Saffman-Taylor fingers and directional solidification at low velocity

Thierry Dombre and Vincent Hakim
Groupe de Physique des Solides de l'Ecole Normale Supérieure, 24 rue Lhomond, F-75231 Paris Cédex 05, France
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We examine the McLean-Saffman equations for viscous fingering in the limit where the finger fills almost completely the Hele-Shaw channel ($\lambda \simeq 1$). We find an infinite countable set of solutions. For each branch of solutions, λ increases toward 1 as $(U-U_n^*)^{3/2}$, when the velocity U of the finger approaches a lower value U_n^* that we calculate. We then discuss the connections of these results with directional solidification at small Péclet numbers. Our analysis does not reveal any sign of wavelength selection for steady-state cells by a solvability condition, contrary to recent numerical findings.

I. INTRODUCTION

Recently, there has been much progress in the understanding of steady-state selection for different problems of interface dynamics. A discrete set of steady states propagating at constant velocity has been obtained numerically for dendritic growth,[1] the Saffman-Taylor problem of viscous fingering,[2] and directional solidification.[3] The selection of needle-crystal shapes[4] and viscous finger widths[5] has also been understood analytically in the high velocity limit, where interfacial tension effects have a small magnitude. In this limit, it amounts to the computation of transcendentally small terms appearing beyond all orders of the regular perturbation expansion. It has at last been understood why the relative finger width λ of a Saffman-Taylor finger approaches $\frac{1}{2}$ when its velocity increases. The first part of the present work is devoted to the solution of Saffman-Taylor equations in the opposite limit where the finger moves at low velocity and fills almost completely the channel. In this case, surface tension is playing a dominant role and cannot be treated as a perturbation. Our strategy is based in part on a previous work of Pomeau.[6] The idea is that when λ approaches 1 the finger can be separated into three different regions: the tip of the finger where curvature effects are important, the trailing part where they are small, and a matching region in between. In the first part of this paper we rederive in an intuitively clear way (we hope) Eqs. (10) and (14) for the interface shape in each of these regions. We then proceed to the matching. Our treatment is rather analogous to the classical analysis of coating films done by Landau and Levich.[7] We summarize our main new results and predictions.

(i) We find an infinite countable set of steady states and give an explicit parametrization of their limiting shapes when λ approaches 1.

(ii) Their widths tend to the channel width for a minimal velocity U_n^* that we compute. Namely $U_n^* = (2n-1)^2 (b^2/a^2)\pi^2 T/12\mu d_0^*$ with $d_0^* = 13.75$.

(iii) When the velocity U of the finger decreases toward U_n^*, the relative finger width λ increases toward 1 as $(U-U_n^*)^{3/2}$.

In the second part of this paper we study the possibility of wavelength selection for steady-state cellular interfaces in directional solidification. Until now, this problem has resisted analytical approaches because of the lack of a zeroth-order solution playing the role of Ivantsov parabolas in dendritic growth. It is, however, well known that Saffman-Taylor fingers and cellular interfaces are related when the diffusion length is much larger than the width of a cell[8] ($P \ll 1$ in the notations below). The cellular interface can then be viewed as a periodic array of fingers with suitable matching far above the tip of the fingers and far below in the trailing part. This is of particular interest because a wavelength selection was found in recent numerical work[3] and the data, when interpreted in the light of the above correspondence, seemed to indicate that when P decreased, the relative width of the corresponding Saffman-Taylor finger increased to 1. Using the results obtained in the first part of this paper, we work out this connection in detail for "fat" fingers ($\lambda \simeq 1$) and can thus study possible wavelength selection inside this restricted family. Unfortunately, our analytical study does not give any sign of a wavelength selection for steady-state cells in disagreement with the numerical work of Ref. 3 and we give arguments to indicate that this should be the case even when λ is not close to 1. We hope that the conflict with the numerics will be resolved in the future.[9] If a continuous family of steady states exists, as we think, there remains the possibility of understanding the selection observed in the experiment by a stability analysis, if it is not due to boundary effects as in convective cells.

