Perturbed solitons in nematic liquid crystals under time-dependent shear

Xu Gang
Department of Physics, Kent State University, Kent, Ohio 44242-0001

Shu Chang-Qing
Institute of Physics, Chinese Academy of Sciences, P.O. Box 603, Beijing, China

Lin Lei
*Department of Physics, City College, City University of New York, New York, New York 10031;**
Department of Physics, Queensborough Community College, City University of New York, Bayside, New York 11364,
and Institute of Physics, Chinese Academy of Sciences, P.O. Box 603, Beijing, China
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Equations of motion for nematic liquid crystals under time-dependent shear are derived. Soliton solutions are investigated. When the soliton velocity is large and the shear varies slowly in time, approximate analytic solutions for single solitons of the A and B types are found with use of multiple-scale analysis. These perturbed solitons move with time-dependent velocities but are constant in shape and carry no tails. The velocity is proportional to the shear rate. Numerical calculations of the director equation of motion are performed and are in agreement with the analytic results. A recent experiment in which dark lines (under white light) excited by a periodically moving plate at one end of a nematic cell are observed is analyzed and interpreted according to our theory. Good agreement between theory and experiment is obtained.

I. INTRODUCTION

The hydrodynamic equations of motion of nematic liquid crystals are highly nonlinear.[1] Under suitable conditions solitary waves (solitons) can exist in nematics (see Ref. 2 for a brief review). They have been extensively studied[3-15] in both theory and experiments. Experimentally, these solitons may be generated by the use of either electric or magnetic field[6] or by shear,[7,10,12,14] and may be observed optically.

Discussion of propagating solitons in *uniform* shearing nematics was first given by Lin.[16] Properties and classification of single solitons of such a case were presented previously.[2,8,9] Relevant experiments[10] are analyzed and interpreted[2,8] as evidence of these solitons. In addition, external-field effects,[9] multisolitons[2,11] and the relation between observed optical-interference patterns and the director distribution[11] have been investigated. Single solitons generated by pressure gradients in long[12] and circular[13,14] cells of nematics, respectively, have also been reported.

In this paper, the case of unsteady shearing nematics is discussed. It is a natural extension of the steady shearing case[8,9] in which solitons are known to exist. As shown in Sec. II the relevant equation is a nonlinear one with time-dependent coefficient. When shear varies slowly in comparison to the large velocity of the wave, analytic perturbed soliton solutions may be obtained using the multiple-scales analysis[17,18] (Sec. III). It constitutes a nice example of perturbed solitons which do not change in shape and have no tails but travel with a varying velocity, proportional to the shear. More importantly, they are physically observable (Sec. VI).

Confirming numerical solutions are given in Sec. V and Sec. VII sums up the results with discussion.

II. EQUATIONS OF MOTION

Let us consider a nematic such that the director $\mathbf{n}$ and velocity $\mathbf{v}$ are given, respectively, by

$$\mathbf{n}=(\sin\theta,\ \cos\theta,0)\ ,$$
$$\mathbf{v}=(v,0,0)\ , \tag{2.1}$$

with

$$\theta=\theta(x,y,t),\quad v=v(x,y,t),\quad P=P(x,y,t)\ , \tag{2.2}$$

where P is the pressure. We further assume that it is incompressible, i.e.,

$$\nabla\cdot\mathbf{v}=0\ . \tag{2.3}$$

For simplicity, the one–elastic-constant approximation, $K_1=K_2=K_3\equiv K$, is adopted where K_i's are elastic constants.

Equations (2.3) and (2.2) imply that $v=v(y,t)$. The Ericksen-Leslie[19] equation[20] for this specific case reduces to

$$M\frac{d^2\theta}{dt^2}=K(\partial_{xx}+\partial_{yy})\theta-\gamma_1\frac{d\theta}{dt}+\tfrac{1}{2}s(\gamma_1-\gamma_2\cos2\theta)=0\ , \tag{2.4}$$

$$\rho\frac{dv}{dt}+\partial_x P+K[\partial_x(\partial_x\theta)^2+\partial_y(\partial_x\theta\partial_y\theta)]-\partial_x\left[\alpha_1 s\cos^3\theta\sin\theta+\left[\alpha_5 s+\gamma_2\frac{d\theta}{dt}\right]\sin\theta\cos\theta\right]$$
$$-\partial_y\left[\tfrac{1}{2}\alpha_4 s+\alpha_1 s\cos^2\theta\sin^2\theta+(\alpha_2\cos^2\theta-\alpha_3\sin^2\theta)\left[\frac{d\theta}{dt}-\tfrac{1}{2}s\right]+\tfrac{1}{2}s(\alpha_5\cos^2\theta+\alpha_6\sin^2\theta)\right]=0\ , \qquad (2.5)$$

and

$$-\rho g+\partial_y P+K[\partial_y(\partial_y\theta)^2+\partial_x(\partial_x\theta\partial_y\theta)]-\partial_y\left[\alpha_1 s\cos^3\theta\sin\theta+\left[\alpha_6 s-\gamma_2\frac{d\theta}{dt}\right]\sin\theta\cos\theta\right]$$
$$-\partial_x\left[\alpha_1 s\cos^2\theta\sin^2\theta+(\alpha_3\cos^2\theta-\alpha_2\sin^2\theta)\left[\frac{d\theta}{dt}-\tfrac{1}{2}s\right]+\tfrac{1}{2}s(\alpha_6\cos^2\theta+\alpha_5\sin^2\theta)\right]=0\ , \qquad (2.6)$$

where M is the moment of inertia density, ρ the density, γ_1, γ_2, α_1 to α_6 viscosity coefficients with $\gamma_1=\alpha_3-\alpha_2$, $\gamma_2=\alpha_6-\alpha_5=\alpha_2+\alpha_3$. $\partial_x\equiv\partial/\partial x$, $\partial_{xx}=\partial^2/\partial x^2$, etc. $d\theta/dt=\partial_t\theta+(\mathbf{v}\cdot\nabla)\theta$, $s\equiv\partial v/\partial y=s(y,t)$.

