PHYSICAL REVIEW A VOLUME 36, NUMBER 5 SEPTEMBER 1, 1987

Classical radiation from a relativistic charge accelerated along a brachistochrone

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(Received 19 November 1986)

The power radiated by an accelerated charge as it moves along a brachistochrone is shown in this paper to be proportional to $(1-v^2/c^2)$. This is quite unlike the behavior described in the literature, where, for example, the power radiated varies as $(1-v^2/c^2)^{-2}$, in the case of a circular orbit. The calculations here are a sequel to a recent paper by Goldstein and Bender [J. Math. Phys. **27**, 507 (1986)], in which the brachistochrone for a charge falling relativistically in a uniform electric field is worked out.

Accelerated charges radiate energy. In the synchrotron for example,[1] the power radiated $P(t)$ at time t is proportional to $(1-v^2/c^2)^{-2}$, so that for extreme relativistic velocities, i.e., $v \sim c$, the loss of energy can be very large. In this paper we examine the power radiated by a charge as it accelerates along a brachistochrone in a uniform electric field.

The brachistochrone problem is well known in the calculus of variations. Briefly, consider a particle which falls from rest in a uniform force field. Then that curve, joining two points A and B over which the time of fall of the particle is a minimum, is the brachistochrone.

The relativistic generalization of this classic problem has recently been presented by Goldstein and Bender.[2] While these authors discuss the problem with uniform electric and uniform gravitational fields, we shall limit our attention to the former only. Remarkably, the power radiated by the charged particle as it moves along the brachistochrone is now proportional to $(1-v^2/c^2)$ and not as mentioned above. It would therefore follow that as v increases towards c, the power radiated becomes less and less; the particle therefore loses much less energy long the brachistochrone compared to a circular orbit in a synchrotron. This is the main result of this paper.

The paper is organized as follows. In Sec. I we present in a concise manner the results of Goldstein and Bender.[2] This is followed in Sec. II, by a brief presentation of relevant formulas for the power radiated $P(t)$ at time t, and the power radiated per unit solid angle $P(\mathbf{n},t)$ in a given direction $\mathbf{n}$ at time t. We find the treatment due to Schwinger[3] quite useful here. In Sec. III we present our results and conclude in Sec. IV.

Two points need to be made. First, Goldstein and Bender have discussed in their paper[4] three separate curves for the brachistochrone. They differ qualitatively from each other; however, the expressions, as well as the discussion for $P(t)$ and $P(\mathbf{n},t)$ given in Sec. III, pertain to two of these three curves $(k^2 \leq 1)$ only. The expression for the third case, namely, $k^2>1$, can be obtained, as will be indicated, through appropriate replacements. As will also be seen in Sec. III, $P(t)$ and $P(\mathbf{n},t)$ can be calculated exactly, without recourse to approximation. This is possible despite the fact that the brachistochrones for these cases do not lend themselves readily to expressions in a closed form. Secondly, for $P(\mathbf{n},t)$ we have for simplicity chosen to discuss the case of instantaneous circular motion, wherein the velocity and acceleration vectors are mutually perpendicular. The more general case is merely a matter of detail and has hence been ignored.

I. THE RELATIVISTIC BRACHISTOCHRONE

Consider a particle of mass m and charge e which falls from rest in a uniform electric field E. With the z axis pointing vertically downwards, in a two-dimensional x-z coordinate system, the conservation of energy requires that

$$mc^2=mc^2\gamma-qEz \ , \tag{1}$$

with $\gamma^{-2}=1-v^2/c^2$. Therefore,

$$v(z)=c[1-1/(1+\alpha z)^2]^{1/2} \ , \tag{1a}$$

with $\alpha=qE/mc^2$. Note that v is independent of x. Now if ds is the arc length of a curve joining two points infinitesimally close to each other, the time of flight along ds is $dt=ds/v(z)$, so that the total time taken by the particle to fall from A to B would be

$$t_B-t_A=\int_{t_A}^{t_B} dt=\int_A^B dx[1+(dx/dz)^2]^{1/2}v^{-1}(z)$$

or, if t_A and A are taken as zero, we get

$$t_B=\int_0^B dz[1+(dx/dz)^2]^{1/2}v^{-1}(z) \ . \tag{2}$$

Clearly t_B is a functional of the path joining the origin to B, so that the trajectory over which the time t_B would be minimum would be a solution of the Euler-Lagrange equation

$$\frac{d}{dz}[x'(z)v^{-1}(z)(1+x'^2)^{-1/2}]=0\ , \tag{3}$$

with $x'=dx/dz$. Integrating (3) we get

$$x(z)=\int_0^z dz\,kv(z)(1-k^2v^2)^{-1/2}\ , \tag{4}$$

k being an integration constant.

