Squeezed states and quantum-mechanical parametric amplification

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The relation of a previous paper on the parametric amplification of a quantum oscillator to squeezed states is described. In particular, we show that in general the amplification factor is also the "squeezing factor" of the final state.

I. INTRODUCTION AND REVIEW

Some time ago, the quantum mechanics of a simple harmonic oscillator with an arbitrary time-dependent spring constant was worked out.[1] In general, this system is a simple model of a parametric amplifier, and in particular it may have some relevance to the g-2 experiment performed at the University of Washington.[2] The present paper is an addendum to the previous paper,[1] whose purpose is to explain how the quantum-mechanical parametric amplification produces a "squeezed state."[3] Although we shall now give a brief review, this paper is not self-contained, and Ref. 1 should be consulted for further details.

The system under consideration is described by the equation of motion

$$\frac{d^2q(t)}{dt^2}+\omega(t)^2q(t)=0 \ , \tag{1.1}$$

where the time-dependent angular frequency $\omega(t)$ becomes constant at remote times,

$$t<-T: \ \ \omega(t)=\omega_- \ , \tag{1.2a}$$

$$t>+T: \ \ \omega(t)=\omega_+ \ , \tag{1.2b}$$

with, in general, different initial and final frequencies ω_- and ω_+. Equation (1.1) is formally akin to the Schrödinger equation for a one-dimensional scattering problem. It proves convenient to exploit this analogy. First one introduces the comparison function $\phi(t)$ which is the WKB approximation to a solution of Eq. (1.1), namely,

$$\phi(t)=[2\omega(t)]^{-1/2}\exp[i\Omega(t)] \ , \tag{1.3}$$

in which

$$\Omega(t)=\int_0^t dt'\omega(t') \ . \tag{1.4}$$

Then two solutions $\psi_\pm(t)$ of Eq. (1.1) are defined by the following boundary conditions:

$$\psi_-(t)=\begin{cases}\phi^*(t)+S_{--}\phi(t), & t\to-\infty\\ S_{+-}\phi^*(t), & t\to+\infty,\end{cases} \tag{1.5a}$$

and

$$\psi_+(t)=\begin{cases}S_{-+}\phi(t), & t\to-\infty\\ \phi(t)+S_{++}\phi^*(t), & t\to+\infty .\end{cases} \tag{1.5b}$$

This procedure defines a "scattering matrix"

$$S=\begin{bmatrix}S_{++} & S_{+-}\\ S_{-+} & S_{--}\end{bmatrix} , \tag{1.6}$$

which is unitary

$$S^\dagger S=1 \tag{1.7}$$

and symmetrical

$$S_{+-}=S_{-+} \ . \tag{1.8}$$

These restrictions imply that the scattering matrix S may be parametrized with hyperbolic functions such that

$$|S_{--}|=|S_{++}|=\tanh\chi \tag{1.9a}$$

and

$$|S_{+-}|=(\cosh\chi)^{-1} \ . \tag{1.9b}$$

The usual methods of scattering theory may be used to construct the S matrix. With this matrix in hand, the asymptotic behavior of classical oscillator coordinate $q_{\rm cl}(t)$ is determined. For example, given the initial energy E^i and phase ϕ^i of the oscillator, the final energy is determined. This final energy depends sensitively on the initial phase. For example, with an optimum choice of ϕ^i one has the maximum amplification with the final energy given by[1]

$$E^f_{\rm max}=e^{2\chi}(\omega_+/\omega_-)E^i \ . \tag{1.10a}$$

On the other hand, changing this optimal phase by $\pi/2$ gives a maximal *de*amplification, and produces the final energy

$$E^f_{\rm min}=e^{-2\chi}(\omega_+/\omega_-)E^i \ . \tag{1.10b}$$

The quantum problem is also solved with the knowledge of the S matrix. To exhibit this solution, we define the Heisenberg-picture annihilation operator

$$a(t)=\left[\frac{\omega(t)}{2}\right]^{1/2}q(t)+i\left[\frac{1}{2\omega(t)}\right]^{1/2}p(t) \ , \tag{1.11}$$

with the creation operator $a^\dagger(t)$ given by the adjoint equation[4] that simply replaces $i\to-i$. Initial and final ground states are defined by

