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Persistence of the Ivantsov continuum despite the existence of a length scale

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We show that the continuous family of steadily advancing, shape-preserving solutions of the Ivantsov problem for dendritic crystal growth is not necessarily broken down to a discrete set by the introduction into the model of an effect which defines a natural length scale for the problem. Specifically, we give an exact solution of a model which includes a special anisotropic heat-loss term in the liquid phase, which can be used to define length and velocity scales for the problem. The resulting needle-crystal solutions for a given undercooling of the liquid form a continuous family. There is a definite relation between tip curvature and growth velocity for these solutions, but the latter is allowed to take any value above a finite lower limit.

The starting point for most studies of dendritic crystal growth is the solution by Ivantsov[1] of the problem of a solid "needle crystal" growing at a constant speed into an undercooled melt, with the solidification front being isothermal. For any given undercooling, this model admits a continuous family of solutions whose members are related by a simultaneous rescaling of length and time; the undercooling determines only the product of the growth velocity and, say, the radius of the needle crystal's tip (or any other length characteristic of the shape of the solidification front). The existence of this continuum of possible steady states is often attributed to the fact that the parameters appearing in the model cannot be combined so as to produce a length or a velocity, so that the dimensional information needed to define a unique growth velocity or a unique tip radius is lacking. Thus one might expect that introducing any effect into the model which provides this information would break this continuous family down to a unique solution, or at most a discrete set of solutions, for each undercooling. Indeed, it has recently been found that including the effect of surface tension on the temperature at the solidification front does this,[2–4] although if the surface tension is isotropic, it produces the drastic but disappointing result of destroying *all* of the steady solutions. Some effect like surface tension (or more general interface kinetics) is needed, at any rate, to stabilize the solutions against short-wavelength perturbations.

In this paper we show that the lack of a length scale is not *solely* responsible for the existence of the continuous infinity of solutions of the Ivantsov model; some effects which define length scales for the problem still lead to continuous families of solutions. We do this by giving an exact solution of a problem in which a heat-loss term is included in the liquid phase, and showing that this problem also admits a continuous family of steady states. Unfortunately, in order to obtain this exact solution, we are forced to give the extra term a special dependence on the angle between the isotherms and the direction in which the dendrite is growing; however, the required angular dependence does not seem *too* unnatural, and it may even be possible to find a physical system for which this "model" is realistic. That the mere introduction of a length scale into the problem does not necessarily break up the Ivantsov continuum is also suggested by a recent result of Lemieux, Liu, and Kotliar,[5] who showed by a singular perturbation calculation that isotropic linear interface kinetics preserve the continuum.

In the *one*-dimensional problem of the thermally controlled solidification of an undercooled melt, it is well known that the solidification front generally advances at a rate which decreases as $t^{-1/2}$. Solutions in which the front moves with a constant velocity exist only[6] when the undercooling is equal to L/c, where L is the latent heat of solidification and c is the specific heat of the liquid. At this undercooling, however, solutions exist for all front velocities. This is actually not very surprising, since a liquid at an undercooling of L/c will have its temperature raised exactly to the melting temperature T_M by the latent heat released when it solidifies, and so will not tend to melt back. The indeterminacy of the front velocity follows from this, together with the fact that the model only allows a single solid-liquid interface; it is also indicated by the fact that the problem has no natural length, time, or velocity scales, since only the diffusion coefficient D has units which include length and time. Since the nonexistence of constant-velocity solutions for undercoolings of less than L/c is a consequence of energy conservation,[6] one should expect that adding a heat-loss term to the model (which would define length and time scales) would remedy this problem. If one includes such a term in the diffusion equation, say $-\Gamma(T-T_\infty)$ where T_∞ is the temperature of the undercooled liquid far from the front, then it is easy to show that for each undercooling between zero and L/c there is a unique steady solution describing a front advancing at a speed[7]

$$v=v_1(\Delta)=\Delta/(1-\Delta)^{1/2} \tag{1}$$

in units of $(D\Gamma)^{1/2}$, where the dimensionless undercooling Δ is given by