With Eq. (1a), we can rewrite Eq. (4) as

$$x(z)=\int_0^z \frac{dz[k^2(1+\alpha z)^2-k^2]^{1/2}}{[(1-k^2)(1+\alpha z)^2+k^2]^{1/2}}\ , \tag{5}$$

wherein the constant c has been absorbed into k. As shown in detail in Ref. 2, there are three distinct minimal curves corresponding to $k^2\le 1$, and $k^2>1$. For $k^2<1$, the above can be put into the form

$$x(z)=\frac{\xi}{\alpha}\int_1^{1+\alpha z} du\frac{(u^2-1)^{1/2}}{(u^2+\xi^2)^{1/2}},\quad \xi^2(1-k^2)=k^2\ . \tag{6a}$$

Similarly, for $k^2=1$ we get

$$x(z)=\frac{1}{\alpha}\int_1^{1+\alpha z} du(u^2-1)^{1/2}\ . \tag{6b}$$

Since the motion of the particle is assumed to be two dimensional, the velocity vector $\mathbf{v}$ has two components and is written as $\mathbf{v}=\dot{x}\mathbf{i}+\dot{z}\mathbf{k}$, with $\mathbf{i},\mathbf{k}$ being unit vectors along x,z directions, respectively, and $\dot{z},\dot{x}$ denote derivatives with respect to time t. From (6a) and (6b) we obtain

$$k^2<1:\quad \dot{x}(z)=\xi\dot{z}\left[\frac{(1+\alpha z)^2-1}{(1+\alpha z)^2+\xi^2}\right]^{1/2}\ , \tag{7a}$$

$$k^2=1:\quad \dot{x}(z)=\dot{z}[(1+\alpha z)^2-1]^{1/2}\ . \tag{7b}$$

It is now easy to show that

$$k^2=1:\quad |\mathbf{v}|=\dot{z}(1+\alpha z)\ , \tag{8a}$$

$$k^2<1:\quad |\mathbf{v}|=\dot{z}(1+\alpha z)\left[\frac{1+\xi^2}{(1+\alpha z)^2+\xi^2}\right]^{1/2}\ . \tag{8b}$$

The corresponding formulas for the acceleration vector $\dot{\mathbf{v}}$ can also be written out. They will be displayed in Sec. III and we now turn to the expression for the power radiated $P(t)$ and $P(\mathbf{n},t)$.

II. THE INSTANTANEOUS POWER RADIATED $P(t)$

We find the treatment due to Schwinger[3] quite convenient for our calculations. Only a summary is given here. In terms of the radiation fields $A^\mu(x)$, the power radiated $P(t)$, by a charge distribution with current and charge density denoted, respectively, by $\mathbf{j}(\mathbf{x},t)$ and $\rho(\mathbf{x},t)$, is given by[5]

$$P(t)=\int d^3x\left[\frac{1}{c}\mathbf{j}\cdot\frac{\partial\mathbf{A}}{\partial t}-\rho\frac{\partial\phi}{\partial t}\right]\ . \tag{9}$$

The radiation fields $A^\mu(x)$ are given by

$$A^\mu(x)=\tfrac{1}{2}(A^\mu_{\text{ret}}-A^\mu_{\text{adv}}) \\ =\frac{1}{8\pi c}\int\frac{d^4x'}{|\mathbf{x}-\mathbf{x}'|}j^\mu(x')[\Theta(x_0-x_0')\delta(x_0-x_0'-|\mathbf{x}-\mathbf{x}'|)-\Theta(x_0'-x_0)\delta(x_0-x_0'+|\mathbf{x}-\mathbf{x}'|)]\ . \tag{10}$$