$$a(t_1)\,|\,0t_1\rangle=0 \ , \tag{1.12a}$$

and

$$\langle 0t_2 | a^\dagger(t_2) = 0 \ . \tag{1.12b}$$

Coherent or classical limit states are built from these according to

$$| zt_1 \rangle = \exp[z a^\dagger(t_1)] \, | 0t_1 \rangle \ , \tag{1.13a}$$

and

$$\langle z^* t_2 | = \langle 0t_2 | \exp[z^* a(t_2)] \ . \tag{1.13b}$$

They are eigenstates of the annihilation and creation operators,

$$a(t_1) | zt_1 \rangle = | zt_1 \rangle z \tag{1.14a}$$

and

$$\langle z^* t_2 | a^\dagger(t_2) = z^* \langle z^* t_2 | \ . \tag{1.14b}$$

For future reference, we note that $a(t_2)$ is represented by the derivative with respect to z^*,

$$\frac{\partial}{\partial z^*} \langle z^* t_2 | = \langle z^* t_2 | a(t_2) \ . \tag{1.15}$$

As shown in Ref. 1, the transformation function for this pair of coherent states, for $t_2 > T$ and $t_1 < -T$ so that the initial and final oscillators are not perturbed [with $\omega(t_2) = \omega_+$ and $\omega(t_1) = \omega_-$], is given by

$$\langle z_2^* t_2 | z_1 t_1 \rangle = (S_{+-})^{1/2} \exp\left[-\frac{i}{2} \int_{t_1}^{t_2} dt \, \omega(t) \right] \times \exp(\tfrac{1}{2} Z^T S Z) \ . \tag{1.16}$$

Here

$$Z^T S Z = S_{++} \exp[-2i\Omega(t_2)] z_2^{*2} + S_{--} \exp[2i\Omega(t_1)] z_1^2 + 2S_{+-} \exp\{-i[\Omega(t_2) - \Omega(t_1)]\} z_2^* z_1 \ . \tag{1.17}$$

Since the coherent states provide a generating function that gives any state, this result provides a complete solution to the quantum problem as discussed at length in Ref. 1. In Sec. II we review the notion of squeezed states and then show in Sec. III how the quantum-mechanical parametric amplification produces such a state.

II. SQUEEZED VERSUS COHERENT STATES

The canonically conjugate, quantum-mechanical coordinate q and momentum p operators obey the commutation relation

$$[q, p] = i \ . \tag{2.1}$$

It follows from this commutator that the squared fluctuations Δq^2 and Δp^2 obey the uncertainty relation

$$\Delta q^2 \Delta p^2 \geq \tfrac{1}{4} \ . \tag{2.2}$$

As is well known,[5] the minimum uncertainty $\Delta q \Delta p = \frac{1}{2}$ is achieved for a family of states $\{ | S > \}$ defined by

$$A_\lambda | S \rangle = | S \rangle z_\lambda \ , \tag{2.3}$$

where

$$A_\lambda = \left[\frac{\lambda}{2} \right]^{1/2} q + i \left[\frac{1}{2\lambda} \right]^{1/2} p \ , \tag{2.4}$$

and z_λ is a complex eigenvalue of this non-Hermitian operator. The arbitrary number λ parametrizes a particular set of states in this family which are then distinguished by the eigenvalue z_λ that runs over all complex numbers. This definition is just that of the coherent or classical limit states of an harmonic oscillator of frequency $\omega = \lambda$ (with mass $m = 1$), and A_λ is the usual destruction operator for this oscillator. However, as we shall soon see, if the arbitrary parameter λ is chosen to differ from the frequency ω of the oscillator under consideration, then the states $| S \rangle$ are squeezed states for this oscillator.

The detailed properties of the general squeezed states are easily worked out. Using the usual notation for the normalized expectation value

$$\langle X \rangle = \frac{\langle S | X | S \rangle}{\langle S | S \rangle} \ , \tag{2.5}$$

it follows immediately from Eq. (2.3) that

$$z_\lambda = \langle A_\lambda \rangle = \left[\frac{\lambda}{2} \right]^{1/2} \langle q \rangle + i \left[\frac{1}{2\lambda} \right]^{1/2} \langle p \rangle \ . \tag{2.6}$$

