Analysis of radiation focusing and steering in the free-electron laser by use of a source-dependent expansion technique

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(Received 16 October 1986)

In the free-electron laser the resonant interaction between the radiation field and electron beam can result in radiation focusing (optical guiding). If the centroid of the electron beam is transversely displaced off axis, the radiation field, under certain conditions, will follow and be steered by the electron beam. The effect of a spatial modulation on the electron-beam envelope can also modify the propagation characteristics of the radiation. These and other phenomena are analytically and numerically studied using a novel source-dependent Laguerre-Gaussian modal representation of the fully three-dimensional radiation field. Unlike the vacuum Laguerre-Gaussian modal expansion, the longitudinal spatial dependence of the radiation waist and curvature are determined and characterized by the source term in the wave equation. Among the advantages of this general source-dependent expansion approach is that few modes are needed to accurately describe the radiation. Hence, fast and accurate numerical solutions of the fully three-dimensional free-electron laser problem can be obtained over distances of many Rayleigh lengths. Furthermore, this expansion enables us to drive an envelope equation for the radiation beam as well as an expression for the centroid of the beam.

I. INTRODUCTION

In the one-dimensional analysis of the free-electron laser (FEL) the radiation field, wiggler field, and electron beam resonantly couple so as to modify the longitudinal wave number of the radiation field.[1–3] This resonant interaction, between the coherent radiation and electron beam in the FEL mechanism, can lead to focusing of the radiation beam. This phenomenon was first analyzed for the low-gain FEL with transverse effects[4] where it was shown that the diffractive spreading of the radiation beam could be overcome by a focusing effect arising from the modified index of refraction. Recently optical guiding in FEL's operating in the exponential-growth regime has been studied in the small-signal, exponential-growth regime,[5–8] for the asymptotic behavior of the radiation beam. This radiation-focusing phenomena has been shown to play a central role in the practical utilization of the FEL,[9] since, in many proposed experiments, the short-wavelength radiation beam will not be confined or guided by a waveguide structure. Furthermore, the interaction length (wiggler length) is usually long compared to the Rayleigh length associated with the radiation beam. Therefore, focusing of the radiation beam, via the resonant interaction with the electron beam, is necessary in order to overcome the natural tendency of the radiation beam to diffract. If diffraction of the radiation field were not fully or partially offset by the focusing effect, the FEL would suffer from reduced gain and efficiency.

The primary objectives of this paper are twofold. The first is to present a general method of formulating and solving problems involving radiation focusing and guiding for mechanisms in which the refractive index is known. The second objective is to apply this approach to the focusing and steering of radiation in FEL's with arbitrary gain. We present a general, self-consistent, fully nonlinear, modal representation formalism which we apply to the phenomena of radiation focusing and guiding in FEL's. The novel aspect of our modal expansion is that the characteristics of the modes are governed by the driving current density, as opposed to a heuristic numerical approach,[10] and hence it is called the "source-dependent expansion" (SDE). Instead of using the usual modal expansion consisting of *vacuum* Laguerre-Gaussian functions,[11] we incorporate the source function (driving current) self-consistently into the functional dependence of (i) the radiation waist and (ii) the radiation wave-front curvature, as well as (iii) the radiation complex amplitude. Because of the source-dependent nature of our modal expansion, the fundamental mode remains dominant throughout the evolution of the radiation field. The SDE scheme appears to have a number of advantages over the conventional vacuum representation. Among the advantages is that the lowest-order term in the expansion is a good approximation to the radiation field even for propagation distances long compared to a Rayleigh length. Hence valuable insight concerning focusing and steering can be obtained analytically. Furthermore, because far fewer modes are needed, compared to the vacuum expansion approach, fast numerical solutions of the fully three-dimensional wave equation can be obtained. Because of the numerical speed of this approach, modeling of the driving current could include electron-beam emittance, energy spread, wiggler gradients, sideband frequencies, etc.

In Sec. II the fully three-dimensional wave equation is solved using the Laguerre SDE approach with a general source function and a system of first-order coupled differential equations, which completely describe the radiation beam, is derived. This formulation is capable of handling a broad range of problems including focusing and guiding of the radiation beam in a FEL. In Sec. III we apply our analysis to a number of situations pertinent to a FEL amplifier. An envelope equation for the radiation field is derived in Sec. III A. In Sec. III B the conditions under which focusing may occur are discussed. For example, in the linear exponential-gain regime a constant radiation beam waist can be achieved. In the high-gain trapped-particle regime, however, a perfectly guided radiation beam, i.e., constant waist, cannot be achieved, since the focusing term in the envelope equation decreases as the radiation amplitude increases. In Sec. III C radiation steering is studied by perturbing the electron beam off axis and an expression for the centroid of the radiation beam is obtained. The effect of modulating the electron-beam envelope on the radiation beam is considered in Sec. III D. In Sec. IV numerical results using the SDE representation are given for a FEL amplifier configuration. We begin this section by showing a comparison between the results obtained from (a) the exact numerical solution of the wave equation, (b) the vacuum modal representation, and (c) the SDE approach. Numerical results for the various aspects of radiation focusing and steering are presented, followed by a conclusion in Sec. V.

II. FORMULATION OF THE SOURCE-DEPENDENT EXPANSION (SDE)

The radiation focusing and guiding configuration for the FEL is shown in Fig. 1. In our model the vector potential of the linearly polarized radiation field is

$$\mathbf{A}_R(r,\theta,z,t)=\frac{A(r,\theta,z)}{2}e^{i(\omega z/c-\omega t)}\hat{\mathbf{e}}_x+\text{c.c.}\ , \tag{1}$$

where $A(r,\theta,z)$ is the complex radiation field amplitude, ω is the frequency and c.c. denotes the complex conjugate.