$$\Delta=c(T_M-T_\infty)/L \ . \tag{2}$$

In a two-dimensional problem with the same isotropic heat-loss term, any needle-crystal solution advancing at velocity v in the x direction would have to be asymptotically flat far from the tip,[8] with its normal making an angle $\cos^{-1}[v_1(\Delta)/v]$ with the x direction in order for the normal velocity of the front to be $v_1(\Delta)$ far from the tip. Thus the velocity of the (putative) needle crystal must be greater than $v_1(\Delta)$. In a previous paper[9] we have shown that the method used by Ivantsov to solve the problem without heat loss fails to yield solutions when an isotropic (linear or nonlinear) heat-loss term is included. However, a solution is possible if the heat-loss term has a special anisotropy, as we now show.

We consider a pure two-dimensional liquid which is initially undercooled, and which solidifies due to heat diffusion without convection. We include in the diffusion equation—in the liquid only—an anisotropic term describing heat loss to the substrate, which has the special form $-\Gamma[(\partial T/\partial x)/|\nabla T|]^2(T-T_\infty)$. The quantity in brackets is the cosine of the angle between the x direction and the normal to the isotherm through the point in space at which the term is being evaluated. The coefficient Γ in this term has the dimensions of reciprocal time, and so can be used to define natural units of length, $(D/\Gamma)^{1/2}$, and velocity, $(D\Gamma)^{1/2}$. In terms of the resulting dimensionless length and time and the dimensionless temperature,

$$u = c(T-T_\infty)/L \ , \tag{3}$$

the steady-state diffusion equation reads

$$0=\nabla^2 u + v\frac{\partial u}{\partial x} - \left[\left(\frac{\partial u}{\partial x}\right)\Big/ |\nabla u|\right]^2 u \ , \tag{4}$$

where v is the velocity of our frame of reference in natural units $(D\Gamma)^{1/2}$. The solidification front is taken to be at the equilibrium melting temperature, $u=\Delta$. Since the new term in the diffusion equation depends only on u and on angles, which are continuous at the front, the Stefan boundary condition at the front is unchanged,

$$v_n = -\hat{\mathbf{n}}\cdot\nabla u \ , \tag{5}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the front and directed into the liquid, v_n is the local normal velocity of the front, and the gradient is evaluated on the liquid side of the front. We have used the fact that the entire solid region is at the melting temperature to eliminate the gradient on the solid side of the front. We are now interested in finding a steady-state solution of this model, which would then describe a needle crystal growing with velocity v in the x direction while keeping its shape fixed.

Using the methods of Ref. 9, with a constant diffusion coefficient (and a change of notation), this problem can be reduced to that of solving the equation

$$0 = 1 + \left(-2\frac{\rho''}{\rho'^2} + v - \rho' u\right)\rho \ , \tag{6}$$

where $\rho(u)/2$ is the value of x at which the isotherm at temperature $T=T_\infty + Lu/c$ crosses the x axis. As in Ref. 9, an angle-dependent term would generally appear added to the ρ outside the parentheses; however, this would then be the only angle-dependent term in the equation, so that it would have to be constant in order for the equation to hold for all angles. As shown previously,[9] this implies that the isotherms are parabolas, and setting this constant to zero places the origin at the common focus of these parabolas. The variable $\rho(u)$ can then be interpreted as the radius of curvature of the isotherm at its tip, where it crosses the x axis. The fact that the isotherms are parabolas is not inconsistent with the fact that, for *isotropic* heat loss, the isotherms must be straight lines far from the tip of the needle crystal, because in this anisotropic situation the heat-loss term vanishes as the isotherms become asymptotically perpendicular to the growth direction far from the tip.