The squared fluctuations are defined by

$$\Delta q^2 = \langle (q - \langle q \rangle)^2 \rangle \ , \tag{2.7a}$$

and

$$\Delta p^2 = \langle (p - \langle p \rangle)^2 \rangle \ . \tag{2.7b}$$

Using the canonical commutation relation (2.1) together with the definition (2.3) of the squeezed states, it is a simple matter to verify that

$$0 = \langle (A_\lambda^\dagger - z_\lambda^*)(A_\lambda - z_\lambda) \rangle = \frac{\lambda}{2} \Delta q^2 + \frac{1}{2\lambda} \Delta p^2 - \frac{1}{2} \ . \tag{2.8}$$

Similarly, one computes

$$0 = \langle (A_\lambda - z_\lambda)^2 \rangle = \frac{\lambda}{2} \Delta q^2 - \frac{1}{2\lambda} \Delta p^2 + i \langle \{ q - \langle q \rangle, p - \langle p \rangle \} \rangle \ , \tag{2.9}$$

where the curly brackets denote the anticommutator,

$$\{ X, Y \} = XY + YX \ . \tag{2.10}$$

The complex conjugate of Eq. (2.9) yields

$$0 = \frac{\lambda}{2} \Delta q^2 - \frac{1}{2\lambda} \Delta p^2 - i \langle \{ q - \langle q \rangle, p - \langle p \rangle \} \rangle \tag{2.11}$$

which, added to Eq. (2.9), informs us that

$$\Delta q^2 = \frac{1}{\lambda^2} \Delta p^2 \ . \tag{2.12}$$

Comparing this result with Eq. (2.8), we learn that

$$\Delta p^2 = \frac{\lambda}{2} \tag{2.13a}$$

and

$$\Delta q^2=\frac{1}{2\lambda}\ , \tag{2.13b}$$

confirming that the squeezed state is indeed a state of minimum uncertainty,

$$\Delta q^2 \Delta p^2=\tfrac{1}{4}\ . \tag{2.13c}$$

We should also note that the difference of Eqs. (2.9) and (2.11) tells us that the coordinate and momentum fluctuations are uncorrelated in the squeezed state:

$$\langle \{(q-\langle q\rangle),(p-\langle p\rangle)\}\rangle=0\ . \tag{2.14}$$

Suppose now that the dynamics of the system is that of an harmonic oscillator of frequency ω. Since the equation of motion is linear, the quantum Heisenberg operators satisfy the classical equations,

$$q(t)=q\cos(\omega t)+(p/\omega)\sin(\omega t)\ , \tag{2.15a}$$

and

$$p(t)=p\cos(\omega t)-q\omega\sin(\omega t)\ . \tag{2.15b}$$

Here $q=q(0)$ and $p=p(0)$ are the operators at the conventional initial time $t=0$, the time we shall assume is referred to in the states $|S\rangle$. Thus the time development of the expectation values $\langle q(t)\rangle$ and $\langle p(t)\rangle$ are given by equations identical to Eqs. (2.15). Moreover, using the results of the previous paragraph, it is easy to check that

$$\begin{aligned}\Delta q(t)^2&=\langle [q(t)-\langle q(t)\rangle]^2\rangle\\ &=\Delta q^2\cos^2(\omega t)+\frac{\Delta p^2}{\omega^2}\sin^2(\omega t)\\ &=\frac{1}{2\lambda}\left[\cos^2(\omega t)+\frac{\lambda^2}{\omega^2}\sin^2(\omega t)\right]\ ,\end{aligned} \tag{2.16a}$$

and

$$\begin{aligned}\Delta p(t)^2&=\langle [p(t)-\langle p(t)\rangle]^2\rangle\\ &=\Delta p^2\cos^2(\omega t)+\Delta q^2\omega^2\sin^2(\omega t)\\ &=\frac{\lambda}{2}\left[\cos^2(\omega t)+\frac{\omega^2}{\lambda^2}\sin^2(\omega t)\right]\ .\end{aligned} \tag{2.16b}$$

If $\lambda=\omega$ so that $|S\rangle$ is the coherent state for the oscillator, then $\Delta q(t)^2=\Delta q^2=1/(2\omega)$ and $\Delta p(t)^2=\Delta p^2=\omega/2$ are both constant in time, and the uncertainty is always minimal, $\Delta q(t)\Delta p(t)=\frac{1}{2}$. On the other hand, if $\lambda>\omega$, $\Delta q(t)$ becomes small at the times $\omega t=0$ (mod π) and $\Delta p(t)$ becomes small at the time $\omega t=\pi/2$ (mod π), while for $\lambda<\omega$ the reverse holds. Note that in general

$$\Delta q(t)^2\Delta p(t)^2=\frac{1}{4}\left[1+\frac{1}{2}\left[\frac{\lambda}{\omega}-\frac{\omega}{\lambda}\right]^2\sin^2(2\omega t)\right]\ , \tag{2.17}$$

so that the uncertainty is minimal for the oscillator starting out in a squeezed state only at the times $\omega t=0$ (mod $\pi/2$).