From (9) and (10), we get, after using the Fourier integral representation of the δ function.

$$P(t)=-\frac{1}{8\pi^2}\int d^3x d^4x'\left[\frac{1}{c}\mathbf{j}(x)\cdot\mathbf{j}(x')-\rho j_0(x')\right] \\ \times d\omega\frac{d^2}{d\lambda^2}\exp(-i\omega\lambda)\frac{d\Omega}{4\pi}\ , \tag{11}$$

where $\lambda=x_0-x_0'-\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')$ and $j_0=c\rho$. In deriving (11) we have used the result

$$\frac{\sin(\omega|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}=\omega\int\frac{d\Omega}{4\pi}\exp(i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}'))\ , \tag{12}$$

where $d\Omega$ is an element of solid angle associated with the direction of the unit vector $\mathbf{n}$.

For a point charge located at the variable position $\mathbf{R}(t)$, we use

$$j_i(\mathbf{x},t)=ev_i(t)\delta[\mathbf{x}-\mathbf{R}(t)]\ ,$$

$$j_0(\mathbf{x},t)=ec\delta[\mathbf{x}-\mathbf{R}(t)]\ ,$$

with $\mathbf{v}(t)=d\mathbf{R}(t)/dt$ to rewrite Eq. (11) as

$$P(t)=\int d\Omega\,P(\mathbf{n},t)\ ,$$

with

$$P(\mathbf{n},t)=\frac{e^2c^2}{16\pi^2}\int dt'\left[1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t')\right]\frac{d^2}{d\lambda^2}\delta(\lambda)\ . \tag{13}$$

Introducing the variable $\tau=t'-t$, it is easy to derive, following Schwinger,[3]

$$P(\mathbf{n},t)=\frac{e^2}{16\pi^2c}\phi(t)\left[\frac{d}{d\tau}\phi(t+\tau)\frac{d}{d\tau}\phi(t+\tau)\left[1-\frac{1}{c^2}\mathbf{v}(t)\cdot\mathbf{v}(t+\tau)\right]\right]_{\tau=0}\ , \tag{14}$$

with

$$\phi^{-1}(t) \equiv 1 - \frac{\mathbf{n}\cdot\mathbf{v}(t)}{c} .$$

There is an extra factor of $(4\pi)^{-1}$ in (14) because of our definition of the radiation fields in Eq. (10). Equation (14) can be recast into

$$P(\mathbf{n},t) = \frac{e^2}{16\pi^2 c^3}[\dot{\mathbf{v}}^2\phi^3(t) + 2(\mathbf{n}\cdot\dot{\mathbf{v}})\phi^4(t)(\mathbf{v}\cdot\dot{\mathbf{v}}/c) - (\mathbf{n}\cdot\dot{\mathbf{v}})^2(1-v^2/c^2)\phi^5(t)] . \tag{15}$$

Deleted in Eq. (13) is an extra term which is of the form of a total time derivative representing the unwanted acceleration energy terms. With the usual definition of the relativistic momentum and energy given by $\mathbf{p}=\gamma m\mathbf{v}$ and $E=\gamma mc^2$, Eq. (15) can be put into the familiar form

$$P(\mathbf{n},t) = \frac{e^2}{16\pi^2 m^2 c^3}\left[\frac{mc^2}{E}\right]^2 \phi^3(t)[\dot{\mathbf{p}}^2 - \dot{E}^2/c^2 - \left[\frac{mc^2}{E}\right]^2 \phi^2(t)(\mathbf{n}\cdot\dot{\mathbf{p}} - \dot{E}/c)^2] . \tag{16}$$

In the approximation of the motion being instantaneously circular, so that $\mathbf{v}\cdot\dot{\mathbf{v}}=0$ and hence $\dot{E}=0$, we obtain with the direction of $\mathbf{v}$ as the z axis

$$P(\mathbf{n},t) = \frac{e^2}{16\pi^2 m^2 c^3}\left[\frac{mc^2}{E}\right]^2 \frac{\dot{\mathbf{p}}^2}{\left[1-\frac{v}{c}\cos\theta\right]^3} \times \left[1 - \left[\frac{mc^2}{E}\right]^2 \frac{\sin^2\theta\cos^2\phi}{(1-\frac{v}{c}\cos\theta)^2}\right] . \tag{16a}$$