The wave equation is

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}_R=-\frac{4\pi}{c}J_x\hat{\mathbf{e}}_x\ , \tag{2}$$

where $J_x(r,\theta,z,t)$ is the driving current density associated with the medium. Substituting (1) into (2) leads to the following reduced wave equation,

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+2i\frac{\omega}{c}\frac{\partial}{\partial z}\right]a(r,\theta,z)=S(r,\theta,z)\ , \tag{3}$$

where $a(r,\theta,z)=|e|A/m_0c^2=|a|\exp(i\phi)$ is the normalized complex radiation field amplitude and we have assumed that $a(r,\theta,z)$ is a slowly varying function of z, i.e., $a^{-1}\partial a/\partial z\ll\omega/c$. The amplitude $|a(r,\theta,z)|$ and phase $\phi(r,\theta,z)$ are real functions expressed in terms of the polar coordinates r, θ, and z. The source function S has the general form

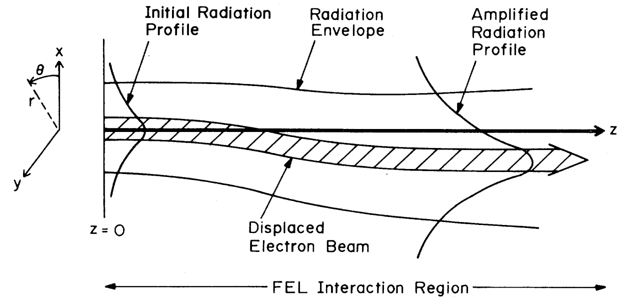

FIG. 1. Schematic of radiation focusing and guiding in a FEL.

$$S(r,\theta,z)=\frac{\omega^2}{c^2}[1-n^2(r,\theta,z,a)]a(r,\theta,z)\ , \tag{4}$$

where $n(r,\theta,z,a)$ is the index of refraction associated with the medium and is in general complex and a function of r,θ,z as well as the radiation field a.

We choose the following representation for $a(r,\theta,z)$ in terms of associated Laguerre polynomials,

$$a(r,\theta,z)=\sum_m\sum_p C_{m,p}(\theta,z)D_m^p(r)\ , \tag{5}$$

where $m=0,1,2,\ldots,\ p=0,1,2,\ldots,$

$$C_{m,p}(\theta,z)=a_{m,p}(z)\cos(p\theta)+b_{m,p}(z)\sin(p\theta)\ , \tag{6a}$$

$$D_m^p(r)=\left[\frac{\sqrt{2}r}{r_s(z)}\right]^p L_m^p\left[\frac{2r^2}{r_s^2(z)}\right]e^{-[1-i\alpha(z)]r^2/r_s^2(z)}\ . \tag{6b}$$

In Eqs. (6a) and (6b) the complex coefficients $a_{m,p}(z)$ and $b_{m,p}(z)$ are functions of z, $r_s(z)$ is the radiation spot size, $\alpha(z)$ is related to the inverse of the radius of curvature of the radiation beam (curvature of wave front), and L_m^p is the associated Laguerre polynomial. Solving for the unknown quantities $a_{m,p}$, $b_{m,p}$, r_s, and α in terms of the source term S allows us to completely describe the radiation dynamics. It will be shown later that the representation in (5) is underspecified, there are more unknown quantities in (5) than available equations. The additional degrees of freedom in our representation allow us to specify particular functional relationships for the unknown quantities r_s and α in such a way that the number of terms (modes needed to accurately describe the radiation beam) is small.

To proceed with the derivation we substitute (5) into (3) and obtain

$$\sum_{m,p}\left[\partial C_{m,p}(\theta,z)/\partial z+C_{m,p}(\theta,z)\left\{\frac{\partial}{\partial z}-\frac{4ic}{\omega r_s^2}\left[\frac{\partial}{\partial\xi}\left[\xi\frac{\partial}{\partial\xi}\right]-\frac{p^2}{4\xi}\right]\right\}\right]D_m^p(\xi)=-\frac{ic}{2\omega}S(\xi,\theta,z)\ , \tag{7}$$

where $\xi=2r^2/r_s^2(z)$. It can be shown that the second term on the left-hand side of (7) can be put into the form

$$\left\{\frac{\partial}{\partial z}-\frac{4ic}{\omega r_s^2}\left[\frac{\partial}{\partial \xi}\left[\xi\frac{\partial}{\partial \xi}\right]-\frac{p^2}{4\xi}\right]\right\}D_m^p(\xi)=A_{m,p}(z)D_m^p(\xi)-i(m+1)B(z)D_{m+1}^p(\xi)-i(m+p)B^*(z)D_{m-1}^p(\xi)\ , \quad (8)$$

where

$$A_{m,p}(z)=r_s'/r_s+i(2m+p+1)[(1+\alpha^2)c/\omega r_s^2-\alpha r_s'/r_s+\alpha'/2]\ , \quad (9a)$$

$$B(z)=-[\alpha r_s'/r_s+(1-\alpha^2)c/\omega r_s^2-\alpha'/2]-i(r_s'/r_s-2\alpha c/\omega r_s^2)\ , \quad (9b)$$

the asterisk denotes the complex conjugate, and the prime denotes a derivative with respect to z, i.e., prime $=\partial/\partial z$.