The three variables θ, v, and P may be solved by the three equations (2.4)–(2.6) with appropriate boundary conditions. However, this is very complicated. In the rest of this paper, we shall consider only the case that (i) $s(y,t)$ is set up externally so that only (2.4) needs to be considered; (ii) a thick liquid crystal cell so that all y dependence in the equations may be dropped and, in particular, $s=s(t)$; (iii) $v\ll c$, the velocity of the soliton, so that $d\theta/dt\simeq\partial_t\theta$. This is experimentally the case (see Ref. 10 and Sec. VI). Under these assumptions, (2.4) becomes

$$M\partial_{tt}\theta=K\partial_{xx}\theta-\gamma_1\partial_t\theta+\tfrac{1}{2}s(t)[\gamma_1-\gamma_2\cos(2\theta)] \qquad (2.7)$$

or, in dimensionless form,

$$(\overline{M}\partial_{TT}+\partial_T-\partial_{XX})\theta=\overline{s}(\epsilon T)[\gamma+\cos(2\theta)]\ , \qquad (2.8)$$

where

$$T\equiv t/\tau,\quad X\equiv x/\lambda,\quad \overline{s}\equiv s/s_0,\quad \overline{M}\equiv M/\tau\gamma_1\ ,$$
$$\tau\equiv 2\gamma/s_0,\quad \lambda\equiv(2K/|\gamma_2|s_0)^{1/2},\quad \gamma\equiv\gamma_1/|\gamma_2|\ . \qquad (2.9)$$

Here s_0 (>0) is a chosen constant shear rate.

For N-(p-methoxybenzylidene)-p-butylaniline (MBBA), $M\sim10^{-12}$ g cm^{-1}, $K\sim10^{-6}$ dyn, $\gamma_1=0.774P$, $\gamma_2=-0.8P$. In the experiments (see Sec. VI), $s_0\sim50$ s^{-1}. We therefore have $\tau\sim0.04$ s, $\lambda\sim2\times10^{-4}$ cm, and $\overline{M}\sim10^{-10}$. The $\overline{M}$ term in (2.8) is then essentially zero and (2.8) reduces to

$$(\partial_T-\partial_{XX})\theta=\overline{s}(\epsilon T)(\gamma+\cos2\theta)\ , \qquad (2.10)$$

where ϵ ($\ll1$) is a small parameter when the shear is assumed to vary slowly in time.

Equations (2.8) and (2.10) are the ones used in the rest of this paper.

III. MULTIPLE-SCALE ANALYSIS

As stated in Sec. II we assume that the time scale on which the shear varies is long compared to that of the wave. We can therefore use the singular perturbation method of multiple scales[18] to analyze Eqs. (2.8) or (2.10).

We make the change of variables (X,T) to (ξ,ϕ) with

$$\xi\equiv\epsilon T,\quad \phi=\phi(X,T)\ , \qquad (3.1)$$

where ξ is a slow variable and ϕ is a fast variable. The functional form of ϕ remains to be determined. At this point, the only requirement is that

$$\phi=X-C_0T,\quad \text{when } \epsilon\to0 \qquad (3.2)$$

where C_0 is a constant, the velocity of the unperturbed traveling wave (or soliton) when $\overline{s}$ is constant.

In general, $C\equiv-\partial_T\phi$, $k\equiv\partial_X\phi$ are functions of ξ and ϕ. Assuming that C and k change slowly when ϵ is small we assume

$$C=C(\xi),\quad k=k(\xi) \qquad (3.3)$$

resulting in $\partial_X C=(\partial_\phi C)\partial_X\phi=0$. But $\partial_X C=-\partial_{XT}\phi=-\partial_T k=-\epsilon\partial_\xi k$. Hence k is constant, independent of ξ. Its value, given by Eq. (3.2), is

$$k=k(\xi=0)=1\ . \qquad (3.4)$$

We now expand θ in a series,

$$\theta(X,T)=\theta^{(0)}(\xi,\phi)+\epsilon\theta^{(1)}(\xi,\phi)+\epsilon^2\theta^{(2)}(\xi,\phi)+\cdots\ . \qquad (3.5)$$

By (3.1), (3.3), and (3.4),

$$\partial_X\theta=\partial_\phi\theta,\quad \partial_{XX}\theta=\partial_{\phi\phi}\theta,\quad \partial_T\theta=\epsilon\partial_\xi\theta-C\partial_\phi\theta\ ,$$
$$\partial_{TT}\theta=\epsilon^2\partial_{\xi\xi}\theta-\epsilon(2C\partial_\xi+\partial_\xi C)\partial_\phi\theta+C^2\partial_{\phi\phi}\theta\ . \qquad (3.6)$$

By expansion,

$$\cos(2\theta)=\cos\{2\theta^{(0)}+2\epsilon[\theta^{(1)}+\epsilon\theta^{(2)}+\cdots]\}$$
$$=\cos(2\theta^{(0)})\cos[2\epsilon(\theta^{(1)}+\epsilon\theta^{(2)}+\cdots)]-\sin(2\theta^{(0)})\sin[2\epsilon(\theta^{(1)}+\epsilon\theta^{(2)}+\cdots)]$$
$$=\left[1-\frac{2^2\epsilon^2}{2!}(\theta^{(1)}+\epsilon\theta^{(2)}+\cdots)^2+\cdots\right]\cos(2\theta^{(0)})$$
$$-\left[2\epsilon(\theta^{(1)}+\epsilon\theta^{(2)}+\cdots)-\frac{2^3\epsilon^3}{3!}(\theta^{(1)}+\epsilon\theta^{(2)}+\cdots)^3+\cdots\right]\sin(2\theta^{(0)})\ . \qquad (3.7)$$