The integral of Eq. (15) over the solid angle of $d\Omega$ results in

$$P(t) = \frac{2}{3}\frac{e^2}{4\pi m^2 c^3}\left[\frac{E}{mc^2}\right]^2 \left[\dot{\mathbf{p}}^2 - \frac{1}{c^2}\dot{E}^2\right] , \tag{17}$$

which is the relativistically invariant version of Larmor's formula. In terms of the velocity and acceleration vectors $\mathbf{v}$ and $\dot{\mathbf{v}}$, respectively, Eq. (17), for example, can be reworked as

$$P(t) = \frac{2}{3}\frac{e^2}{4\pi c^3}\left[\frac{E}{mc^2}\right]^6 \left[\dot{\mathbf{v}}^2 - \left[\frac{\mathbf{v}}{c}\times\dot{\mathbf{v}}\right]^2\right] . \tag{18}$$

We use Eq. (18) extensively below with the expressions for $\mathbf{v}$ and $\dot{\mathbf{v}}$ derived from Sec. I.

III. RESULTS FOR THE BRACHISTOCHRONE CASE

From Eq. (17) of Sec. II, it is clear that the power radiated depends on $(\dot{\mathbf{p}}^2 - 1/\dot{E}^2/c^2)$, so that if these quantities can be computed from kinematical considerations as, for example, in the case of circular orbits, the power radiated is known easily. We discuss this below.

The acceleration vector $\dot{\mathbf{v}} = \ddot{x}\mathbf{i} + \ddot{z}\mathbf{k}$, so that using Eq. (7b), we can write for the case when $k^2=1$,

$$\dot{\mathbf{v}} = \ddot{z}\mathbf{k} + \left[\ddot{z}[(1+\alpha z)^2-1]^{1/2} + \frac{\alpha\dot{z}^2(1+\alpha z)}{[(1+\alpha z)^2-1]^{1/2}}\right]\mathbf{i} . \tag{19}$$

Using Eqs. (1a) and (7a), we arrive at

$$|\dot{\mathbf{v}}(t)| = \frac{\alpha c^2}{(1+\alpha z)^3} = \alpha c^2(1-v^2/c^2)^{3/2} . \tag{20}$$

We thus see that the instantaneous acceleration is a decreasing function of $v(t)$; of course, $v(t)$ increases as z increases [see (1a)]. The acceleration vector for $k^2<1$ is easily worked out as

$$\dot{\mathbf{v}} = \ddot{z}\mathbf{k} + \left[\xi\ddot{z}\left[\frac{(1+\alpha z)^2-1}{(1+\alpha z)^2+\xi^2}\right]^{1/2} + \frac{\xi\alpha\dot{z}^2(1+\alpha z)(1+\xi^2)}{[(1+\alpha z)^2-1]^{1/2}[(1+\alpha z)^2+\xi^2]^{3/2}}\right]\mathbf{i} , \tag{19'}$$

Again it is easy to convince oneself that despite the difference in Eq. (19) and (19'), the modulus $|\dot{\mathbf{v}}|$ from (19') still works out to that given in Eq. (20). The magnitude of the acceleration vector is thus k^2 independent.

We also record below a few other formulas that are useful.

$$k^2=1: \quad \left|\frac{\mathbf{v}}{c}\times\frac{d\mathbf{v}}{dt}\right| = \alpha\dot{z}^3 v^{-1} , \tag{21a}$$

$$k^2<1: \quad \left|\frac{\mathbf{v}}{c}\times\frac{d\mathbf{v}}{dt}\right| = \frac{\alpha\dot{z}^3\xi(1+\xi^2)}{v[(1+\alpha z)^2+\xi^2]^{3/2}} . \tag{21b}$$