In obtaining (8) the following identities were used,

$$\xi U_m^p=(2m+p+1)U_m^p-(m+1)U_{m+1}^p-(m+p)U_{m-1}^p\ ,$$

$$2\xi\partial U_m^p/\partial\xi=(2m+p-\xi)U_m^p-2(m+p)U_{m-1}^p\ ,$$

$$\xi\partial^2U_m^p/\partial\xi^2+\partial U_m^p/\partial\xi=\tfrac{1}{4}[\xi+p^2/\xi-2(2m+p+1)]U_m^p\ ,$$

where $U_m^p(\xi)=\xi^{p/2}L_m^p(\xi)\exp(-\xi/2)$. Substituting (8) into (7) and performing the operation

$$\int_0^{2\pi}(\cos(p'\theta),\sin(p'\theta))d\theta/2\pi$$

on the resulting equation yields

$$\sum_{m=0}^{\infty}\left[D_m^p(\partial/\partial z+A_{m,p})\times\begin{Bmatrix}a_{m,p}\\ b_{m,p}\end{Bmatrix}-i[(m+1)BD_{m+1}^p+(m+p)B^*D_{m-1}^p]\times\begin{Bmatrix}a_{m,p}\\ b_{m,p}\end{Bmatrix}\right.$$

$$=\frac{-ic}{2\pi\omega}\int_0^{2\pi}d\theta\, S(\xi,\theta,z)\times\begin{Bmatrix}(1+\delta_{p,0})^{-1}\cos(p\theta)\\ \sin(p\theta)\end{Bmatrix}\ , \quad \begin{matrix}(10a)\\(10b)\end{matrix}$$

where $\delta_{p,0}$ is the Kronecker delta. Multiplying (10) by $(D_n^p)^*$ and integrating over ξ from 0 to ∞ yields

$$\left\{\frac{\partial}{\partial z}+A_{m,p}(z)\right\}\times\begin{Bmatrix}a_{m,p}\\ b_{m,p}\end{Bmatrix}-imB(z)\times\begin{Bmatrix}a_{m-1,p}(z)\\ b_{m-1,p}(z)\end{Bmatrix}-i(m+p+1)B^*(z)\times\begin{Bmatrix}a_{m+1,p}(z)\\ b_{m+1,p}(z)\end{Bmatrix}=-i\times\begin{Bmatrix}F_{m,p}(z)\\ G_{m,p}(z)\end{Bmatrix}\ , \quad \begin{matrix}(11a)\\(11b)\end{matrix}$$

where

$$\begin{Bmatrix}F_{m,p}(z)\\ G_{m,p}(z)\end{Bmatrix}=\frac{c}{2\pi\omega}\frac{m!}{(m+p)!}\int_0^{2\pi}d\theta\int_0^{\infty}d\xi\, S(\xi,\theta,z)[D_m^p(\xi)]^*\times\begin{Bmatrix}(1+\delta_{p,0})^{-1}\cos(p\theta)\\ \sin(p\theta)\end{Bmatrix}\ . \quad \begin{matrix}(12a)\\(12b)\end{matrix}$$

In obtaining (11) we used the orthogonality relation

$$\int_0^{\infty}D_m^p(\xi)[D_n^p(\xi)]^*d\xi=\frac{(n+p)!}{n!}\delta_{m,n}\ .$$

The function $B(z)$ is arbitrary and is not specified. The equations for $a_{m,p}$ and $b_{m,p}$ in (11) are underdetermined, since the function $B(z)$ can be shown to be arbitrary. If we choose $B(z)=0$, for example, we would in effect be expanding the radiation field in the conventional vacuum Laguerre-Gaussian modes.[11] We will show later that, in general, expansion in terms of the vacuum modes, $B=0$, would require far too many modes to accurately describe the radiation beam over distances of many Rayleigh lengths. A more appropriate choice for $B(z)$ will depend on the particular problem under consideration. Let us consider one of the most common situations where the radiation beam at $z=0$ is known and has a Gaussian radial profile symmetric about the z axis. In this case the complex radiation amplitude at $z=0$ is given by

$$a(r,\theta,0)=a_{0,0}\exp\{-[1-i\alpha(0)]r^2/r_s^2(0)\}$$

and is independent of θ. Let us further assume that for $z>0$ the radiation-beam profile remains approximately Gaussian with a nearly circular cross section. That is, the dominant part of the source $S(r,\theta,z)$ has an r and z dependence and the θ-dependent part is weak. In this case we expect the magnitude of the coefficients, $a_{m,p}(z)$ and $b_{m,p}(z)$, to become progressively smaller as m and p take on larger values, i.e., $|a_{m,p}|\gg|a_{m+1,p}|$, $|a_{m,p+1}|$ and $|b_{m,p}|\gg|b_{m+1,p}|$, $|b_{m,p+1}|$. The lowest-order approximation to the radiation beam is given by the $a_{0,0}(z)$ mode. Hence, if the $a_{0,0}$ mode gives a rough approximation to the radiation field we may solve for $a_{0,0}(z)$, $r_s(z)$, and $\alpha(z)$ using (11a). From (11a) we find that only the $m=0,1$ and $p=0$ equations are relevant and yield

$$(\partial/\partial z + A_{0,0})a_{0,0} = -iF_{0,0} \ , \tag{13a}$$

$$Ba_{0,0} = F_{1,0} \ . \tag{13b}$$

We now have a specific expression for $B(z)$, from (13b), in terms of one of the moments, $F_{1,0}$, of the source term. Substituting (9b) into $B(z)=F_{1,0}(z)/a_{0,0}(z)$ yields the following first-order coupled differential equations for r_s and α,

$$r_s' - 2c\alpha/\omega r_s = -r_s(F_{1,0}/a_{0,0})_I \ , \tag{14a}$$

$$\alpha' - 2(1+\alpha^2)c/\omega r_s^2 = 2[(F_{1,0}/a_{0,0})_R - \alpha(F_{1,0}/a_{0,0})_I] \ , \tag{14b}$$

where $()_{R,I}$ denotes the real and imaginary part of the enclosed function. Since $r_s(z)$ and $\alpha(z)$ are now known from (14a) and (14b) we may solve for $A_{m,p}(z)$ using (9a),

$$A_{m,p}(z) = 2c\alpha/\omega r_s^2 - (F_{1,0}/a_{0,0})_I + i(2m+p+1)[2c/\omega r_s^2 + (F_{1,0}/a_{0,0})_R] \ . \tag{15}$$

Using $B(z)=F_{1,0}(z)/a_{0,0}(z)$ and the resulting equations for r_s and α in (14) allows us to solve for $a_{m,p}$ and $b_{m,p}$ in (11a) and (11b).