Putting (3.6) and (3.7) into (2.8) and equating the coefficient of each order of ϵ to zero we have

$$m\ddot{\theta}^{(0)}+C\dot{\theta}^{(0)}+\bar{s}[\gamma+\cos(2\theta^{(0)})]=0\ , \tag{3.8}$$

$$m\ddot{\theta}^{(n)}+C\dot{\theta}^{(n)}-\bar{s}[2\sin(2\theta^{(0)})]\theta^{(n)}=F^{(n)}\ ,\quad n=1,2,3,\ldots \tag{3.9}$$

where

$$F^{(1)}=-\overline{M}(2C\partial_\xi+\partial_\xi C)\dot{\theta}^{(0)}+\partial_\xi\theta^{(0)}\ , \tag{3.10}$$

$$F^{(2)}=-\overline{M}(2C\partial_\xi+\partial_\xi C)\dot{\theta}^{(1)}+\partial_\xi\theta^{(1)}+\overline{M}\partial_{\xi\xi}\theta^{(0)}+\bar{s}(2\cos 2\theta^{(0)})\theta^{(1)2}\ , \tag{3.11}$$

$$F^{(3)}=-\overline{M}(2C\partial_\xi+\partial_\xi C)\dot{\theta}^{(2)}+\partial_\xi\theta^{(2)}+\overline{M}\partial_{\xi\xi}\theta^{(1)}+\bar{s}[2\cos(2\theta^{(0)})]\theta^{(1)}\theta^{(2)}-\bar{s}[2\sin(2\theta^{(0)})]\tfrac{4}{3}\theta^{(1)3}\ , \tag{3.12}$$

with

$$\dot{\theta}^{(n)}\equiv\partial_\phi\theta^{(n)},\quad n=0,1,2,\ldots$$
$$m\equiv 1-\overline{M}C^2\ . \tag{3.13}$$

Here m, C, $\bar{s}$ and the right-hand side of (3.9) are all functions of ξ. The structure of these equations is such that (3.8) and (3.9) can be considered as ordinary differential equations with variable ϕ, while ξ is treated as a parameter.

Equation (3.9) can be formally solved to be (see Appendix A)

$$\theta^{(n)}(\phi)=f^{(n)}(\phi)\dot{\theta}^{(0)}(\phi),\quad n=1,2,\ldots \tag{3.14}$$

where

$$f^{(n)}(\phi)=\int_{\phi_n}^{\phi}d\phi'\exp(-C\phi'/m)\dot{\theta}^{(0)-2}(\phi')\times\left[\int_{\phi_0}^{\phi'}d\phi''\exp(C\phi''/m)\times\dot{\theta}^{(0)}(\phi'')F^{(n)}(\phi'')/m+g_n\right]. \tag{3.15}$$

Here and in Sec. IV (as well as in Appendix A) the ξ dependence in $\theta^{(n)}$ and $f^{(n)}$ are suppressed explicitly. In (3.15), ϕ_0, ϕ_n, and g_n are constants of integration and functions of ξ. Once $\theta^{(0)}$ from (3.8) is known, $\theta^{(n)}$ is determined by (3.14).

The results in this section are applicable for $\overline{M}$ and C of any order of magnitude such that $m\neq 0$, i.e., $C\neq\overline{M}^{-1/2}$.

IV. FAST SOLITONS—ANALYTIC RESULTS

Generally, Eqs. (3.8) and (3.14) cannot be solved analytically. We therefore consider the case $C\gg 1$ in which asymptotic solutions can be obtained by using multiple-scales perturbation once again. More importantly, this case is physically relevant.

Equation (3.8) can be rewritten as

$$\frac{d^2\theta^{(0)}}{dZ^2}+a\frac{d\theta^{(0)}}{dZ}+f(\theta^{(0)})=0\ , \tag{4.1}$$

where

$$Z\equiv\phi(\bar{s}/m)^{1/2}\ ,\quad a\equiv C(\bar{s}m)^{-1/2}\ ,\quad f(\theta^{(0)})\equiv\gamma+\cos(2\theta^{(0)})\ . \tag{4.2}$$

Equation (4.1) is identical to Eq. (1.1) of Ref. 18. Asymptotic solutions which are finite at $Z=\pm\infty$ for $a\gg 1$ are given by

$$\theta^{(0)}=A_0(u)+a^{-2}A_1(u)+\cdots\ , \tag{4.3}$$

where

$$u\equiv Z/a=(\bar{s}/C)\phi\ . \tag{4.4}$$

A_0 and A_1 are given by

$$\frac{dA_0}{du}+f(A_0)=0\ , \tag{4.5}$$

$$A_1(u)=f(A_0)(a_1+\ln|f(A_0)|)\ , \tag{4.6}$$

where $a_1=a_1(\xi)$ is a constant of integration.

Equation (4.3) is valid for $a\gg 1$ and $m\neq 0$, which are equivalent to

$$C\gg\bar{s}/(1+\overline{M}\bar{s})^{1/2}\ \text{and}\ C\neq\overline{M}^{-1/2}\ . \tag{4.7}$$

When $\overline{M}\ll 1$ Eq. (4.7) becomes

$$\overline{M}^{-1/2}\neq C\gg\bar{s}^{1/2}\ (\simeq 1)\ . \tag{4.8}$$

For $\gamma<1$ there exist four different types of single solitons of (4.5).[2,8] The A soliton corresponds to $-(\theta_0+n\pi)\le A_0\le\theta_0+n\pi$ and is given by

$$A_0=-\tan^{-1}\{w\tanh[(1-\gamma^2)^{1/2}(u-u_0)]\}+n\pi\ , \tag{4.9}$$

where

$$\pi/4<\theta_0\equiv\tfrac{1}{2}\cos^{-1}(-\gamma)<\pi/2\ ,\quad w\equiv[(1+\gamma)/(1-\gamma)]^{1/2}\ . \tag{4.10}$$

The B soliton corresponds to $\theta_0+n\pi\le A_0\le\pi-\theta_0+n\pi$ and is given by

$$A_0=-\cot^{-1}\{w^{-1}\tanh[(1-\gamma^2)^{1/2}(u-u_0)]\}+n\pi\ . \tag{4.11}$$

In (4.10) and (4.11), $n=0,\pm1,\pm2,\ldots$. The function A_1 can be determined by (4.6). To the order of a^{-2}, (4.3) is equivalent to the solutions obtained in (13) and (16) of Ref. 9.