If we change variables in Eq. (6) from u to ρ, we arrive at the linear equation

$$2u'' + \left(v + \frac{1}{\rho}\right)u' - \frac{u}{2} = 0 \ , \tag{7}$$

where primes now denote derivatives with respect to ρ rather than with respect to u as above. The boundary conditions on this equation are, first, that $u(\rho)$ must vanish at infinity, and, second, that we must have $u=\Delta$ and $du/d\rho = -v/2$ at some positive value of ρ, which then represents the position of the tip of the needle crystal. Since the equation is linear, these last two conditions are equivalent to the condition that the logarithmic derivative of u,

$$G = \frac{d(\ln u)}{d\rho} \ , \tag{8}$$

must take on the value $-v/2\Delta$ for some positive value of ρ. Differentiating $G(\rho)$ and using (7), we find that G obeys the first-order equation

$$2G' + 2G^2 + \left(v + \frac{1}{\rho}\right)G - \tfrac{1}{2} = 0 \ . \tag{9}$$

We then obtain the asymptotic result

$$G \rightarrow G_\infty = -[v + (v^2+4)^{1/2}]/4 \tag{10}$$

for large ρ. Note that G_∞ is negative.

From Eq. (9) we see that $G(\rho)$ increases with ρ for $G_\infty \geq G(\rho) > G_0(\rho) = -\{(v+\rho^{-1}) + [(v+\rho^{-1})^2+4]^{1/2}\}/4$, and decreases for $G(\rho) < G_0(\rho)$. Thus in order to reach G_∞ as $\rho \rightarrow \infty$, $G(\rho)$ must always stay between G_∞ and $G_0(\rho)$; if $G(\rho)$ ever leaves this range it can never reenter it. In addition, we see from (9) that G must diverge as $-\rho^{-1/2}$ for small ρ. Thus G must increase monotonically from $-\infty$ as $\rho \rightarrow 0$ to G_∞ as $\rho \rightarrow \infty$. In other words, G takes on every value below G_∞ for some positive ρ. This implies that the condition on G at the solidification front will be satisfied for some ρ provided we have $v/2\Delta > |G_\infty|$. Rearranging this inequality, we find that steady-state needle crystals exist in this model which grow with all velocities satisfying

$$v \geq v_1(\Delta) \ . \tag{11}$$

The lower limit represents the one-dimensional solution discussed above, which is also a solution to the present problem. Curved needle crystals, which bulge into the liquid, advance with faster velocities; the monotonicity of $G(\rho)$ implies that the growth velocity of a dendrite increases as its tip curvature increases.

As a result of this calculation, we see that introducing a length scale into the Ivantsov problem by means of this peculiar anisotropic heat-loss term does not break the continuous family of needle-crystal solutions to a discrete set whose velocities and tip radii are given by definite multiples of the natural velocity and length scales in the problem. Rather, these scales have a far weaker effect: they merely set a nonzero lower limit on the band of allowed velocities of needle crystals. They do not even set limits on the radii of the tips. Note that the original Ivantsov problem, which lacks a velocity scale entirely, is, of course, incapable even of setting finite limits on this band. To see how the lower limit on the growth velocity disappears when the heat-loss term is taken away, it suffices to notice that the inequality (9) gives the minimum growth velocity in units of $(D\Gamma)^{1/2}$, which vanishes for $\Gamma \to 0$. Thus we see that the continuous family of needle crystals predicted by the Ivantsov model of dendritic growth is a bit more robust than is often supposed; although it can be broken by the "right" physical effects, the simple introduction of a length scale into the problem does not necessarily produce discrete steady states.

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[1]G. P. Ivantsov, Dokl. Akad. Nauk SSSR **58**, 567 (1947); G. Horvay and J. W. Cahn, Acta Met. **9**, 695 (1961).

[2]D. A. Kessler, J. Koplik, and H. Levine, Phys. Rev. A **33**, 3352 (1986).

[3]D. I. Meiron, Phys. Rev. A **33**, 2704 (1986).

[4]A. Barbieri, D. C. Hong, and J. S. Langer, Phys. Rev. A **35**, 1802 (1987).

[5]M. A. Lemieux, J. Liu, and B. G. Kotliar (unpublished).

[6]J. S. Langer, Revs. Mod. Phys. **52**, 1 (1980).

[7]This result is correct if the heat loss occurs only from the melt. If there is also heat loss from the solid, then the steady front velocity is $v = [4\Delta/(1-\Delta)]^{1/2}$.

[8]J. D. Weeks and W. van Saarloos, Phys. Rev. A **32**, 3001 (1986).

[9]D. A. Kurtze, Phys. Rev. A **36**, 232 (1987).