It is useful at this point to consider the simple case of propagation of a radiation beam in vacuum (no source term). To illustrate this well-known limit we evaluate $a_{m,p}$, $b_{m,p}$, r_s, and α in the source-free case, $F_{m,p}=G_{m,p}=B=0$. Equations (14a) and (14b) become $r_s''=(2c/\omega)^2 r_s^3$ and $\alpha=(\omega/2c)r_s r_s'$ and have the solutions

$$r_s(z) = r_s(0)(1+z^2/z_R^2)^{1/2} \ , \tag{16a}$$

$$\alpha(z) = z/z_R \ , \tag{16b}$$

where $r_s(0)$ is the minimum radiation spot size at $z=0$, $z_R=(\omega/2c)r_s^2(0)=\pi r_s^2(0)/\lambda$ is the Rayleigh length, and $\lambda=2\pi c/\omega$ is the wavelength. From (10a) we find that $A_{m,p}(z)=2[\alpha(z)+i(2m+p+1)]c\omega r_s^2(z)$ which allows us to solve for $a_{m,p}$ and $b_{m,p}$ using (11)

$$\begin{bmatrix} a_{m,p}(z) \\ b_{m,p}(z) \end{bmatrix} = \begin{bmatrix} a_{m,p}(0) \\ b_{m,p}(0) \end{bmatrix} [r_s(0)/r_s(z)] \times e^{-i(2m+p+1)\tan^{-1}(z/z_R)} \ . \tag{17}$$

Equations (16a), (16b), and (17), together with the representations in (5), (6a) and (6b), are in agreement with the conventional vacuum Gaussian-Laguerre form.

III. RADIATION FOCUSING AND STEERING IN FEL's

A. Radiation-beam envelope equation

We first consider the dynamics of an axially symmetric radiation field in the FEL. For a linearly polarized wiggler field and axially symmetric electron beam having a Gaussian density profile, the appropriate index of refraction for the FEL mechanism[4,5,9,12] is

$$n(r,z,a) = 1 + \frac{1}{2}\frac{\omega_b^2(r,z)}{\omega^2}\left\langle \frac{e^{-i\psi}}{\gamma} \right\rangle \frac{a_w}{|a(r,z)|} \ , \tag{18}$$

where

$$\omega_b^2(r,z) = \omega_{b0}^2[r_{b0}/r_b(z)]^2 \exp[-r^2/r_b^2(z)] \ ,$$

$r_b(z)$ is the electron-beam radius, $r_{b0}=r_b(0)$, $\omega_{b0}=(4\pi|e|^2 n_{b0}/m_0)^{1/2}$ is the initial beam plasma frequency on axis, n_{b0} is the initial beam density on axis, $a_w=|e|B_w/k_w m_0 c^2$ is the normalized wiggler amplitude, B_w is the wiggler magnetic field strength, k_w is the wiggler wave number, γ is the electron's Lorentz factor, ψ is the electron's phase in the ponderomotive wave potential, and $\langle\ \rangle$ denotes the ensemble average over all electrons. Substituting (18) into (4) and noting that $|1-n| \ll 1$ gives the FEL source function

$$S(r,z) = \frac{-\omega_b^2(r,z)}{c^2} a_w \left\langle \frac{e^{-i\psi}}{\gamma} \right\rangle \frac{a(r,z)}{|a(r,z)|} \ . \tag{19}$$

Since the electron-beam radius r_b may not be matched with respect to the focusing fields (wiggler gradients) and defocusing effects (beam emittance), we allow r_b to be a function of z (this case is considered in Sec. III C). To proceed with the analysis we assume that in the source function the complex radiation field amplitude in (5) can be approximated by the lowest-order mode, $a_{0,0}(z)\exp[-(1-i\alpha)r^2/r_s^2]$. With this assumption the source function can be written as

$$S(\xi,z) = -4\nu(a_w/r_b^2)\frac{a_{0,0}}{|a_{0,0}|}\left\langle \frac{e^{-i\psi}}{\gamma} \right\rangle \times e^{-(r_s^2/r_b^2 - i\alpha)\xi/2} \ , \tag{20}$$

where $\nu=(\omega_{b0}r_{b0}/2c)^2=I_b/17\times 10^3$ is Budker's constant and I_b is the electron-beam current in amperes. The moments of the source function, $F_{m,p}(z)$, are given by (12a)

$$F_{m,0} = -4\frac{c}{\omega}\nu(a_w/r_b^2)\frac{a_{0,0}}{|a_{0,0}|}\left\langle \frac{e^{-i\psi}}{\gamma} \right\rangle \frac{(r_s^2/r_b^2-1)^m}{(r_s^2/r_b^2+1)^{m+1}} \ , \tag{21}$$

where we have assumed ψ to be constant across the electron beam. Since we are considering an axially symmetric electron beam and radiation field we note that $a_{m,p}=F_{m,p}=G_{m,p}=0$ for $p>0$. Substituting (21) into (14a), (14b), and (15) yields

$$r_s r_s' - 2c\alpha/\omega = -2\frac{c}{\omega}C(z)\langle \sin\psi \rangle \ , \tag{22a}$$

$$r_s^2\alpha' - 2(1+\alpha^2)c/\omega = -4\frac{c}{\omega}C(z)(\langle \cos\psi \rangle + \alpha\langle \sin\psi \rangle) \ , \tag{22b}$$

$$A_{m,0}(z) = \frac{2c}{r_s^2\omega}\{\alpha + i(2m+1) - C(z)[\langle \sin\psi \rangle + i(2m+1)\langle \cos\psi \rangle]\} \ , \tag{22c}$$

where $C(z)=(2\nu/\gamma)H(z)a_w/|a_{0,0}(z)|$, $H(z)=(1-F)/(1+F)^2$, and $F(z)=r_b^2/r_s^2$ is the filling factor. The function $C(z)$ measures the coupling between the radiation and electron beam and decreases as the radiation grows.