To determine $\theta^{(1)}$, one has to use (3.14) together with (3.10) and (3.11) (see Appendix B). The procedure can be much simplified under the conditions

$$\overline{M}<C^{-2}\bar{s}\ll\epsilon\ , \tag{4.12}$$

which are, in fact, satisfied in the experimental situations.

Under the conditions (4.8) and (4.12) and after some lengthy calculations (see Appendix B) one obtains

$$\theta^{(0)}=A_0(u)\ , \tag{4.13}$$

$$\theta^{(1)}=d_1\frac{dA_0}{du}\ , \tag{4.14}$$

$$C=(C_0/s')\overline{s}(\epsilon T)\ , \tag{4.15}$$

and

$$u=(s'/C_0)\left[X-\frac{C_0}{s'}\int_0^T\overline{s}(\epsilon T)dT-X_0\right]\ , \tag{4.16}$$

where d_1 and X_0, constants of integration, are pure numbers. To first order in ϵ, by (3.5), (4.13), and (4.14), one has

$$\theta(X,T)=\theta^{(0)}(u+\epsilon d_1)\ . \tag{4.17}$$

By (4.9), (4.11), (4.13), and (4.17), the perturbed A soliton is

$$\theta(X,T)=\theta_A(X,T)\equiv-\tan^{-1}\left\{w\tanh\left[(1-\gamma^2)^{1/2}\frac{s'}{C_0}\left[X-\frac{C_0}{s'}\int_0^T\overline{s}(\epsilon T)dT-X_0+\epsilon d_1\frac{C_0}{s'}\right]\right]\right\}\ . \tag{4.18}$$

Similarly, the perturbed B soliton is

$$\theta(X,T)=\theta_B(X,T)\equiv-\cot^{-1}\left\{w^{-1}\tanh\left[(1-\gamma^2)^{1/2}\frac{s'}{C_0}\left[X-\frac{C_0}{s'}\int_0^T\overline{s}(\epsilon T)dT-X_0+\epsilon d_1\frac{C_0}{s'}\right]\right]\right\}\ . \tag{4.19}$$

The solution (4.17), in particular (4.18) and (4.19), represents a perturbed soliton which has constant shape and no tail but a time-dependent velocity C which is proportional to $\overline{s}(\epsilon T)$ [see (4.15)]. It has a constant phase shift $(\epsilon d_1 C_0/s')$ in comparison to the unperturbed soliton $\theta^{(0)}$. C_0 is the constant velocity of $\theta^{(0)}$.

In physical units the velocity of the perturbed soliton is, by (4.15) and (2.9),

$$\begin{aligned}c&=C\lambda/\tau\\&=(C_0/s')s(t)(K/|\gamma_2|s_0)^{1/2}\gamma^{-1}\ .\end{aligned} \tag{4.20}$$

V. NUMERICAL SOLUTIONS

Given $\theta(X,0)$, (2.8) and (2.10) can be solved numerically. For nematics like MBBA, $\overline{M}\sim10^{-10}$, Eqs. (2.8) and (2.10) are practically identical numerically. In this section (2.10) is solved, but the results are equally applicable to (2.8) within the numerical accuracy of our calculations.

For simplicity, the free-boundary conditions

$$(\partial_X\theta)_{X=0}=(\partial_X\theta)_{X=L}=0 \tag{5.1}$$

are adopted where $[0,L]$ is the domain of X. The parabolic equation (2.10) is replaced by a difference equation.

In the numerical calculation, (2.10) with $\gamma=0.96$, the parameter for MBBA at room temperature,

$$\overline{s}=10^{-2}[1+0.8\sin(\epsilon T)] \tag{5.2}$$

and $\epsilon=\pi/50$ are used. The choice of (5.2) is explained in Sec. VI. The initial condition $\theta(X,0)$ is taken to be the unperturbed A soliton θ_A of (4.18) with $\overline{s}=s'=10^{-2}$, $C_0=2.5$, $X_0=d_1=0$, $\gamma=0.96$. $L=10^4$, $0\le T\le250$. L is chosen such that the center of the soliton is far enough away from the boundaries to minimalize any end effects. The intervals in the X and T mesh used are both equal to one.

Figure 1 shows the resulting $\theta(X,T)$ curves where $\theta_0=81.87°$ corresponding to $\gamma=0.96$ in (4.10). As shown, the shape of the $\theta\sim X$ curve remains unchanged as T increases but the velocity does change periodically, as evidenced from the change of the spacing of the lines at $\theta=0$. Note that there is no tail, in agreement with the analytic result of (4.18).

Figure 2 shows the time variation of the velocity of the perturbed soliton, $C(T)$. The solid line represents the analytic result from (4.15) and (5.2); C_0 and s' are the same as in Fig. 1. The crosses represent $V\equiv(dX/dT)_{\theta=0}$ ob-

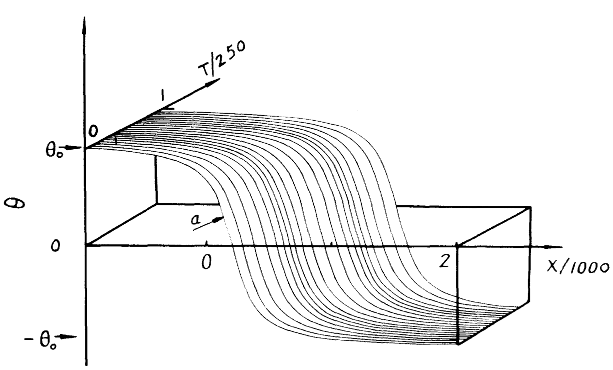

FIG. 1. Numerical solutions of Eq. (2.10). $\theta_0=81.87°$, $\gamma=0.96$. Curve a corresponds to $T=0$.