II. SAFFMAN-TAYLOR FINGERS IN THE SMALL-$(1-\lambda)$ LIMIT

A. Description of the outer and inner regions

In the small-$(1-\lambda)$ limit, the finger profile separates into two regions where two different types of simple and useful approximations apply.

(i) In the tip region (called hereafter the outer region) the viscous fluid is weakly perturbed by the existence of

the thin layer of width $1-\lambda$ left on the sides of the advancing finger. To leading order in $1-\lambda$, it behaves as if it were completely expelled and advected by the finger; the flow is therefore uniform, with a velocity U equal to that of the finger.

(ii) In the trail region (called hereafter the inner region) we can no more neglect the tangential velocity of the viscous fluid relative to the finger. But since the finger profile becomes nearly flat, the flow is essentially parallel to the finger. The pressure will therefore be taken to be constant across the thickness of the layer and equal to its value on the interface. This approximation allows us to derive an ordinary differential equation for the shape of the finger [Eq. (14)] which then has to be matched to the "meniscus" shape in the outer region. The whole approach is very similar to the classical analysis of coating flows by Landau and Levich.[7]

Before presenting the equations which follow from the preceding remarks, let us recall the basic formulation of the Saffman-Taylor problem. If $\mathbf{v}(x,y)$ denotes the two-dimensional average velocity in the viscous fluid and $P(x,y)$ the pressure field, the flow outside the finger is governed by the following equations:

$$\mathbf{v}(x,y)=-(b^2/12\mu)\nabla P \ , \tag{1}$$

$$\nabla^2 P=0 \ , \tag{2}$$

where b is the plate spacing of the Hele-Shaw cell, μ the viscosity of the driven fluid (as usual, in the following, the viscosity of the pushing fluid will be assumed to be zero). For a steady-shape finger advancing at constant velocity U along the x direction, the set of boundary conditions on the interface is

$$(v_n)_{\rm int}=U\cos\theta \ , \tag{3}$$

$$(P)_{\rm int}=P_0-T\kappa \ , \tag{4}$$

where θ is the angle between the local normal vector $\hat{n}$ on the finger and the x direction, P_0 is the constant pressure inside the finger, T the surface tension between the two fluids, and κ the local curvature. On the side walls of the channel free-slip boundary conditions are imposed,

$$v_y=0 \ , \tag{5}$$

a will denote in the following the half-width of the channel, and $\Phi=-(b^2/12\mu)P$ the velocity potential.

In the outer region, as explained before, we are interested in interfaces going from one side of the channel to the other. The obvious solution of Eqs. (1)–(5) is therefore a uniform flow at the finger velocity

$$\Phi=Ux \ . \tag{6}$$

The interface profile then has to satisfy the condition

$$-(b^2/12\mu U)(P)_{\rm int}=-(b^2/12\mu U)(P_0-T\kappa)$$
$$=(x)_{\rm int} \ , \tag{7}$$

which means that the pressure difference between two points on the viscous fluid side of the interface is exactly compensated by the difference in curvature between these two points. Equation (7) is a differential equation for the interface profile.

We can now make contact with the results previously obtained in Ref. 6. We obtain a pendulum equation by differentiating Eq. (7) with respect to the dimensionless curvilinear abscissa s, taking a as unit length,

$$(b^2/12\mu Ua^2)\partial^2\theta/\partial s^2=-\sin\theta \ . \tag{8}$$

The general solution of Eq. (8) may be written as

$$\kappa=\partial\theta/\partial s=\pi[(2/d_0)(\cos\theta+C)]^{1/2} \ , \tag{9}$$

where we have introduced[10] as a dimensionless control parameter the combination $d_0=\pi^2 b^2 T/12\mu Ua^2$. This finally gives the following parametrization for the shape[6]

$$x(\theta)=-(d_0/2\pi^2)^{1/2}\int_0^\theta d\theta'\sin\theta'/(\cos\theta'+C)^{1/2} \ .$$
$$y(\theta)=(d_0/2\pi^2)^{1/2}\int_0^\theta d\theta'\cos\theta'/(\cos\theta'+C)^{1/2} \ . \tag{10}$$