Equations (22a) and (22b) can be combined to give the following envelope equation for the radiation beam,

$$r_s''+K^2(z,r_b,r_s,a_{0,0})r_s=0 , \qquad (23a)$$

where the initial condition on r_s' is found from (22a) and

$$K^2=(2c/\omega)^2(-1+C^2\langle\sin\psi\rangle^2+2C\langle\cos\psi\rangle +(\omega/2c)r_s^2C'\langle\sin\psi\rangle)r_s^{-4} . \qquad (23b)$$

The first term on the right-hand side of (23b) is defocusing and corresponds to the usual diffraction expansion, the second and third terms are always focusing while the last term is usually a defocusing contribution.

B. Radiation focusing

Focusing occurs when $K^2\geq 0$. In the high-gain trapped-particle regime, the condition for a perfectly guided beam ($K=0$) cannot be maintained since K^2 decreases as the radiation grows. In the small-signal, exponential-gain regime the quantities $\langle\sin\psi\rangle$ and $\langle\cos\psi\rangle$ may be calculated from the linearized orbit equations. The envelope equation may then be solved to determine r_s as a function of distance along the wiggler. One finds that in this regime conditions for a perfectly guided radiation beam can be achieved.[13]

Using (11a) or (13a) we find that the magnitude of $a_{0,0}(z)$ evolves according to

$$[\partial/\partial z+(A_{0,0}+A_{0,0}^*)]|a_{0,0}|^2 =-i(F_{0,0}a_{0,0}^*-F_{0,0}^*a_{0,0}) . \qquad (24)$$

Substituting (21) and (22c) with $m=0$ into (24) and using (22a) yields

$$\frac{\partial}{\partial z}(r_s|a_{0,0}|)=\frac{4c}{\omega}\frac{\nu}{\gamma}a_w\frac{r_s\langle\sin\psi\rangle}{(r_s^2+r_b^2)} , \qquad (25)$$

where $(r_s|a_{0,0}|)^2$ is proportional to the radiation power, $P(z)=2.15\times 10^{10}(|a_{0,0}(z)|r_s/\lambda)^2$ W. Equation (25) should be solved together with (23a) and show that the maximum rate of increase in power occurs when $r_s=r_b$.

C. Radiation steering in the FEL

In the FEL the centroid of the electron beam may be displaced off axis by a misalignment, a redirection of the beam, or because of the oscillations in the wiggler field. To determine the degree to which the radiation beam will follow or be steered by the electron beam, we consider the case where the electron-beam centroid is displaced transversely in the x direction. The index of refraction in this case is given by (18) with $\omega_b^2(r,z)$ replaced by

$$\omega_b^2(r,\theta,z)=\omega_{b0}^2(r_{b0}/r_b)^2e^{-r^2/r_b^2}[1+(2rx_b(z)/r_b^2)\cos\theta] , \qquad (26)$$

where $x_b(z)$ is the displacement of the electron-beam centroid and $|x_b|\ll r_b$. In the source term, given by (19), we consider only the lowest-order symmetric and antisymmetric mode with respect to the x axis,

$$a(r,\theta,z)\sim(a_{0,0}+a_{0,1}\xi^{1/2}\cos\theta)\exp[-(1-i\alpha)\xi/2] .$$

With this assumption the moments of the source function, $F_{m,p}(z)$, for $p=0,1$ are

$$F_{m,p}=-8\frac{c}{\omega}\nu(a_w/r_b^2)\left\langle\frac{e^{-i\psi}}{\gamma}\right\rangle\frac{a_{0,0}}{|a_{0,0}|}\times\left[\epsilon(z)+i\left[\frac{a_{0,1}}{a_{0,0}}\right]_I\right]^p\frac{(r_s^2/r_b^2-1)^m}{(r_s^2/r_b^2+1)^{m+p+1}} , \qquad (27)$$

where $\epsilon(z)=2^{1/2}x_b(z)r_s(z)/r_b^2$ and $G_{m,p}=0$. For small displacements of the electron-beam centroid it is easy to show that the centroid of the radiation beam is given by

$$x_L(z)=\frac{r_s(z)}{\sqrt{2}}\left[\frac{a_{0,1}}{a_{0,0}}\right]_R , \qquad (28)$$

where x_L is defined so that $|a|$ is proportional to

$$\exp\{-[(x-x_L)^2+y^2]/r_s^2\} .$$

D. Effect of modulated electron beam

The electron-beam envelope in the FEL can undergo modulations. The modulation is symmetric about the z

TABLE I. Parameters used in numerical examples.

Electron beam	
Current	$I_b=2kA$ ($\nu=0.118$)
Energy	$E_b=50$ MeV ($\gamma=100$)
Radius	$r_{b0}=0.3$ cm
Radiation beam	
Wavelength	$\lambda=10.6$ μm
Input power	$P(z=0)=230MW$ ($|a(0,0)|=1.84\times 10^{-4}$)
Spot size	$r_s(0)=0.6$ cm ($z_R=10.7$ m)
Wiggler field	
Wavelength	$\lambda_w=8$ cm
Wiggler strength	$B_w=2.3kG$ ($a_w=1.716$)
Resonant phase	$\psi_R=0.358$ rad

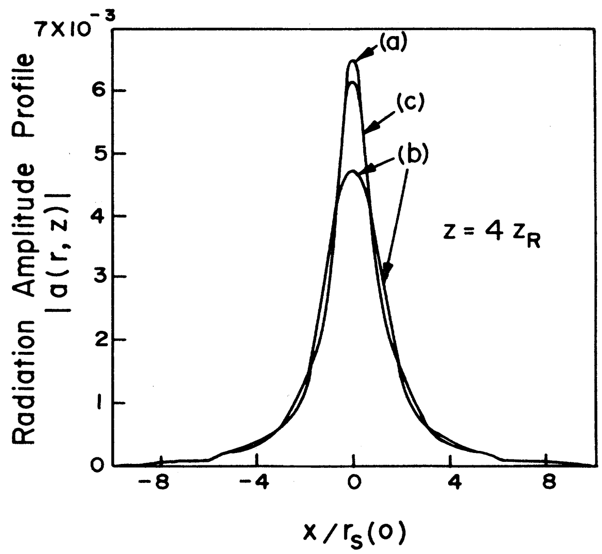

FIG. 2. Radiation amplitude profile $|a(r,z)|$ for (a) exact numerical solution (64×64 Fourier modes), (b) vacuum modal expansion solution (ten modes), and (c) SDE solution (ten modes) at a distance of $z=4z_R=42.8$ m.