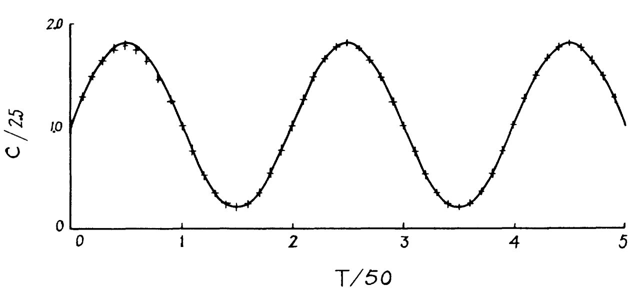

FIG. 2. Time-dependent velocity C of the perturbed soliton. The solid line is the analytic result. The crosses are numerical results obtained from Fig. 1 (see text).

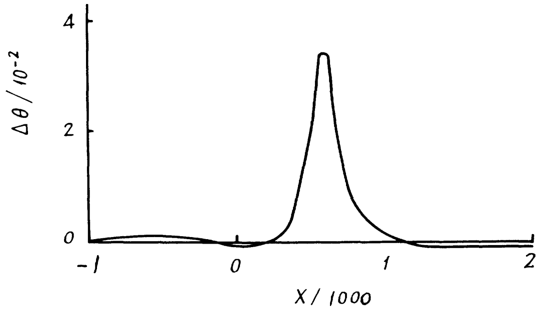

FIG. 3. The difference between the analytic and numerical $\theta(X,T)$, $\Delta\theta$, for $T=250$.

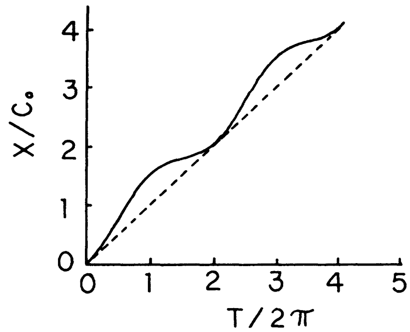

FIG. 4. Theoretical result from Eq. (6.7) for the time variation of the dark-line position X. $\epsilon=0.5$, $e_1/(e_0\epsilon C_0)=0.5$.

tained from each of the curves in Fig. 1, the numerical results. By (4.18) and (4.15) it is easy to see that $C=V$. There is an obvious agreement between the analytic and numerical results.

There, in fact, is a slight difference between the analytic and numerical $\theta(X,T)$ curves. In Fig. 3, the difference between the analytic and numerical $\theta(X,T)$ for $T=250$ is plotted. The analytic θ is obtained from (4.18) with $X_0=d_1=0$. The numerical one is from Fig. 1. For both cases, (5.2) and $C_0=2.5$, $\gamma=0.96$ are used. Comparing Fig. 1 and Fig. 3 we see that $\Delta\theta$ is maximum near $\theta=0$ but $\Delta\theta$ is always less than 0.04 (or $\sim 4\%$). This difference can be reduced to $\sim 0.4\%$ if d_1 is suitably chosen.

We therefore conclude that our analytic results of Sec. IV are in agreement with numerical calculations within the accuracy of 10^{-2}. Similar results are obtained when θ_A is replaced by θ_B.

VI. COMPARISON WITH EXPERIMENTS

A. Analytic results for periodic $s(t)$

The results in previous sections are for a general $s(t)$. In the special case of a periodic $s(t)$ with period t_p,

$$s(t+t_p)=s(t) . \tag{6.1}$$

$s(t)$ may be expanded in a Fourier series,

$$s(t)=e_0+\sum_{n=1}^{\infty} e_n \sin(n\omega t+\psi_n) , \tag{6.2}$$

where e_0, e_n, and ψ_n are constants independent of time, $\omega=2\pi/t_p$. When higher harmonics are ignored and without loss of generality $\psi_1=0$ is chosen, $s(t)$ may be approximated by

$$\begin{aligned} s(t)&=e_0+e_1\sin(\omega t)\\ &=e_0+e_1\sin(\epsilon T) , \end{aligned} \tag{6.3}$$

where $\epsilon=\omega\tau$. We then have

$$\bar{s}=s/s_0=1+(e_1/e_0)\sin(\epsilon T) \tag{6.4}$$

with the choice of

$$s_0=e_0 , \tag{6.5}$$

i.e., $s'=1$.

The trajectory of the point with $\theta=0$, for the A soliton, is obtained by setting $\theta=0$ in (4.18). With $X_0=d_1=0$ one has

$$X-\frac{C_0}{s'}\int_0^T \bar{s}(\epsilon T)dT=0 . \tag{6.6}$$

Putting (6.4) into (6.5) we have

$$X=C_0T+(e_1/e_0\epsilon)[1-\cos(\epsilon T)] . \tag{6.7}$$

Also, by (4.15) and (6.4)

$$C=C_0+(e_1/e_0)\sin(\epsilon T) , \tag{6.8}$$

which is depicted in Fig. 2 as the solid line.

In Fig. 4 the variation of the center of A soliton ($\theta=0$) X as a function of T according to (6.7) is plotted. The broken line is the result of steady shear ($\omega=0$).