The integration constant C as well as d_0 are at this stage still arbitrary, but we find a relation between them by imposing that the meniscus completely fills the channel width, which means

$$y(\theta=\pi/2)-y(\theta=0)$$
$$=(d_0/2\pi^2)^{1/2}\int_0^{\pi/2}d\theta\cos\theta/(\cos\theta+C)^{1/2}=1 \ . \tag{11}$$

Note that in writing Eq. (11) we have favored solutions which come tangentially to the side walls (zero contact angle for $y=1$). This will turn out to be exactly true only for the limit shape $\lambda=1$ and more generally the upper bound of integration in Eq. (11) will also have to be considered as an unknown function, close but not equal to $\pi/2$ (see below Sec. II B).

Among all solutions of Eqs. (9) and (11), the one corresponding to $C=0$ plays clearly a special role. Whereas for $C>0$ the curvature never vanishes, for $C=0$, $dy/dx=-\cot\theta$ and κ vanish together on the side walls. This particular solution will turn out to be our $\lambda=1$ solution, since small curvature and small contact angle will be required for the matching to the inner region.

We proceed now to the discussion of the inner region. Let $\hat{\Phi}=\Phi-Ux$ denote the velocity potential in the frame of the finger. From Eqs. (2), (3), and (5) it is easily seen that the relative flux of the viscous fluid flow normally to the direction of propagation of the finger is a conserved quantity, i.e.,

$$\varphi(x)=\int_{y(x)}^1 dy'\,\partial_x\hat{\Phi}(x,y')=\varphi(x=-\infty) \ . \tag{12}$$

In the inner region we approximate $\varphi(x)$ by the value of $\partial_x\hat{\Phi}$ on the interface multiplied by the local thickness of the layer $h(x)=1-y(x)$. This gives

$$\varphi(x)=Uh(x)[(d_0/\pi^2)d^3h/dx^3-1] \ . \tag{13}$$

To derive Eq. (13) we have used the fact that in the inner region $|dh/dx|\ll 1$ and set

$$\kappa(x)=(d^2h/dx^2)[1+(dh/dx)^2]^{-3/2}\simeq d^2h/dx^2 \ .$$

On the other hand, as x goes $-\infty$, (i.e., far behind the finger tip), $\hat{\Phi}$ behaves as $-Ux$, expressing the fact that the viscous fluid is at rest in the laboratory frame, and $h(x)$ tends by definition to its limiting value $1-\lambda$. Equations (12) and (13) lead therefore to

$$h(x)[(d_0/\pi^2)d^3h/dx^3-1]=-(1-\lambda) . \tag{14}$$

By rescaling the variable x and the function h as $x=[(1-\lambda)d_0/\pi^2]^{1/3}\tilde{x}$ and $h=(1-\lambda)\tilde{h}$, we get the following parameterless equation, constituting our inner problem,

$$d^3\tilde{h}/d\tilde{x}^3=(\tilde{h}-1)/\tilde{h} . \tag{15}$$

As pointed out by Pomeau,[6] this equation is similar to the equation of Landau and Levich[7] for the problem of the drained film and has been derived here in an analogous manner (the difference resides only in the denominator of the right-hand side where we have h instead of h^3). Our scalings and equations are identical to those given in Ref. 6. They have been derived directly in real space for the sake of simplicity. Equation (15) has to be integrated with the boundary condition

$$\tilde{h}\to 1, \quad d\tilde{h}/d\tilde{x} \text{ and } d^2\tilde{h}/d\tilde{x}^2\to 0 \quad \text{as } \tilde{x}\to-\infty . \tag{16}$$

We are now ready to proceed to the matching between the two regions.