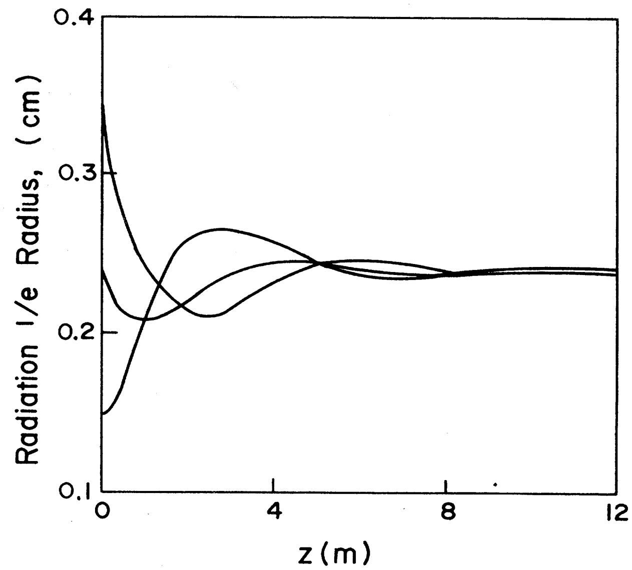

FIG. 4. Spatial evolution of the radiation spot size in the exponential-gain regime for initial spot sizes (a) 0.35 cm, (b) 0.24 cm, and (c) 0.15 cm.

axis and can be caused by improper values for the beam emittance, radius, and/or current injected into the wiggler region. For small perturbations about the matched beam radius, r_{b0}, we find from the electron-beam envelope equation that $r_b(z)=r_{b0}[1+\Delta\sin(K_B z)]$ where $r_{b0}=(2\epsilon_n/a_w k_w)^{1/2}$, $K_B=a_w k_w/\sqrt{2}\gamma$ is the betatron wave number, due to the weak focusing effect of wiggler gradients, ϵ_n is the normalized emittance, $a_w=|e|B_w/(k_w m_0 c^2)$, and $\Delta \ll 1$. The modulation of the electron-beam envelope may be included in the source term, Eq. (19), through the electron-beam plasma frequency $\omega_b(r,z)$. The effect of a modulated electron beam on the radiation beam is illustrated in Sec. IV.

In cases where the electron-beam centroid or envelope is displaced or modulated with a spatial period close to the wiggler period, it becomes necessary to include in the source function, Eq. (19), the rapidly varying part of the phase ψ. This rapidly oscillating contribution to the phase, $[a_w^2/(4+2a_w^2)]\sin(2k_w z)$, arises from the linearly polarized wiggler field.

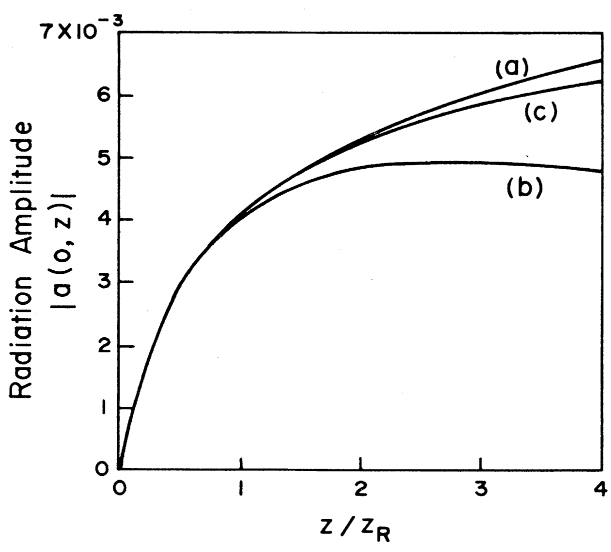

FIG. 3. Radiation amplitude on axis, $|a(0,z)|$, for (a) exact numerical solution (64×64 Fourier modes), (b) vacuum modal expansion solution (ten modes), and (c) SDE solution (ten modes).

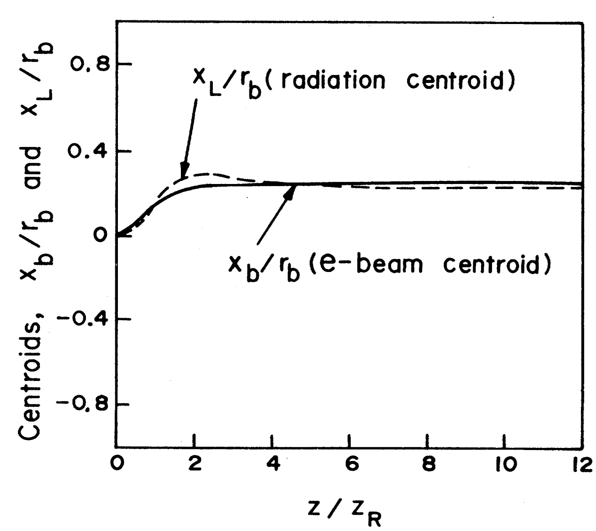

FIG. 5. Electron- and radiation-beam centroids x_b and x_L for a displaced electron beam, $x_b=x_c[1-\mathrm{sech}(k_c z)]$ with $x_c=r_b/4$ and $\lambda_c=2\pi/k_c=4z_R$.

