math. Phys. **21**, 2780 (1980).
[5]A. Hautot, J. Comput. Phys. **39**, 72 (1981).
[6]G. P. Fiessas, Phys. Lett. **83A**, 121 (1981).
[7]V. S. Varma, J. Phys. A **14**, L489 (1981).
[8]R. R. Whitehad, A. Watt, G. P. Flessas, and M. A. Nagarajan, J. Phys. A **15**, 1217 (1982).
[9]C. S. Lai and H. E. Lin, J. Phys. A **15**, 1495 (1982).
[10]J. Heading, J. Phys. A **15**, 2355 (1982).
[11]M. Znojil, J. Phys. A **16**, 293 (1983).
[12]N. Bessis, G. Bessis, and G. Hadinger, J. Phys. A **16**, 497 (1983).

[13]R. N. Chaudhuri and B. Mukherjee, J. Phys. A **16**, 4031 (1983).
[14]G. Marcilhacy and R. Pons, J. Phys. A **18**, 2441 (1985).
[15]H. Risken and H. D. Vollmer, Z. Phys. **201**, 323 (1967).
[16]H. Haken, *Laser Theory,* in Vol. XXV/9c of *Handbuch der Physik,* edited by S. Flugge (Springer-Verlag, New York, 1970).
[17]S. N. Biswas, K. Datta, R. P. Saxena, P. K. Srivastava, and V. S. Varma, J. Math. Phys. **14**, 1190 (1973).
[18]S. N. Biswas, G. P. Malik, and E. C. G. Sudarshan, Phys. Rev. D **7**, 2884 (1973).
[19]L. Mlodinow and N. Papanicolaou, Ann. Phys. (N.Y.) **128**, 314 (1980); **131**, 1 (1981).
[20]L. Mlodinow and M. P. Shatz, J. Math. Phys. **25**, 943 (1984).
[21]G. Moreno and A. Zepeda, J. Phys. B **17**, 21 (1984).
[22]A. Chatterjee, J. Phys. A **18**, 1193 (1985); **18**, 2403 (1985).
[23]U. Sukhatme and T. Imbo, Phys. Rev. D **28**, 418 (1983).
[24]T. Imbo, A. Pagnamenta, and U. Sukhatme, Phys. Rev. D **29**, 1669 (1984).
[25]R. Dutt, U. Mukherji, and Y. P. Varshni, Phys. Rev. A **34**, 777 (1986).
[26]R. Dutt, U. Mukherji, and Y. P. Varshni, J. Phys. B **19**, 3411 (1986).
[27]B. Roy, Phys. Rev. A **34**, 5108 (1986).
[28]A. Z. Tang and F. T. Chan, Phys. Rev. A **35**, 911 (1987).
[29]S. Flugge, *Practical Quantum Mechanics I* (Springer-Verlag, New York, 1971); R. A. Buckingham, in *Quantum Theory,* edited by D. R. Bates (Academic, New York, 1961), Vol. I.
[30]M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (Wiley, New York, 1955).