B. Matching and predictions

We consider first the inner problem. As in the Landau-Levich calculation, the prescription (16) is enough to fix entirely the solution, up to a translation on the x axis. This can be understood by linearizing Eq. (15) around $\tilde{h}=1$. One finds that the possible asymptotic behaviors of $\tilde{h}$ are of the form

$$\tilde{h}=1+\alpha\exp(\mu\tilde{x}) \quad \text{as } \tilde{x}\to-\infty ,$$

with $\mu^3=1$. The only acceptable solution is the convergent one ($\mu=1$), the constant α corresponding to the translational degree of freedom with respect to x. From Eq. (15), it is also clear that $\tilde{h}$ diverges like $\tilde{x}^3/6$ as $\tilde{x}\to+\infty$. Its leading behavior may be written in this limit as

$$\tilde{h}(\tilde{x})=\tilde{x}^3/6+\beta\tilde{x}^2/2+\gamma\tilde{x}+\delta-\ln(\tilde{x})/2+o(1/\tilde{x}) , \tag{17}$$

where β,γ,δ are numbers of order one which have to be calculated numerically and which naturally depend on the choice of origin in x. Note, however, that the quantity $\beta^2-2\gamma$ is translationally invariant. We have found numerically

$$\beta^2-2\gamma=6.0\pm0.1 . \tag{18}$$

This indicates that γ is always greater than $\gamma^*=3.0\pm0.05$, a value which is attained when β vanishes.

In the outer region $h(x)$ takes the general form

$$h(x)=\sum_n \epsilon^n h^n(x) , \tag{19}$$

where the small expansion parameter ϵ turns out to be $(1-\lambda)$ (Ref. 11) and the zeroth-order term $h^0(x)$ belongs to the class of functions described by Eqs. (9) and (11) of Sec. II A. We choose as the origin of the x coordinate the point where the outer meniscus touches the side wall [i.e., $h^0(x)=0$]. Since $\tilde{x}$ is the rapid variable, we then have to match the large-x asymptotics of $\tilde{h}(\tilde{x})$ given by Eq. (17) to the small-x expansion of $h(x)/1-\lambda$ around $x=0$. As a simplifying feature, we observe that powers of x coming from higher-order corrections to $h^0(x)$ in (19), once expressed with the inner variable $\tilde{x}$ and divided by $1-\lambda$, all vanish in the limit $(1-\lambda)\to 0$. They can be discarded in our leading order matching procedure. Writing now

$$h^0(x)/(1-\lambda)=\sum_p a_p/(1-\lambda)x^p$$
$$=\sum_p (1-\lambda)^{p/3-1}a_p(d_0/\pi^2)^{p/3}\tilde{x}^p , \tag{20}$$

we note that derivatives of h^0 of order greater than 3 also give negligible contributions in the limit $(1-\lambda)\to 0$. Finally, what we have to do to leading order, is to match the first three terms of the inner expansion [Eq. (17)] to the first three ones of the right-hand side of Eq. (20) [see note (Ref. 11) for the logarithmic term]. Going back to formulas (9) and (10) and calling θ_M and κ_M, the angle and the curvature of the outer solution at $x=0$, one easily gets

$$dh^0/dx_{x=0}=\cot\theta_M , \tag{21a}$$

$$d^2h^0/dx^2_{x=0}=\kappa_M/\sin^3\theta_M , \tag{21b}$$

$$d^3h^0/dx^3_{x=0}=3\cos\theta_M\,\kappa_M^2/\sin^5\theta_M + (\pi^2/d_0)/\sin^3\theta_M . \tag{21c}$$

If the linear and squared terms in Eqs. (17) and (20) are to be identified, it is clearly seen from Eq. (21) that $\pi/2-\theta_M$ and κ_M have to be respectively of order $(1-\lambda)^{2/3}$ and $(1-\lambda)^{1/3}$. These two conditions once met, the matching of the cubic terms follows automatically [up to corrections of the order $(1-\lambda)^{4/3}$]. At this point two choices are possible: either the constant C in Eq. (9) is taken positive of the order $(1-\lambda)^{2/3}$ and $\theta_M=\pi/2$, or C is taken negative and then $\theta_M=\arccos C$, $\kappa_M=0$. It can be shown that every other choice, compatible with the preceding requirements on θ_M and κ_M, is reducible to one of these two cases at the price of a translation of order 1 in the inner variable and corrections of order $1-\lambda$ which are beyond the scope of this paper. The first choice would lead, as can be seen from Eq. (10), to a parameter d_0 increasing with $1-\lambda$ from $\lambda=1$, which clearly does not correspond to what is physically expected. Fortunately (and undoubtedly in a nonaccidental way), the numerical result given in Eq. (18) is negative and forces us to adopt the second assumption. We deduce from Eqs. (17), (20), and (21) the relation