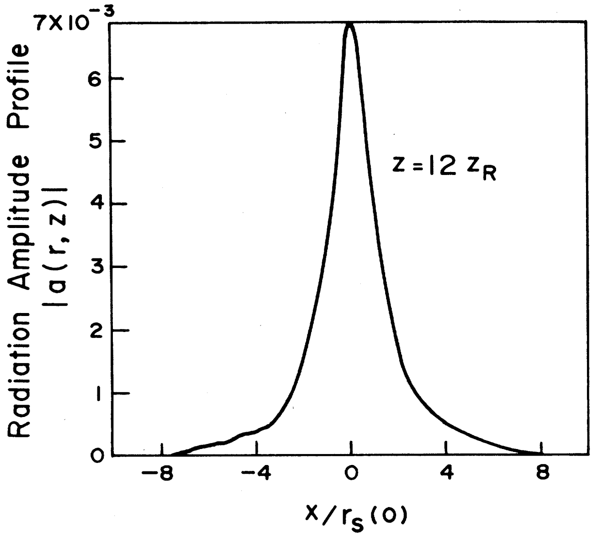

FIG. 6. Radiation amplitude profile at $z = 12z_R$ for a displaced electron beam, $x_b = x_c[1\text{-sech}(k_c z)]$ with $x_c = r_b/4$ and $\lambda_c = 2\pi/k_c = 4z_R$.

IV. NUMERICAL RESULTS

In this section we apply the SDE formulation, given by (11) together with (12) to the FEL. Using the source term given in (4) and (18) we first present a comparison between (a) the exact numerical solution of the wave equation in (4), (using 64×64 Fourier modes), (b) the solution obtained using a vacuum Laguerre modal expansion ($B = 0$, using ten modes), and (c) the solution obtained from the Laguerre SDE approach ($B = F_{1,0}/a_{0,0}$, using ten modes). The FEL parameters used in these illustrations are similar to those used in Ref. 14 and are given in Table I where the resonant phase approximation, $\langle \exp(-i\psi) \rangle = \exp(-i\psi_R)$, is used for demonstration purposes. Propagation distances are measured in terms of the Rayleigh length, $z_R = \pi r_s^2(0)/\lambda$ where λ is the wavelength, and $r_s(0)$ is the minimum spot size.

For an axially symmetric configuration, Fig. 2 shows the radiation magnitude, $|a(r,z)|$ as a function of r at four Rayleigh lengths ($z = 4z_R = 42.8$ m) for the (a), (b), and (c) methods of solution. The SDE solution (c) shows excellent agreement with solution (a), while solution (b) is in poor agreement. To continue this comparison we show in Fig. 3 the evolution of the radiation-beam am-

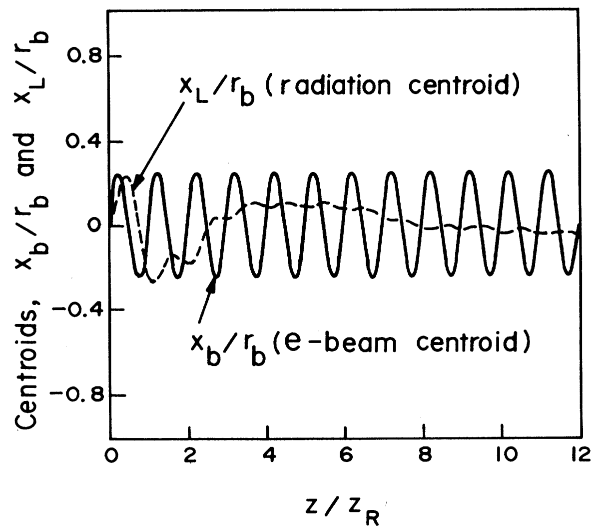

FIG. 8. Electron- and radiation-beam centroids, x_b and x_L for an oscillating electron beam, $x_b = x_c \sin(k_c z)$ with $x_c = r_b/4$ and $\lambda_c = 2\pi/k_c = z_R$.

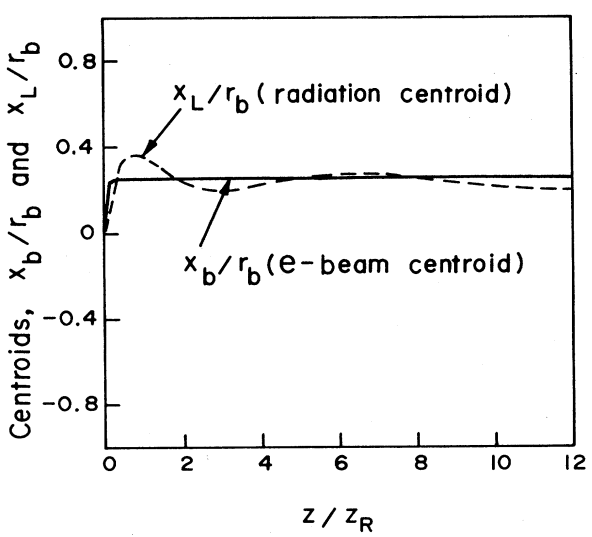

FIG. 7. Electron- and radiation-beam centroids, x_b and x_L for a displaced electron beam, $x_b = x_c[1\text{-sech}(k_c z)]$ with $x_c = r_b/4$ and $\lambda_c = 2\pi/k_c = z_R/4$.

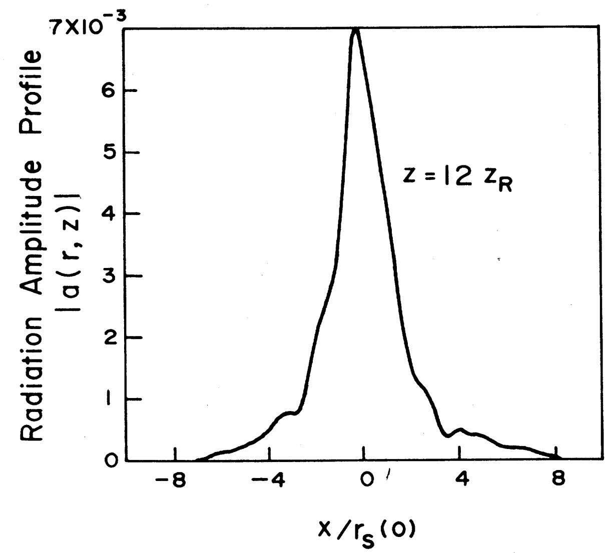

FIG. 9. Radiation amplitude profile at $z = 12z_R$ for an oscillating electron beam, $x_b = x_c \sin(k_c z)$ with $x_c = r_b/4$ and $\lambda_c = 2\pi/k_c = z_R$.