With Eqs. (19) and (21a), the power radiated can be easily arrived at for $k^2=1$, from Eq. (18). We obtain

$$P(t) = \frac{2}{3}\frac{e^2\alpha^2}{4\pi}c\left[1-\frac{v^2(t)}{c^2}\right]\left[1+\frac{v^2(t)}{c^2}\right] . \tag{22}$$

It is thus clear that as $v(t)$ increases towards c the power radiated $P(t)$ would decrease. Of course, central to this result is Eq. (20) which shows that the acceleration is a decreasing function of $v(t)$. Equation (22) is true for a general orbit; however, for the case where the orbit is instantaneously circular, so that $\mathbf{v}\cdot\dot{\mathbf{v}}=0$, it is easy to show that (since $\dot{E}=0$)

$$P(t) = \frac{2}{3}\frac{e^2\alpha^2}{4\pi}c\left[1-\frac{v^2(t)}{c^2}\right] . \tag{22a}$$

The corresponding formulas for $k^2<1$ are also easily derived from Eqs. (19') and (21b). We obtain for a general orbit, i.e., $\mathbf{v}\cdot\dot{\mathbf{v}}\neq 0$,

$$P(t)=\frac{2}{3}\frac{e^2\alpha^2}{4\pi}c\left[1-\frac{v^2}{c^2}\right]\left[\frac{\xi^2}{\xi^2+1}\frac{v^2}{c^2}+\frac{(1+\alpha z)^2+\xi^2}{1+\xi^2}\right]. \tag{22b}$$

Clearly, as $\xi\to\infty$ ($k^2\to 1$), we recover $P(t)$ given by Eq. (22). For the case with $k^2>1$, the relevant formulas are obtained through the replacements $(\xi^2+1)\Longrightarrow(1-\xi^2)$ and $[\xi^2+(1+\alpha z)^2]\Longrightarrow[(1+\alpha z)^2-\xi^2]$. Equation (22b) also shows the characteristic decrease of $P(t)$ as $v(t)$ increases.

A word about $P(\mathbf{n},t)$ before we conclude this section. For the two-dimensional motion that we have considered here, one can put $\phi=0$ in Eq. (16). The maximum value of $P(\mathbf{n},t)$ for the case of instantaneously circular motion can then be worked out. Since the acceleration $|\dot{\mathbf{v}}|$ is the same for $k^2=1$ and $k^2<1$ (hence $k^2>1$), $P(\mathbf{n},t)$ as given by Eq. (16) will be the same for all these cases. It is easy to see that $\theta=0$ is the favored angle for a maximum of $P(\mathbf{n},t)$. It is also clear that the acceleration and velocity vector are aligned such that

$$k^2=1:\quad \mathbf{v}\cdot\dot{\mathbf{v}}=c^2v\alpha(1-v^2/c^2)^2\ , \tag{23a}$$

$$k^2<1:\quad \mathbf{v}\cdot\dot{\mathbf{v}}=c^2v\alpha\left[\frac{(1+\alpha z)^2+\xi^2}{1+\xi^2}\right]^{1/2}(1-v^2/c^2)^2\ . \tag{23b}$$

Thus as $v\to c$, the motion of the charge becomes increasingly similar to the instantaneously circular path. The remarks made above for $P(\mathbf{n},t)$ are therefore of relevance in this limit ($v\to c$).

IV. CONCLUSION

We have examined in this paper the power radiated by a relativistic, charged particle as it is accelerated along a brachistochrone. It is seen that the velocity $|\mathbf{v}|$ and acceleration $|\dot{\mathbf{v}}|$ are independent of the nature of the brachistochrone. The instantaneous power radiated in each case decreases with an increase in the velocity $v(t)$, and this is quite unlike the behavior described in the literature.[1,3]

[1]See J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), pp. 661 and 662.

[2]H. F. Goldstein and C. M. Bender, J. Math. Phys. **27**, 507 (1986).

[3]J. Schwinger, Phys. Rev. **75**, 1912 (1949).

[4]Goldstein and Bender (Ref. 2) also discuss a fourth, namely, the proper-time brachistochrone in their paper. We have ignored it here.

[5]As noted by Schwinger (Ref. 3), there are unwanted acceleration energy terms in Eq. (9). These will be discarded, as indicated in Eq. (15).