$$\cot\theta_M\simeq|C|=\gamma^*(\pi^2/d_0)^{1/3}(1-\lambda)^{2/3} , \tag{22}$$

where $\gamma^*\simeq 3$. This result in turn enables us to link $1-\lambda$ to d_0 as $1-\lambda$ tends to zero when we put it into condition (11),

$$1=(1/\pi)(d_0/2)^{1/2} \times \int_0^{\arccos C} d\theta \cos\theta/(\cos\theta-|C|)^{1/2} . \quad (23)$$

The maximal value of d_0 corresponding to $1-\lambda=0$ is obtained by setting $C=0$ in the last equation, which yields

$$d_0^* = 2\pi^2 \Big/ \left[\int_0^{\pi/2} d\theta (\cos\theta)^{1/2} \right]^2 = (1/4\pi)[\Gamma(1/4)]^4 \simeq 13.75 . \quad (24)$$

It is worth noting that this value lies well outside the region of instability of the planar interface ($d_0<1$) and significantly below the Kadanoff-Shraiman upper bound[12] $d_0=2\pi^2$ for the existence of a fingerlike interface. From Eqs. (22) and (23) we conclude that d_0 approaches its maximal value as $1-\lambda$ goes to zero like

$$(d_0^*-d_0)=\gamma^*(\pi^2/d_0^*)^{1/3}(\partial d_0/\partial C)_{C=0}(1-\lambda)^{2/3} . \quad (25)$$

Calling $I(C)$ the integral on the right-hand side of Eq. (23), we get

$$(\partial d_0/\partial C)_{c=0} = -2d_0^* I'(0)/I(0) = (1/\pi)(d_0^*)^{3/2} K(1/\sqrt{2}) , \quad (26)$$

where $K(1/\sqrt{2})$ is a complete elliptic integral of the first kind.[13] The numerical coefficient in front of $(1-\lambda)^{2/3}$ in Eq. (25) is found to be of the order of 81.

Unpublished numerical work by McLean[14] which extends the results presented in Ref. 15 seems to corroborate our prediction. He investigated values of $1-\lambda$ as small as 10^{-4}. For his last reliable data $1-\lambda=5.8\times10^{-4}$, he gets for d_0 the value 13.302 which is in good agreement with our estimation (24). Moreover, the $\frac{2}{3}$ power law appears to be well established in this range of values of $1-\lambda$ [although there seems to be a discrepancy of 20% concerning the limiting value of the ratio $(d_0^*-d_0)/(1-\lambda)^{2/3}$, whose origin is not clear to us].

To end up this section, we stress that the existence of higher branches of solutions for small $1-\lambda$ may also be easily understood in our approach. In our discussion of the external region, essentially described by a pendulum equation, we have considered up to now solutions where the "pendulum" performs one single oscillation between $-\theta_M$ and θ_M (from one side of the channel to the other one). The matching to the inner solution was then made possible by the vanishing of the curvature κ_M. Obviously, other solutions where the pendulum performs an odd number of oscillations greater than one are equally admissible. Hence, in the limit of small $1-\lambda$ we deduce the existence of a countable set of solutions characterized by the appearance of n bumps and $n-1$ holes. For the nth branch Eq. (23) transforms into

$$1/(2n-1)=1/\pi(d_0/2)^{1/2} \times \int_0^{\arccos C} d\theta \cos\theta/(\cos\theta-|C|)^{1/2} , \quad (27)$$

where C is still given by Eq. (22). The upper bounds d_n^* for these successive branches behave therefore like $d_0^*/(2n-1)^2$. This means that the lowest velocity attainable on the nth branch increases with n as $(2n-1)^2$. Such kinds of solutions have been recently observed numerically[16] in the related symmetric model of dendritic growth in a channel (with a diffusion length much larger than the radius of curvature of the tip).