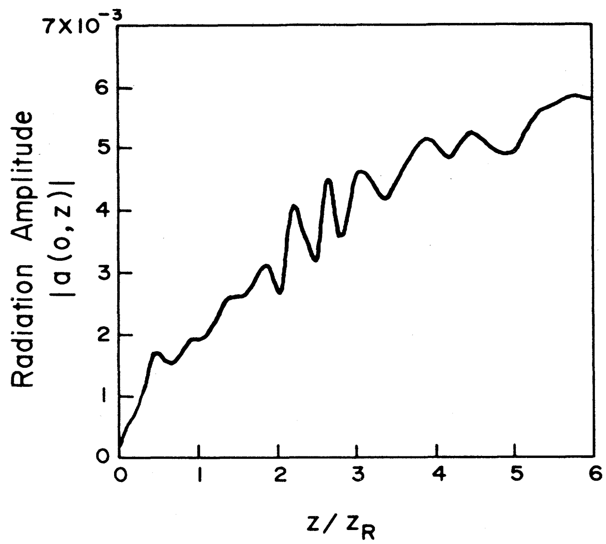

FIG. 10. Radiation amplitude on axis, $|a(0,z)|$, for a modulated electron beam, $r_b = r_{b0}[1+\Delta\sin(K_B z)]$ with $r_{b0}=0.3$ cm, $\Delta=0.1$, and $\lambda_B = 2\pi/K_B = 4.66$ m.

plitude on axis obtained from methods (a), (b), and (c) as a function of propagation distance. The SDE solution (c) is again in good agreement with solution (a) where as solution (b), beyond a Rayleigh length, grossly deviates from (a) and (c). The results in Figs. 2 and 3 clearly show the improved accuracy of the SDE approach over the conventional vacuum expansion method. As an example of radiation focusing, for an FEL in the small-signal, exponential-gain regime, the radiation beam radius is found to asymptotically approach a matched perfectly guided value as shown in Fig. 4. For this example, the parameters of Table I were used and five modes employed.

We now use the SDE method to illustrate the steering of the radiation beam when the electron beam is displaced off axis. In these numerical illustrations ten radial modes ($m=0,\ldots,9$) and two angular modes ($p=0,1$) were used. In the first example the electron-beam centroid is displaced off axis according to $x_b = x_c$ [1-sech$(k_c z)$]. Figure 5 shows the electron- and radiation-beam centroids x_b and x_L for $x_c = r_b/4 = 0.075$ cm and $\lambda_c = 2\pi/k_c = 4z_R = 42.8$ m. The radiation centroid follows and oscillates about the electron-beam centroid. Figure 6 shows the radiation profile at 12 Rayleigh lengths ($z=12z_R$). The asymmetry of the radiation profile is apparent. Figure 7 shows another illustration of steering where the electron beam is displaced more abruptly, with $\lambda_c = z_R/4 = 2.7$ m. After an initial transient, the radiation centroid is again steered by and oscillates about the electron-beam centroid. In the next example we take the electron-beam centroid to be oscillating about the z axis, $x_b = x_c \sin k_c z$, with amplitude $x_c = r_b/4$ and period $\lambda_c = z_R = 10.7$ m. Figure 8 shows the electron and radiation centroids x_b and x_L, as a function of z/z_R. Because of the high gain in the radiation field, the radiation centroid eventually follows the average position of the electron-beam centroid. Figure 9 shows the distortion of the radiation profile due to the oscillating electron beam at 12 Rayleigh lengths ($z=12z_R$). In the case where the electron-beam centroid oscillation is due to the wiggler field, $x_c = a_w/\gamma k_w$ and $k_c = 2\pi/\lambda_w$, no noticeable change in the evolution of the radiation field (compared to the case for $x_c=0$) is observed.

The last illustration is for the case where the electron beam envelope is spatially modulated. Using the parameters in Table I we find that $\epsilon_n = 0.06$ cm rad and $\lambda_B = 2\pi/K_B = 4.66$ m. Figue 10 shows the amplitude of the radiation field on axis as a function of propagation distance when the electron-beam envelope is not matched, $r_b = r_{b0}[1+\Delta\sin(K_B z)]$, where $r_{b0}=0.3$ cm and $\Delta=0.1$.

V. CONCLUSION

In this paper a technique for solving the three-dimensional wave equation with a driving current density has been developed. Using this source-dependent expansion technique, a number of effects associated with radiation focusing and steering in the FEL have been illustrated. The formalism is used to derive a general envelope equation for the radiation beam. Using the envelope equation we find that it is possible to have a stable guided optical beam in the exponential-gain (small-signal) regime but not in the high-gain trapped-particle regime. We also considered the effects on the radiation beam when the electron-beam centroid is transversely displaced and when the electron-beam envelope is modulated. The source-dependent expansion approach lends itself to fast and accurate numerical solutions as well as to a better analytical description of focusing and steering in the FEL. We conclude by noting that this approach can be readily generalized to include both spatial and temporal variations in the radiation field in order to study sideband generation and focusing effects simultaneously in the FEL. Recently there has been indirect[15] as well as direct[16] experimental evidence indicating optical guiding in FEL's.

ACKNOWLEDGMENTS

We acknowledge useful discussions with B. Hafizi. This work was sponsored by the U.S. Army Strategic Defense Command and Department of Energy.

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