Dynamical behavior of stochastic systems of infinitely many coupled nonlinear oscillators exhibiting phase transitions of mean-field type: H theorem on asymptotic approach to equilibrium and critical slowing down of order-parameter fluctuations

Masatoshi Shiino

Department of Applied Physics, Faculty of Science, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152, Japan

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It is shown that statistical-mechanical properties as well as irreversible phenomena of stochastic systems, which consist of infinitely many coupled nonlinear oscillators and are capable of exhibiting phase transitions of mean-field type, can be successfully explored on the basis of nonlinear Fokker-Planck equations, which are essentially nonlinear in unknown distribution functions. Results of two kinds of approaches to the study of their dynamical behavior are presented. Firstly, a problem of asymptotic approaches to stationary states of the infinite systems is treated. A method of Lyapunov functional is employed to conduct a global as well as a local stability analysis of the systems. By constructing an H functional for the nonlinear Fokker-Planck equation, an H theorem is proved, ensuring that the Helmholtz free energy for a nonequilibrium state of the system decreases monotonically until a stationary state is approached. Calculations of the second-order variation of the H functional around a stationary state yield a stability criterion for bifurcating solutions of the nonlinear Fokker-Planck equation, in terms of an inequality involving the second moment of the stationary distribution function. Secondly, the behavior of critical dynamics is studied within the framework of linear-response theory. Generalized dynamical susceptibilities are calculated rigorously from linear responses of the order parameter to externally driven fields by linearizing the nonlinear Fokker-Planck equation. Correlation functions, together with spectra of the fluctuations of the order parameter of the system, are also obtained by use of the fluctuation-dissipation theorem for stochastic systems. A critical slowing down is shown to occur in the form of the divergence of relaxation time for the fluctuations, in accordance with the divergence of the static susceptibility, as a phase transition point is approached.

I. INTRODUCTION

The study of dynamical behavior of systems exhibiting thermodynamic phase transitions has been of considerable interest for many years.[1-3] It is well known that in a thermodynamic system undergoing phase transitions critical anomaly such as critical slowing down is generally expected to occur at its transition points. Recently the concept of phase transition has been extended to include nonthermodynamic or nonequilibrium phase transitions.[4-6] A problem arises of how the dynamical behavior of nonequilibrium phase transitions compares with the one for phase transitions in thermodynamic systems. To discuss this sort of problem, stochastic approaches or models have been extensively employed,[5,7,8] since such stochastic methods as using Langevin equation models are often considered to be capable of simulating the dynamical behavior of phase transitions both in thermodynamic and in nonequilibrium systems.

In particular, the Langevin equation of the form[9(a)]

$$\dot{x}=\gamma x-x^3+f(t)\,,$$
$$\langle f(t)f(t')\rangle=\sigma^2\delta(t-t') \tag{1.1}$$

has been one of the most popular models used to discuss the dynamical behavior of systems undergoing phase transitions involving symmetry-breaking instabilities with change in certain control parameters as expressed by γ in Eq. (1.1). We must however be cautious of the use of this equation. Although the ordinary differential equation obtained by omitting the random force $f(t)$ in Eq. (1.1) can exhibit a bifurcation at $\gamma=0$, the stochastic differential equation (1.1) has nothing to do with bifurcations nor phase transitions in that a stationary distribution for the random variable x is always uniquely determined irrespective of the values of γ and σ. This is because the corresponding linear Fokker-Planck equation

$$\frac{\partial}{\partial t}p(t,x)=-\frac{\partial}{\partial x}(\gamma x-x^3)p(t,x)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p(t,x) \tag{1.2}$$

is confirmed to have the property of global stability with respect to its uniquely determined stationary solution, as will be noted later. Thus, one cannot expect any critical divergence at $\gamma=0$ for such physical quantities as the variance or relaxation time for the variable x in Eq. (1.1). In fact, the finiteness of the variance is easily checked with use of the stationary solution of Eq. (1.2) and the absence of critical divergence of the relaxation time was shown through the investigation of the eigenvalues of Eq. (1.2).[9(b)] In view of the fact that changes in the shape of the stationary distribution for Eq. (1.2) surely occur at $\gamma=0$, however, the term "phase transition" seems to have been extensively used in a somewhat wider sense to describe a qualitative change in most probable values as well as shapes of a distribution function accompanied by changes in control parameters, in the study of symmetry-breaking instabilities observed in far-from-equilibrium systems. On the other hand, we want to be concerned with genuine phase transitions involving stability exchanges. It may be worth noting that the concept of phase transition makes sense for a system with infinitely many degrees of freedom and that the occurrence of phase transition is associated with bifurcations of solutions to underlying equations of the system which are in most cases nonlinear. In fact, as is well known, thermodynamic phase transitions in thermodynamic systems can occur only under the condition usually referred to as the thermodynamic limit. Thus, the dynamical critical behavior must be studied on a carefully selected model capable of exhibiting genuine phase transitions caused by the presence of nonlinearities and by taking the thermodynamic limit.