Our finding suggests that the infinite discrete family of solutions found near $\lambda=\frac{1}{2}$ persists for all values of λ between $\frac{1}{2}$ and 1. It is also amusing to point out that it may explain simply in the opposite limit recent numerical results concerning the stability of the various branches in the $\lambda\simeq\frac{1}{2}$ limit.[17] It has been found that solutions on the nth branch are unstable against $(n-1)$ deformation modes localized around the finger tip. In the $\lambda\simeq1$ limit, it is clear that on the nth branch each bump may grow to the detriment of others. This yields n unstable modes to which we have to subtract the zero frequency mode corresponding to a global translation of the finger. Therefore, we also expect $n-1$ unstable modes on the nth branch.

III. DIRECTIONAL SOLIDIFICATION

We turn now to an apparently different problem, namely, the mechanism of wavelength selection for directional solidification cells. In most common directional solidification experiments, one is drawing at constant velocity a dilute binary mixture across a linear temperature gradient. Above a critical pulling speed, the interface between the liquid and solid parts assumes a cellular pattern, where deep liquid grooves separate periodically spaced fingers of solid.[18] The interesting problem, yet unsolved, is to understand the mechanism of wavelength selection for this pattern. This question is treated here because, when the diffusion length is much larger than the width of a cell, a Saffman-Taylor finger of a given width corresponds to solidification cells of a particular wavelength in a sense that is made precise by a matching procedure, as explained below. A recent numerical prediction of solidification cell wavelength[3] translates into the fact that the relative width of the corresponding finger approaches unity. This particular limit is now under control by using the techniques and results of the first part of this paper. We can therefore perform analytically the above-mentioned numerical calculation. However, this analysis leads us to conclude that a continuous family of steady-state cells exists for a given value of the pulling speed in contradiction with the numerical findings of Ref. 3.

We consider here the one-sided model of directional solidification (diffusion only in the liquid part). As in Ref. 3, we will restrict ourselves to the somewhat unrealistic case of a partition coefficient $K=(dCeq^{sol}/dT)/(dCeq^{liq}/dT)$ equal to 1. However, some of the subsequent analysis could be applied with slight modifications to the more general case of K different from 1. We refer to the reader to Ref. 3 for more details on the notations or definitions and to the review paper by Langer[19] on crystal growth for a discussion of the physics contained in the model. The equations to be solved are

$$\nabla^2 u + P\partial u/\partial x = 0 ,$$
$$P\cos\theta = -(\hat{\mathbf{n}}\cdot\nabla u)_{\text{int}} , \tag{28}$$
$$u_{\text{int}} = 1-(P/2\nu)x-(2V/P)\kappa .$$

The diffusion field $u(x,y)$ measures the excess solute concentration in the liquid, which has to vanish for $x\to+\infty$. The Péclet number P is defined as the ratio of $2a$, the cell spacing (chosen in the following to be the unit of length rather than a as in Secs. II and III) over the diffusion length $l=2D/U$. We will be interested in the limit of small Péclet numbers, which is experimentally relevant. The two other dimensionless parameters ν and V in Eqs. (28) are defined, respectively, as the ratios of the thermal length and the capillary length over the diffusion length. Note that $\nu=\infty$ means the absence of temperature gradient along the x direction. The problem of wavelength selection, in this formulation, amounts to seeing whether in Eqs. (28) the Péclet number P is determined as a function of ν and V.