III. FINAL SQUEEZED STATE

To examine the nature of the final quantum state produced in the parametric amplification process, it is convenient to write the transformation function as a Schrödinger-picture wave function

$$\langle z_2^* t_2\,|\,z_1 t_1\rangle=\langle z_2^*\,|S\rangle\ . \tag{3.1}$$

Thus $|S\rangle$ is the state that is observed by measurements performed at the final time t_2. Correspondingly, we have

$$z^*\langle z^* t_2\,|\,z_1 t_1\rangle=\langle z^*\,|a^\dagger|S\rangle\ , \tag{3.2a}$$

and

$$\frac{\partial}{\partial z^*}\langle z^* t_2\,|\,z_1 t_1\rangle=\langle z^*\,|a|S\rangle\ , \tag{3.2b}$$

where here $a^\dagger=a(t_2)$ and $a=a(t_2)$. Now, according to the results (1.16) and (1.17) for the transformation function,

$$\left[\frac{\partial}{\partial z_2^*}-S_{++}e^{-2i\Omega(t_2)}z_2^*\right]\langle z_2^* t_2\,|\,z_1 t_1\rangle =S_{+-}e^{-i[\Omega(t_2)-\Omega(t_1)]}z_1\langle z_2^* t_2\,|\,z_1 t_1\rangle\ . \tag{3.3}$$

Therefore

$$(a-S_{++}e^{-2i\Omega(t_2)}a^\dagger)\,|S\rangle =|S\rangle S_{+-}e^{-i[\Omega(t_2)-\Omega(t_1)]}z_1\ . \tag{3.4}$$

So long as $t_2>T$, where T is the time at which $\omega(t)$ has ceased to vary, and $\omega(t)=\omega_+$, the time t_2 may be chosen arbitrarily, for the time development for the later times $t_2>T$ is a simple harmonic motion with frequency ω_+. We shall soon see that choosing t_2 appropriately makes $|S\rangle$ a squeezed state with a minimum uncertainty $\Delta q\Delta p=\frac{1}{2}$. As we just saw in Sec. II [Eq. (2.17)], a squeezed state is a minimum uncertainty state only at certain specific times. It is convenient to define the state at one of these times and then describe the system at other times by the Heisenberg operators $q(t),p(t)$ referred to that standard time. Thus we choose t_2 such that

$$S_{++}e^{-2i\Omega(t_2)}=-|S_{++}|\ , \tag{3.5}$$

such that the time-dependent phase cancels the phase of S_{++}. Thus we have

$$a+|S_{++}|a^\dagger\equiv(\cosh\chi)^{-1}A_\lambda\ . \tag{3.6}$$

Recalling Eqs. (1.9a) and (1.11), we see that this defines

$$A_\lambda=e^{+\chi}\left[\frac{\omega_+}{2}\right]^{1/2}q+ie^{-\chi}\left[\frac{1}{2\omega_+}\right]^{1/2}p\ , \tag{3.7}$$

which is just the annihilation operator for a harmonic oscillator with frequency

$$\lambda=e^{2\chi}\omega_+\ . \tag{3.8}$$

Since, in view of Eq. (1.9b),

$$S_{+-}=(\cosh\chi)^{-1}e^{i\alpha}\ , \tag{3.9}$$

we now have

$$A_\lambda\,|S\rangle=|S\rangle z_\lambda\ , \tag{3.10}$$

in which

$$z_\lambda = e^{i[\alpha-\Omega(t_2)+\Omega(t_1)]} z_1$$
$$\equiv e^{i\theta} |z_1| \; . \tag{3.11}$$

Therefore $|S\rangle$ is indeed a squeezed state: An initial classical limit, coherent state $|z_1 t_1\rangle$ is changed into a squeezed state by any arbitrary parametric amplification process with a time-dependent angular frequency $\omega(t)$.