Recently, stochastic systems described by a set of infinitely many coupled Langevin equations of the form (1.1), which can exhibit genuine phase transitions,[10,11] have been investigated rigorously to see the structure of the phase transitions by employing analyses of a nonlinear Fokker-Planck equation, which is nonlinear in an unknown probability distribution function.[12,13] The models have dealt with a system of coupled nonlinear oscillators in the presence of external white noise, which was originally introduced by Kometani and Shimizu[14] to study such self-organization processes in biological science as muscle contraction. As shown below, a nonlinear Fokker-Planck equation with its drift term characterized by a certain type of potential, together with mean-field interaction, can exhibit bifurcations of solutions leading to the existence of phase transitions of an infinitely many particle system. The present author has succeeded in obtaining a criterion for the determination of the stability of the bifurcating solutions of a nonlinear Fokker-Planck equation on the basis of analyses of the second-order variation of an H functional constructed for the nonlinear Fokker-Planck equation.[15] Our studies have shown usefulness of the nonlinear Fokker-Planck equation approach to a mean-field treatment of nonthermodynamic or nonequilibrium phase transitions in synergetical systems. It is worth noting that the nonlinear Fokker-Planck equation, which is termed as such due to the presence of the first moment in its drift term and is a kind of nonlinear diffusion equation,[16(a)] belongs to the concept of a nonlinear Markov diffusion process (in the sense of Mckean)[16(b)] and in general no longer ensures the ergodicity of the process. It is hence on the same level as a nonlinear master equation,[16(c)] which was developed by Malek Mansour and Nicolis to study the onset of instabilities of local fluctuations in a reacting medium in a mean-field treatment.

It is very interesting to investigate the dynamical behavior of such stochastic systems of infinitely many particles described by nonlinear Fokker-Planck equations. The dynamical behavior of a system, in general, can be studied in two different kinds of ways. Firstly, one is supposed to deal with a relaxation process to a stationary (or equilibrium) state of the system from a nonequilibrium thermodynamical point of view. A second approach is to observe a dynamical response of the system to a time-dependent external perturbation. As is well known, the two approaches are in some respects interrelated with each other in a linear regime, that is, in the case of sufficiently small perturbation for linear-response theory to be applied.

The problem of an irreversible approach to an equilibrium state constitutes a realm of nonequilibrium thermodynamics or statistical mechanics and has attracted considerable attention from many chemists and physicists. In the nonequilibrium statistical mechanics, the concept of entropy plays a very significant role, as seen in the case of Boltzmann's H theorem or the law of entropy production rate minimum, which was established by Glansdorff and Prigogine[17] as a stability criterion for time evolutions of nonequilibrium states. A similar idea was developed for the study of an asymptotic approach to an equilibrium state for Fokker-Planck dynamics by Green[18] and Graham and Haken,[19] who constructed an H functional and proved an H theorem to ensure a convergence of time-dependent solutions to a uniquely determined equilibrium solution. As far as we know, any rigorous treatment of an asymptotic approach to equilibrium in terms of such thermodynamical languages as free energy and entropy production cannot be found for systems allowed to undergo phase transitions. In a previous paper[15] we presented a brief report on the validity of an H theorem even for infinite systems exhibiting phase transitions. The present paper is intended to give a detailed description of the H theorem and stability analysis for nonlinear Fokker-Planck equations giving rise to bifurcations corresponding to mean-field-type phase transitions in the system of infinitely many coupled nonlinear oscillators.