B. Analysis of experiments

An experiment has been performed by Zhu[21] in which the experimental setup is the same as in Ref. 10. White incident light was used. In this case the pushing plate at one end of the homeotropic MBBA cell was set in vibration in the direction of the long axis of the cell. The vibrating frequency was 1 and 2 Hz, respectively. As in Ref. 10 which corresponds to the case of 0 Hz, three dark lines were observed under white light; however, they now move with time-dependent and seemingly periodic velocities (Fig. 5).

This experiment may be understood as follows. Since the pushing plate does not move very fast, a shear $s(t)$ is created in the fluid which is now time dependent. The period of $s(t)$, t_p, may be assumed to be the same as that of the plate. The center of each of the dark lines corresponds to molecules being vertical, i.e., $\theta=0$.[8]

For MBBA, $K=10^{-6}$ dyn, $\gamma_2=-0.8P$, $\gamma=0.96$. Using the experimental parameters of Ref. 10 as given in Ref. 8, $v\sim 0.5$ mm/s and $d=10$ μm, we obtain $v/d\sim 50$ s^{-1}. We therefore take $s_0=e_0=50$ s^{-1} and use (2.9) to obtain $\tau=0.04$ s and $\lambda=2\times 10^{-4}$ cm. For $t_p^{-1}=1$ and 2 Hz we have $\epsilon\equiv\omega\tau=2\pi\tau/t_p=0.25$ and 0.5, respectively, which are both less than one. The results in Sec. VI A are

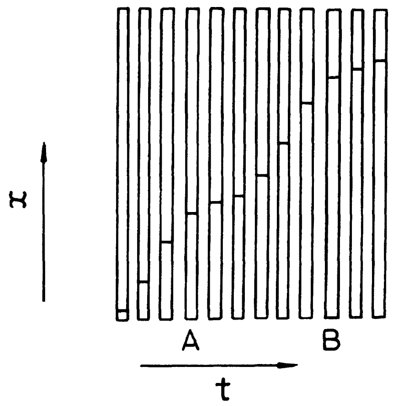

FIG. 5. Sketch of the observed dark line under white light at equal time interval. The two strips A and B correspond to the dark line differing by a phase of 2π in its periodic variation.

then applicable to this experiment.

According to (6.3) and (6.7) the dark-line position X varies with t with the same period t_p. This point may be checked with experiments as follows. In Fig. 5 the time variation of each of the three dark lines in the experimental photographs is sketched. Each strip represents the photograph of the liquid-crystal cell taken at equal time intervals. The time axis runs from left to right. The curve x versus t is obtained by connecting the centers of the dark-line curves in a periodic way. The period of this x versus t curve is given by the time difference between two points corresponding to strips A and B in Fig. 5, which differ by a phase of 2π. It is equal to the product of the time interval between two neighboring strips and the number of strips from A to B. We have checked the experimental photographs and found that at low frequencies ($t_p^{-1}=1$ and 2 Hz), the periods of the dark lines under white light and that of the pushing plate are equal to each other, in agreement with our theory. Moreover, we note the following.

(i) The experimental curve (sketched in Fig. 5) and the theoretical one (in Fig. 4) for the dark line, the x versus t curve, are similar in shape.

(ii) By (6.8) the time-averaged velocity $\bar{C}$ is given by

$$\bar{C}=C_0 \tag{6.9}$$

or, with units included,

$$\bar{c}=C_0\lambda/\tau=C_0\gamma^{-1}(e_0K/2\,|\,\gamma_2\,|\,)^{1/2}\ , \tag{6.10}$$

which is independent of the period of $s(t)$. For a given nematic $\bar{c}\propto e_0^{1/2}$ where e_0 is the steady-state value of $s(t)$.

We have checked the average speed $\bar{c}$ of the dark lines for different t_p from the experimental photographs and found them to be independent of the frequency, as predicted by our theory. The dependence of $\bar{c}$ on the material parameters shown in (6.10) remains to be checked with future experiments.

(iii) Since the shape of the solitons (the dark lines under white light) remains unchanged during motion, the relation between velocity and width of the dark line obtained in the steady shear case[8] is still valid, i.e., the thicker the dark line, the larger its propagating velocity. An exception exists for the unsteady shear case here, where the velocity should be understood to be the averaged velocity. Unfortunately, there is no systematic experimental data available to check this point.

VII. DISCUSSION

(1) In this paper we have assumed an unsteady shear which is slowly varying in time, i.e.,

$$\bar{s}=\bar{s}(\epsilon T) \tag{7.1}$$

and

$$\epsilon \ll 1\ . \tag{7.2}$$

If ω^{-1} is the characteristic time of the flow then $\epsilon=\omega\tau$. For a periodic $s(t)$, ω is the angular frequency. It should be pointed out that (2.8) and (2.10) are invariant under the transformation

$$\begin{aligned}&T\to T'\equiv r^2T,\quad X\to X'\equiv rX,\quad \bar{M}\to\bar{M}'\equiv r^2\bar{M}\ ,\\ &\bar{s}\to\bar{s}'\equiv\bar{s}/r^2,\quad \epsilon\to\epsilon'\equiv\epsilon/r^2\ .\end{aligned} \tag{7.3}$$

In many cases it is possible to use (7.3) to make (7.2) satisfied in the equation of motion and then use the results obtained in this paper. In so doing the region of space and time in which the results are valid may be reduced.

(2) Generally,[22-25] in soliton equations with small parameters the perturbation manifests itself in three ways: (i) the parameters of the unperturbed soliton are changed, albeit slowly; (ii) the shape of the unperturbed soliton is changed; (iii) a tail appears. Usually, these three effects appear simultaneously. In Ref. 24, using the inverse scattering method, the condition for the nonexistence of the tail is derived for the Korteweg–de Vries equation, the modified Korteweg–de Vries equation and the nonlinear Schrödinger equation, respectively. See Refs. 26 and 27 for further discussions on this point. Yet, to our knowledge, there is no general theorem on the nonexistence of tails due to perturbation for nonlinear equations. Our perturbed solitons presented in this paper constitute another example that is tailess. And being tailess and shape preserved simultaneously during motion they are the first examples of this type that we are aware of.