It is useful to introduce a new field $w(x,y)=[2\nu/(2\nu-1)](1-u/P-x/2\nu)$ and to rewrite Eqs. (28) in terms of w,

$$\nabla^2 w + P\partial w/\partial x = -P/(2\nu-1) ,$$
$$\cos\theta = (\hat{\mathbf{n}}\cdot\nabla w)_{\text{int}} , \tag{29}$$
$$w_{\text{int}} = [2\nu/(2\nu-1)](2V/P^2)\kappa .$$

In the small-P limit three regions may be distinguished in the space occupied by the liquid part.

(i) Well downstream the interface ($x\to+\infty$), the flow of u is directed essentially along the x direction and decays exponentially as

$$u\sim u_{\text{tip}}\exp -P(x-x_{\text{tip}}) .$$

(ii) On length scales of order 1 around the finger tip, the terms linear in P are negligible compared to the Laplacian in the conservation equation. Up to corrections of order P, Eqs. (29) are identical to the Saffman-Taylor equations, with the quantity $\sigma=[2\nu/(2\nu-1)](2V/P^2)$ playing the same role as $d_0/4\pi^2$ in Sec. II. If we call the capillary length l_{ca} and remember that the instability threshold of the planar interface occurs for $V\ll 1$ at $\nu_c=\frac{1}{2}+O(V^{1/3})$, this control parameter σ can be written for small V as

$$\sigma = l_{\text{ca}}D/[4a^2(U-U_c)] , \tag{30}$$

where U_c is the critical velocity. In this region the shape of the interface will be very much like that of a Saffman-Taylor finger characterized by a value of λ uniquely determined by the value of σ ($\frac{1}{2}<\lambda<1$ for $0<\sigma<0.35$). At this stage λ is arbitrary and nothing fixes P. If we were able to derive another relation between λ and P, then the mechanism of selection would be understood.

(iii) Well behind the tip region ($x\to-\infty$, $\kappa\to 0$), the presence of a temperature gradient constrains the grooves to become infinitely narrow. The asymptotic profile of the cell ($-\frac{1}{2}\le y\le\frac{1}{2}$) in the cusp region takes the form[3] for $y\to\frac{1}{2}$

$$\tfrac{1}{2}-y = A\exp[Px/(2\nu-1)] . \tag{31}$$

Clearly this phenomenon occurs on length scales of order $1/P$, large compared with 1 and one expects the matching of this asymptotic behavior to a Saffman-Taylor finger to be possible for any value of λ. We now show it explicitly in the $\lambda\simeq 1$ limit. To match regions (ii) and (iii), we write an equation generalizing in the case of finite P Eq. (13) of Sec. II. As before, we consider the flux

$$\varphi(x) = \int_{y(x)}^{1/2} dy\,\partial_x(w-x) \simeq h(\sigma\, d^3h/dx^3 - 1) .$$

$\varphi(x)$ is no more a conserved quantity but obeys the equation

$$\partial\varphi(x)/\partial x + P\varphi(x) = -P[2\nu/(2\nu-1)]h(x) , \tag{32}$$

which gives for $h(x)=\frac{1}{2}-y(x)$, the following fourth-order differential equation

$$\sigma h\, d^4h/dx^4 = (dh/dx)(1-\sigma\, d^3h/dx^3) - P[1/(2\nu-1)+\sigma\, d^3h/dx^3]h . \tag{33}$$

The asymptotic behavior of Eq. (31) is easily recovered by balancing the linear terms in Eq. (33). It appears, moreover, that the solutions of Eq. (33) possessing this asymptotic behavior belong to a two-dimensional set. We find indeed that they can be parametrized by two independent constants

$$h(x) = A\exp[Px/(2\nu-1)] + \cdots + B\exp(-[3(2\nu-1)/P(\sigma A)^{1/3}]\{\exp[-Px/3(2\nu-1)]-1\}) , \tag{34}$$

where we have discarded higher-order corrections in P which might be expressed in terms of A and B. Note that for $|Px|\ll 1$ and by assuming $A=(1-\lambda)/2$, $B=\alpha(1-\lambda)/2$, Eq. (34) reduces to