We are also concerned in the present paper with dynamical responses of the system to external time-dependent perturbations near the phase-transition points along with the fluctuational behavior within the framework of linear-response theory and the fluctuation-dissipation theorem. Though linear-response theory for finite dimensional Fokker-Planck dynamics has been discussed by many authors,[20–22] the theory for infinite dimensional Fokker-Planck equations or for nonlinear Fokker-Planck equations seems to be lacking. In particular, the problem of how critical slowing down for the order parameter is described is our primary concern. Dawson[12] investigated critical fluctuations for a stochastic infinite system associated with nonlinear Fokker-Planck equations from a point of view of the central-limit theorem. We study the dynamical behavior of fluctuations and critical slowing down of the order parameter in an infinite system from a different point of view. Our present approach, which is based on linear-response theory, may be compared with the method used in the study of critical dynamics of mean-field Ising spin systems.[23] To obtain an idea of critical slowing down of the order-parameter fluctuations in ferromagnetic materials, a mean-field treatment of the Glauber model[24] has been conveniently used to simulate the spin dynamics. Such a simple analysis results in a single-exponential-type relaxation for the order-parameter fluctuations with a relaxation time diverging at the phase-transition point. In our present treatment, based on nonlinear Fokker-Planck equations, non-single-exponential-type relaxation of fluctuations has been obtained and an appropriately defined relaxation time has been shown to diverge as the transition point is approached.

The present paper is organized as follows. In Sec. II we give a description of a mean-field model of stochastic infinite systems exhibiting phase transitions in terms of coupled nonlinear Langevin equations. With a heuristic derivation of a nonlinear Fokker-Planck equation, phase transitions in the system are shown to be well described by the equation, whose stationary solutions undergo bifurcations with changes in parameters such as intensity of external white noise. Section III deals with an asymptotic approach of time-dependent solutions to stationary solutions of the nonlinear Fokker-Planck equation on the basis of an H theorem. A detailed description of a proof of the H theorem and of the stability analysis of bifurcating solutions is presented with use of a constructed H functional. Section IV is devoted to describing the fluctuation-dissipation theorem in Fokker-Planck dynamics to relate linear responses with correlation functions of the fluctuations for the present infinite system. Since the fluctuation-dissipation theorem in Fokker-Planck dynamics is not so popular as in thermodynamic systems, we give a detailed description to make the present article self-contained. In Sec. V we calculate linear responses of the order parameter to a periodically oscillating external force with use of the nonlinear Fokker-Planck equation to obtain generalized susceptibilities and correlation functions of the order-parameter fluctuations. Critical slowing down is shown to manifest itself in the form of the divergence of relaxation time on approaching phase-transition points. Finally, in Sec. VI, followed by four Appendixes, we shall give a brief summary and discussions of the present results.

II. PHASE TRANSITION AND NONLINEAR FOKKER-PLANCK EQUATION

We begin with presenting an N-particle Markov system which consists of a collection of anharmonic oscillators (which in reality are overdamped in the limit of high friction) in a potential $\tilde{\Phi}(x)$, interacting with each other via an attractive linear coupling under the influence of white noise. The set of stochastic differential equations describing the system is assumed to be given by[15]

$$\frac{dx_i}{dt} = -\frac{d\tilde{\Phi}(x_i)}{dx_i} + \sum_{k=1}^{N} \frac{\epsilon}{N}(x_k - x_i) + \sigma \frac{dB_i(t)}{dt} ,$$
$$\left\langle \frac{dB_i(t)}{dt}\frac{dB_j(t')}{dt'} \right\rangle = \delta_{ij}\delta(t-t'), \quad i=1,2,\ldots,N \tag{2.1}$$

with $\epsilon > 0$ denoting the strength of attractive mean-field-type coupling and $\sigma > 0$ the intensity of the statistically independent white noise added on each oscillator. We note that when $N=1$ and $\tilde{\Phi}(x) = -(\gamma/2)x^2 + x^4/4$, the above equation reduces to Eq. (1.1). One can easily write down a corresponding Fokker-Planck equation for the joint probability distribution function $p(x_1, x_2, \ldots, x_N; t)$ as well as its stationary solution, if necessary. Since, in general, as in the present case, a stationary distribution for an N-dimensional Fokker-Planck equation was shown to be uniquely determined and to be asymptotically approached after a long time for most cases with physical significance,[8] one cannot expect any bifurcations of solutions to occur in such a finite particle system. For certain types of phase transitions to take place in the present coupled oscillator system, the thermodynamic limit ($N \to \infty$) should be taken with an appropriate choice for potential $\tilde{\Phi}(x)$. Thus we are led to consider an infinite system obtained by taking the limit $N \to \infty$ in Eq. (2.1) to discuss the dynamical behavior of the systems exhibiting phase transitions.