(3) The three dark lines under white light observed in experiments discussed in Sec. VI should be understood as a multisoliton, as in the steady shear case.[2,9,11] The properties of each dark line, of course, can be identified and represented by those of single soliton as is done here and elsewhere.

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APPENDIX A: DERIVATION OF EQ. (3.14)

Differentiate (3.8) with respect to ϕ once. We have

$$m\dddot{\theta}^{(0)}+C\ddot{\theta}^{(0)}-\bar{s}[2\sin(2\theta^{(0)})]\dot{\theta}^{(0)}=0 \ . \tag{A1}$$

If $\dot{\theta}^{(0)}$ is identified with $\theta^{(n)}$ in (3.9) we see that (A1) is the homogeneous equation of (3.9). We therefore try the solution (3.14). Putting (3.14) into (3.9) and using (3.8) one obtains

$$m\dot{g}^{(n)}(\phi)+Cg^{(n)}(\phi)=\dot{\theta}^{(0)}(\phi)F^{(n)}(\phi), \quad n=1,2,\ldots \tag{A2}$$

where

$$g^{(n)}(\phi)=f^{(n)}(\phi)(\dot{\theta}^{(0)})^2 \tag{A3}$$

and the overdot represents ∂_ϕ. The solution of (A2) is

$$g^{(n)}(\phi)=\exp(-C\phi/m)\times\left[\int_{\phi_0}^{\phi}d\phi'\,\dot{\theta}^{(0)}(\phi')\times\exp(C\phi'/m)F^{(n)}(\phi')/m+g_n\right], \tag{A4}$$

where $\phi_0=\phi_0(\xi)$ and $g_n=g_n(\xi)$ are constants of integration. Putting (A4) into (A3) and integrating once we have

$$f^{(n)}(\phi)=\int_{\phi_n}^{\phi}d\phi'\,g^{(n)}(\phi')[\dot{\theta}^{(0)}(\phi')]^{-2} \ , \tag{A5}$$

where $\phi_n=\phi_n(\xi)$ is a constant of integration.

Combining (A4) and (A5) we obtain (3.15).

APPENDIX B: DERIVATION OF EQS. (4.14)–(4.16)

We first note that the terms containing $a^{-2}(\sim C^{-2}\bar{s})$ in (4.3) and $\bar{M}$ in (3.10)–(3.12) will result in terms of the same order in $\theta^{(1)}$, which are negligible compared to ϵ under the assumption (4.12). Hence, in this appendix, we will take $\bar{M}=0$ (i.e., $m=1$) in (3.10)–(3.12) from the beginning and retain only $A_0(u)$ for $\theta^{(0)}$ in (4.3).

We now prove the following two lemmas.

Lemma 1. For

$$u\to\pm\infty \ , \qquad \theta^{(1)}=\dot{\theta}^{(0)}f^{(1)}(u) \ , \tag{B1}$$

where

$$f^{(1)}(u)=(C/\bar{s})(pu^2/2-qu+d_1) \ , \quad p\equiv\bar{s}^{-1}\partial_\xi[\ln(\bar{s}/C)],\ q\equiv C^{-1}\partial_\xi\phi_0 \tag{B2}$$

and $\phi_0=\phi_0(\xi)$, $d_1=d_1(\xi)$ are constants of integration.

Lemma 2. For

$$u\to\pm\infty \ , \qquad \theta^{(2)}=\dot{\theta}^{(0)}f^{(2)}(u) \tag{B3}$$

where

$$\begin{aligned}f^{(2)}(u)=(C/\bar{s})\{&\mp2(1-\gamma^2)^{1/2}\bar{a}_3u^4/4\\&+[a_2\mp2(1-\gamma^2)^{1/2}\bar{a}_2]u^3/3\\&+[a_1\mp2(1-\gamma^2)^{1/2}\bar{a}_1]u^2/2\\&+[a_0\mp2(1-\gamma^2)^{1/2}\bar{a}_0]u+d_2\} \ ,\end{aligned}$$

$$\begin{aligned}&\bar{a}_3\equiv\tfrac{1}{2}p^2, \quad \bar{a}_2\equiv-(\tfrac{3}{2})pq \ ,\\&\bar{a}_1\equiv q^2+pd_1,\ \bar{a}_0\equiv-qd_1 \ ,\\&a_2\equiv p^2+(\partial_\xi p)/2\bar{s}, \quad a_1\equiv-pq-(\partial_\xi q)/\bar{s} \ ,\\&a_0\equiv q^2+(\partial_\xi d_1)/\bar{s}\end{aligned} \tag{B4}$$

and $d_2=d_2(\xi)$ is a constant of integration.

Proof. Let us define

$$\begin{aligned}I_n(u)&\equiv\int^u(dA_0/du)^2u^n\exp(a^2u)du \ ,\\J_n(u)&\equiv\int^u(dA_0/du)^3u^n\exp(a^2u)du \ ,\\P_{mn}(u)&\equiv\int^u2\sin(2A_0)(dA_0/du)^m u^n\exp(a^2u)du \ ,\\Q_{mn}(u)&\equiv\int^u2\cos(2A_0)(dA_0/du)^m u^n\exp(a^2u)du \ ,\end{aligned} \tag{B5}$$

where $n=0,1,2,\ldots$, $m=1,2,\ldots$.