At our selected time t_2, the squeezed state gives the coordinate and momentum expectation values

$$\langle q\rangle = e^{-\chi}\left[\frac{2}{\omega_+}\right]^{1/2} |z_1| \cos\theta \; , \tag{3.12a}$$

and

$$\langle p\rangle = e^{\chi}(2\omega_+)^{1/2} |z_1| \sin\theta \; . \tag{3.12b}$$

The corresponding uncertainties are given by

$$\Delta q^2 = \frac{1}{2\omega_+} e^{-2\chi} \; , \tag{3.13a}$$

and

$$\Delta p^2 = \frac{\omega_+}{2} e^{2\chi} \; . \tag{3.13b}$$

Thus the parametric amplification has "squeezed" the uncertainty in Δq by the factor $e^{-\chi}$ relative to the uncertainty in Δq produced by a coherent, classical limit state for an oscillator of frequency ω_+. At the same time the corresponding uncertainty in Δp is increased by the factor e^{χ}. According to Eqs. (2.16), if we increase the time t_2 with $\omega_+ t_2 \rightarrow \omega_+ t_2 + \pi/2 \pmod{\pi}$, then instead Δp is squeezed by the factor $e^{-\chi}$ and Δq increased by e^{χ}.

The final energy is given by

$$\begin{aligned}\langle H\rangle^f &= \langle \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_+^2 q^2\rangle \\ &= \tfrac{1}{2}(\langle p\rangle^2 + \Delta p^2) + \tfrac{1}{2}\omega_+^2(\langle q\rangle^2 + \Delta q^2) \\ &= \omega_+[e^{2\chi}(|z_1|^2 \sin^2\theta + \tfrac{1}{4}) \\ &\quad + e^{-2\chi}(|z_1|^2\cos^2\theta + \tfrac{1}{4})] \; .\end{aligned} \tag{3.14}$$

On the other hand, the energy of the initial coherent state is given by

$$\langle H\rangle^i = \omega_-(|z_1|^2 + \tfrac{1}{2}) \; . \tag{3.15}$$

Hence

$$\begin{aligned}\langle H\rangle^f &= (\omega_+/\omega_-)(e^{2\chi}\sin^2\theta + e^{-2\chi}\cos^2\theta)\langle H\rangle^i \\ &\quad + \tfrac{1}{2}\omega_+ \sinh(2\chi)\cos(2\theta) \; .\end{aligned} \tag{3.16}$$

Thus, if the phase of the initial oscillation is chosen so as to make $\theta = \pi/2$, there is a maximal amplification, with

$$\langle H\rangle^f_{\max} = (\omega_+/\omega_-)e^{2\chi}\langle H\rangle^i - \tfrac{1}{2}\omega_+ \sinh\chi \; , \tag{3.17a}$$

while choosing $\theta = 0$ gives a maximum deamplification,

$$\langle H\rangle^f_{\min} = (\omega_+/\omega_-)e^{-2\chi}\langle H\rangle^i + \tfrac{1}{2}\omega_+ \sinh(2\chi) \; . \tag{3.17b}$$

Except for the zero-point energy additions, these are just the classical results displayed in Eqs. (1.10). One sees that the energy amplification factor is just the squeezing factor.

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[1] L. S. Brown and L. J. Carson, Phys. Rev. A **20**, 2486 (1979).

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[3] Squeezed states are reviewed by D. F. Walls, Nature **306**, 141 (1983).

[4] In terms of these annihilation and creation operators, the time evolution of the oscillator is governed by the Hamiltonian

$$H(t) = \omega(t)[a^\dagger(t)a(t) + \tfrac{1}{2}] \; .$$

On the other hand, one could work with operators defined in terms of the initial "spring constant" with

$$a_-(t) = \left[\frac{\omega_-}{2}\right]^{1/2} q(t) + i\left[\frac{1}{2\omega_-}\right]^{1/2} p(t) \; ,$$

and its adjoint $a_-^\dagger(t)$. They put the Hamiltonian in the form

$$H(t) = \frac{\omega^2(t)+\omega_-^2}{2\omega_-} a_-^\dagger(t)a_-(t) + \frac{\omega^2(t)-\omega_-^2}{4\omega_-}[a_-(t)^2 + a_-^\dagger(t)^2] + \text{const} \; .$$

This is of the general form of the Hamiltonian used in the earlier work on squeezed or "two-photon" states by H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[5] See, for example, the exposition in Sec. 24 of K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966).