Noting that an average of the x_k's

$$\lim_{N\to\infty} \left[\frac{1}{N} \sum_{k=1}^{N} x_k(t) \right] = \bar{x}(t) \tag{2.2}$$

can be postulated not to fluctuate and to behave in a deterministic way due to the law of large numbers, each stochastic differential equation in Eq. (2.1) turns out to be reduced to an independent and identical stochastic differential equation of the form

$$\frac{dx}{dt} = -\frac{d\tilde{\Phi}(x)}{dx} - \epsilon x + \epsilon \bar{x}(t) + \sigma \frac{dB}{dt} . \tag{2.3}$$

The Fokker-Planck equation corresponding to this Langevin equation is formally given as

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[-\frac{d\tilde{\Phi}(x)}{dx} - \epsilon x + \epsilon \bar{x}(t) \right] p + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} p . \tag{2.4}$$

We can interpret $p(t,x)$ along with this Fokker-Planck equation as follows. When $\bar{x}(t)$ is considered to be given *a priori,* the probability distribution for each oscillator, which originates from various realizations of white noise in Eq. (2.3), evolves with time according to Eq. (2.4). Since we are dealing with infinitely many oscillators at one time whose time evolutions are governed by an identical Langevin equation as given by Eq. (2.3), the probability distribution appearing in Eq. (2.4) is considered to be actually realized in real time by the present infinite system. In this way the probability distribution $p(t,x)$ can be viewed as an empirical distribution, which denotes the probability density of finding oscillators with x being in $[x, x+dx]$ at time t in the system. More precisely $p(t,x)$ can be written as

$$p(t,x)dx = \lim_{N\to\infty} \left[\frac{1}{N} \sum_{i=1}^{N} 1_{[x,x+dx]}(x_i(t)) \right] . \tag{2.5}$$

Here, $1_{[x,x+dx]}(\ )$ denotes the indicator function of the set $[x, x+dx] \subset \mathbb{R}$. Since $\bar{x}(t)$ can then be expressed in terms of $p(t,x)$ as

$$\bar{x}(t) = \int_{\mathbb{R}} x p(t,x) dx , \tag{2.6}$$

we arrive from Eq. (2.4) at a self-consistent nonlinear Fokker-Planck equation for the empirical distribution function:

$$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{d\Phi(x)}{dx} + \epsilon\int_{\mathbb{R}} xp(t,x)dx\right]p(t,x) + D\frac{\partial^2}{\partial x^2}p(t,x) ,$$

$$\Phi(x) = -\tilde{\Phi}(x) - \tfrac{1}{2}\epsilon x^2, \quad D = \tfrac{1}{2}\sigma^2 . \tag{2.7}$$

We see that the original infinitely-many-body problem has been reduced to a one-body problem as a result of the mean-field-type interaction in the present system. The effect of the mean-field-type interaction manifests itself as a feedback effect expressed in terms of the first moment of the time-dependent distribution function in the above equation. In fact, Eq. (2.7) is of an integro-differential type with nonlinearity in the unknown function $p(t,x)$, which can be responsible for the occurrence of bifurcations of the solutions under a certain choice of potentials $\tilde{\Phi}(x)$ as seen below. Since potentials giving rise to phase transitions are of our particular concern, we assume in what follows that $\tilde{\Phi}(x)$ can induce certain types of phase transitions. For example, the present system with such a symmetric bistable potential as a well-known one, $\tilde{\Phi}(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$, was shown to exhibit a phase transition bearing a close resemblance to thermodynamic phase transitions in ferromagnets.[11,12] For this reason, we hereafter refer to the potentials of the form $\tilde{\Phi} = \gamma_1 x^2 + \gamma_2 x^4 (\gamma_1 < 0, \gamma_2 > 0)$ as ferromagnetic potentials.

The stationary distribution function $p_{st}(x)$ obtained by putting $\partial p/\partial t = 0$ reads

$$p_{st}(x,x_0) = e^{D^{-1}[\Phi(x)+x_0 x]} \Big/ \int_{\mathbb{R}} e^{D^{-1}[\Phi(x)+x_0 x]}dx , \tag{2.8}$$

with a self-consistent equation for the determination of the value of the order parameter x_0/ϵ being

$$x_0/\epsilon = \int_{\mathbb{R}} xp_{st}(x,x_0)dx . \tag{2.9}$$

This equation for x_0 plays a key role in the existence of bifurcations of stationary solutions to the nonlinear Fokker-Planck equation (2.7).