By (4.2) and (4.5) we have

$$\begin{aligned}dA_0(u)/du&=-[\gamma+\cos(2A_0)]\\&=\begin{cases}-(1-\gamma^2)/\{\cosh[2(1-\gamma^2)^{1/2}(u-u_0)]+\gamma\}, & \text{for } A \text{ soliton}\\-(1-\gamma^2)/\cosh[2(1-\gamma^2)^{1/2}(u-u_0)], & \text{for } B \text{ soliton}\end{cases}\\&\to-2(1-\gamma^2)\exp[\mp2(1-\gamma^2)^{1/2}(u-u_0)], \quad \text{for } u\to\pm\infty \ .\end{aligned} \tag{B6}$$

Putting (B6) into (B5), for both A and B solitons, when $u\to\pm\infty$ we have

$$\begin{aligned}I_n(u)=&\,4(1-\gamma^2)^2a^{-2}\exp(a^2u)\exp[\mp4(1-\gamma^2)^{1/2}(u-u_0)]\{1+O(\exp[\mp2(1-\gamma^2)^{1/2}(u-u_0)])\}\\&\times\{u^n[1+O(a^{-2})]+u^{n-1}O(a^{-2})+u^{n-2}O(a^{-4})+\cdots\} \ ,\end{aligned} \tag{B7}$$

$$\begin{aligned}J_n(u)=&-8(1-\gamma^2)^3a^{-2}\exp(a^2u)\exp[\mp6(1-\gamma^2)^{1/2}(u-u_0)]\{1+O(\exp[\mp2(1-\gamma^2)^{1/2}(u-u_0)])\}\\&\times\{u^n[1+O(a^{-2})]+u^{n-1}O(a^{-2})+u^{n-2}O(a^{-4})+\cdots\} \ ,\end{aligned} \tag{B8}$$

$$P_{mn}(u)=\pm(-2)^{m+1}(1-\gamma^2)^{m+1/2}a^{-2}\exp(a^2u)\exp[\mp 2m(1-\gamma^2)^{1/2}(u-u_0)]$$
$$\times\{1+O(\exp[\mp 2(1-\gamma^2)^{1/2}(u-u_0)])\}\{u^n[1+O(a^{-2})]+u^{n-1}O(a^{-2})+\cdots\}\ , \quad \text{(B9)}$$
$$Q_{mn}(u)=(-2)^m(1-\gamma^2)^m(-2\gamma)a^{-2}\exp(a^2u)\exp[\mp 2m(1-\gamma^2)^{1/2}(u-u_0)]$$
$$\times\{1+O(\exp[\mp 2(1-\gamma^2)^{1/2}(u-u_0)])\}\{u^n[1+O(a^{-2})]+u^{n-1}O(a^{-2})+\cdots\}\ . \quad \text{(B10)}$$

By (3.10) one has

$$F^{(1)}=\bar{s}(pu-q)dA_0/du\ , \quad \text{(B11)}$$

where p and q are defined in (B2).

Inserting (B11) into (A4) we have

$$g^{(1)}=(\bar{s}/m)\exp(-a^2u)[pI_1(u)-qI_0(u)]\ ,$$
$$\text{for } u\to\pm\infty\ . \quad \text{(B12)}$$

For $u\to\pm\infty$, by (B7) and (B12) one has

$$g^{(1)}\simeq 4(\bar{s}/C)^2(1-\gamma^2)^{1/2}\exp[\mp 4(1-\gamma^2)^{1/2}(u-u_0)]$$
$$\times(pu-q)[1+O(a^{-2})]$$
$$\times\{1+O(\exp[\mp 2(1-\gamma^2)^{1/2}(u-u_0)])\}$$
$$+g_1\exp(-C\phi/m)\ , \quad \text{(B13)}$$

where g_1 is a constant of integration which should be taken to be zero to satisfy (B14) (since it produces a term of the order of $\exp[-C\phi/m\pm 4(1-\gamma^2)^{1/2}(u-u_0)]$ for $u\to\pm\infty$ in the calculation of $f^{(1)}(u)$ below and is obviously a secular term).

Putting (B13) into (A5) we get (B1). Similarly, by using (B8)–(B10) to calculate $f^{(2)}(u)$ we obtain (B3). Q.E.D.

We can now discuss the two limits, $\lim_{u\to\pm\infty}|\theta^{(1)}/\theta^{(0)}|$ and $\lim_{u\to\pm\infty}|\theta^{(2)}/\theta^{(1)}|$, and the coefficients of their expansions. Since $\dot{\theta}^{(0)}=\dot{A}^{(0)}$ tends to zero exponentially for $u\to\pm\infty$ [see (B6)] it can be verified that $\theta^{(1)}$ satisfies

$$\lim_{u\to\pm\infty}|\theta^{(1)}/\theta^{(0)}|<\infty\ . \quad \text{(B14)}$$

By requiring

$$\lim_{u\to\pm\infty}|\theta^{(2)}/\theta^{(1)}|=\lim_{u\to\pm\infty}|f^{(2)}(u)/f^{(1)}(u)|<\infty \quad \text{(B15)}$$

we get

$$\bar{a}_3=0 \quad \text{(B16)}$$

and hence $p=0$ by (B4), resulting in $C(\xi)\propto\bar{s}(\xi)$ and hence (4.15) in which the proportional constant C_0/s' is obtained from the requirement that $\bar{s}\to s'$, $C\to C_0$ when $\epsilon\to 0$.

By (B16) and (B4) we also have $a_2=\bar{a}_2=0$. Using the definitions of $f^{(1)}$ and $f^{(2)}$ from (B2) and (B4), respectively, and then (B15) we have

$$a_1=\bar{a}_1=0 \quad \text{(B17)}$$

and hence $q=0=\partial_\xi\phi_0(\xi)$. Therefore, ϕ_0 is a pure number independent of ξ. Using (B15) once again we have

$$a_0=\bar{a}_0=0 \quad \text{(B18)}$$

resulting in $\partial_\xi d_1(\xi)=0$, i.e., d_1 is also a pure number independent of ξ.

Using the above results, (3.3), and (4.4) and integrating (4.14) once we obtain (4.16).

*Correspondence address.

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