$$h(x)/(1-\lambda) = \tfrac{1}{2}+(\alpha/2)\exp\{[2/(1-\lambda)\sigma]^{1/3}x\} , \tag{35}$$

which is the result obtained in Sec. II B for the asymptotic behavior of the Saffman-Taylor problem in the inner region ($x\to-\infty$). Since the choice of A and B is arbitrary, we conclude that Eq. (33) cannot provide the supplementary relation between λ and P that we are looking for. From Eq. (34) we deduce also that is is safe to identify, to leading order in P, the constant A with $1-\lambda$, provided that the origin $x=0$ in Eq. (34) is chosen in the region where the Saffman-Taylor description applies. In other words, any Saffman-Taylor finger shape in region (ii) can be continued smoothly in region (iii)

with the help of Eq. (33). For λ close to 1, the terms proportional to P can be neglected in the matching region and Eq. (33), then, integrated once and rescaled to give Eq. 15 (it is easily seen in the rescaled variables that the neglected terms are indeed negligible). The perturbation of h, the solution of Eq. (15), by the neglected terms can be taken into account by usual multiple-scale analysis[20] and at first order amounts to replacing $(1-\lambda)$ by $(1-\lambda)\exp[Px/(2\nu-1)]$ everywhere in the expression of h.

The matching between region (i) and region (ii) gives a relation between $1-\lambda$ and the position of the interface that we can also easily obtain by invoking the global conservation law of the solute in the liquid. It leads in this case to the following integral condition on the interface:

$$0=\int_0^{1/2} dy(1-u)/P = \int_0^{1/2} dy\{x/2\nu+[(2\nu-1)/2\nu]\sigma\kappa\} . \quad (36)$$

The cusp contributes in this equation by a finite quantity, equal to leading order in P to $(1-\lambda)/2P[(2\nu-1)/2\nu]$, which one has to add to the contribution of the Saffman-Taylor finger. Let us introduce a point (x_e,y_e) located at the boundary between the finger-tip region and the exponential tail. Then Eq. (36) can be expressed as

$$[(2\nu-1)/4\nu](1-\lambda)/P-x_e/4\nu = [(2\nu-1)/2\nu]\sigma+(1/2\nu)\int_0^{ye} dy(x-x_e) . \quad (37)$$

We first note that this relation, in the case of a uniform temperature ($\nu=\infty$), reduces to $1-\lambda=2P\sigma$, which is the result obtained for dendritic growth in a channel with an undercooling equal to 1 (Ref. 21). But at finite ν the undeterminacy on $1-\lambda$ remains. The right-hand side of Eq. (37) is a quantity of order 1 since it depends only on the shape of the Saffman-Taylor finger of width λ. Therefore, solutions corresponding to a nonvanishing $(1-\lambda)$ in the $P\ll 1$ limit are necessarily accompanied by displacements of the interface of order $1/P$ relatively to the origin of x imposed by the temperature gradient. Finite displacements of the interface occur, on the contrary, if $1-\lambda$ vanishes in the small-P limit, at least as fast as P. Within this assumption, our outer solution for $\lambda-1$ described in Sec. II gives a fair approximation to the finger-tip region and Eq. (37) takes the more simple form, valid to leading order in P,

$$[(2\nu-1)/4\nu](1-\lambda)/P-x_e/4\nu=\sigma^*=0.35 , \quad (38)$$

where x_e now designates the position of the rear of the meniscus. Here again $1-\lambda$ remains indeterminate.

In this paper we have studied Saffman-Taylor fingers and directional solidification cells at low velocity and have obtained many new results. It would be worth testing numerically our predicted shapes for the fundamental and higher branches of Saffman-Taylor fingers as well as our values for minimal velocities. For solidification cells, in contrast to the numerics, we do not find analytically any selection of P by a microscopic solvability principle but indeed a continuum of solutions where P (i.e., the cell width) is related to the position of the interface [via Eqs. (31) and (37)] (Ref. 22). Clearly, further work is needed to elucidate this discrepancy and to decide whether something goes wrong with the numerics,[9] or whether some subtle effect is missing in our analysis.

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