It is somewhat convenient to introduce a graphical representation of Eq. (2.9) in order to have a firsthand understanding for the occurrence of bifurcations. Figure 1 shows a typical example of such representations for systems endowed with a symmetric potential such as a ferromagnetic one. All of the self-consistently-determined values for x_0 are given by the intersection points of the two curves in Fig. 1 representing the right-hand and left-hand sides of Eq. (2.9). For the case of symmetric potentials, $x_0 = 0$ is easily seen to satisfy the self-consistent equation (2.9). It can also be shown, by asymptotic evaluations of integrals for a wide class of symmetrical potential functions, that the initial slope defined as

$$\frac{\partial}{\partial x_0}\int_{\mathbb{R}} xp_{st}(x,x_0)dx \,\Big|_{x_0=0} \tag{2.10}$$

vanishes in the large noise limit ($D \to \infty$) for ϵ fixed, and that it becomes greater than the value $1/\epsilon$ of the slope of the straight line, when D approaches 0 (Appendixes A and B). Accordingly, under the conditions implied in Appendixes A and B for the potentials, it follows that there exists $D_c(\epsilon)$ at which the two curves for Eq. (2.9) intersect with each other at origin tangentially, leading to the existence of bifurcations of solutions. For systems with previously mentioned ferromagnetic potentials, the occurrence of phase transitions or bifurcations associated with change in D was rigorously proved.[12] Namely, for ϵ fixed, there exists a critical value $D_c(\epsilon)$ such that for $D \geqq D_c(\epsilon)$ there is a unique stationary distribution function with $x_0 = 0$, whereas for $D < D_c(\epsilon)$ two others with $x_0 \neq 0$ appear besides the former. Figure 2 illustrates a bifurcation scheme for such systems with x_0/ϵ (order parameter) schematically plotted against the control parameter D for arbitrarily chosen $\epsilon > 0$. The state characterized by $x_0 = 0$ corresponds to a paramagnetic state and the two bifurcating solutions with $x_0 \neq 0$ can be viewed as a ferromagnetic state below the Curie point of a ferromagnet, when one identifies D as the temperature of a thermodynamic system.

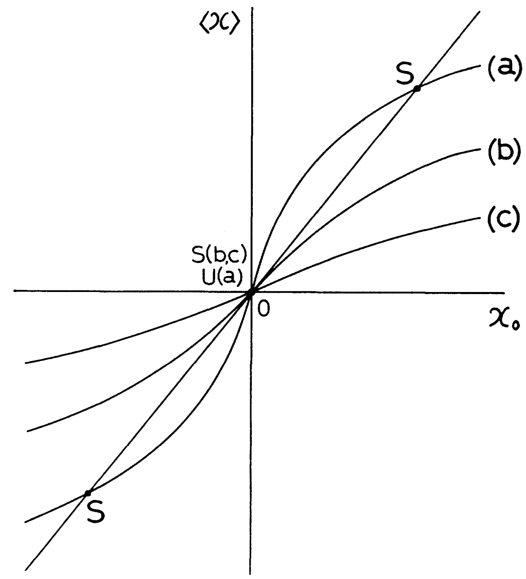

FIG. 1. Schematic plots of the averages $\langle x \rangle$ corresponding to the rhs (straight line) and lhs of Eq. (2.9) [(a), (b), and (c)] as functions of x_0 for a ferromagnetic model. The value of the order parameter $\langle x \rangle (= x_0/\epsilon)$ associated with a stationary distribution function p_{st} is determined by the intersection of the two curves. Three typical cases are shown according to the values of D for fixed ϵ. (a) $D < D_c(\epsilon)$ (ordered or ferromagnetic states corresponding to the two stable points with $\langle x \rangle \neq 0$), (b) $D = D_c(\epsilon)$ (bifurcation point), (c) $D > D_c(\epsilon)$ (disordered or paramagnetic state with $\langle x \rangle = 0$). Stable (S) and unstable (U) points are distinguished by the stability criterion [Eq. (3.52)] in Sec. III.

As seen in this case, a condition for the occurrence of phase transitions of any kind will, in general, be given by the requirement that the two curves representing the left-hand and right-hand sides of Eq. (2.9) intersect with each other with the same tangent at certain values of ϵ and D:

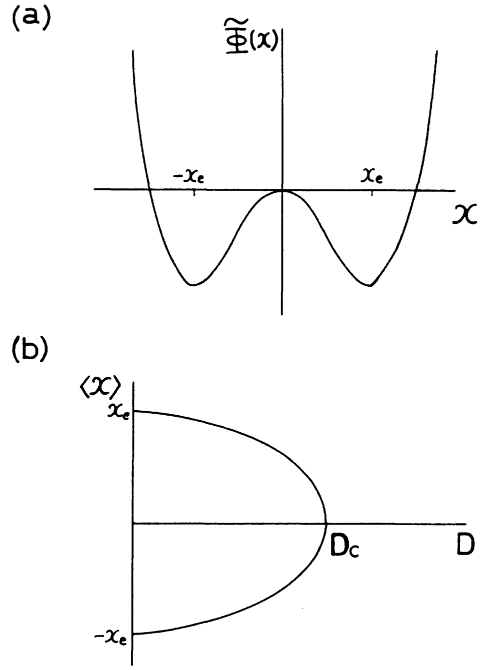

FIG. 2. (a) When $\tilde{\Phi}(x)$ takes the form of a bistable (ferromagnetic) potential, (b) the system undergoes a pitchfork bifurcation as the external noise power D changes.

$$\frac{\partial}{\partial x_0}\left[\frac{x_0}{\epsilon}\right]-\frac{\partial}{\partial x_0}\int_{\mathbf{R}} x p_{\rm st}(x,x_0)dx=0 . \tag{2.11}$$

In fact, when the difference of the slopes of the two curves at an intersection point changes its sign at the point corresponding to Eq. (2.11), a bifurcation or phase transition of second order takes place as a result of stability exchange of the stationary solutions of Eq. (2.8), as will be shown later in the following section. Since the following equality holds,

$$\frac{\partial}{\partial x_0}\int_{\mathbf{R}} x p_{\rm st}(x,x_0)dx=\frac{1}{D}\langle (x-\langle x\rangle)^2\rangle , \tag{2.12}$$

the difference of the slopes of the two curves at an intersection point can be written as

$$\frac{\partial}{\partial x_0}\left[\frac{x_0}{\epsilon}\right]-\frac{\partial}{\partial x_0}\langle x\rangle=\frac{1}{\epsilon}\left[1-\frac{\epsilon}{D}\langle (x-\langle x\rangle)^2\rangle\right] . \tag{2.13}$$

Here $\langle\ \rangle$ denotes the average over the stationary distribution $p_{\rm st}(x,x_0)$. Thus it follows that the bifurcation point is characterized by the following condition expressed in terms of the second moment of the fluctuations $x-\langle x\rangle$ with respect to $p_{\rm st}$:

$$1-\frac{\epsilon}{D}\langle (x-\langle x\rangle)^2\rangle=0 . \tag{2.14}$$

It is noted that $D_c(\epsilon)$ for the ferromagnetic models is given by[12,13]

$$1-\frac{\epsilon}{D}\langle x^2\rangle=0 , \tag{2.15}$$

because $\langle x\rangle$ vanishes when $x_0=0$. Since, when x_0 is viewed as an external perturbational field, Eq. (2.12) is reminescent of a thermodynamic relation between the fluctuations and the linear-response functions of physical quantities, the equality (2.14) holding at the phase-transition points is suggestive of the divergence of the static susceptibility associated with the order parameter at the transition points. Although we can give an expression for the static susceptibility as well as its divergence at the bifurcation points on the basis of a perturbational treatment for $\langle x\rangle$,[13] we shall taken an alternative and more systematic approach to study these problems in Sec. V.

III. *H* THEOREM AND STABILITY ANALYSIS FOR NONLINEAR FOKKER-PLANCK EQUATION

As stated in the preceding section, nonlinear Fokker-Planck equations of the form given by Eq. (2.7) can yield bifurcations of their stationary solutions for appropriately chosen potentials, when D is viewed as a control parameter. For such nonlinear systems displaying bifurcations, it is very important to investigate the asymptotic approach of a time-dependent solution to equilibrium and to conduct a stability analysis of bifurcating solutions to determine which state is stable. In this section we study the dynamical behavior of the system with regard to how the system that is set far out of equilibrium decays to a stationary or equilibrium state, and conduct an asymptotic stability analysis of the system described by Eq. (2.7). Progress in studies of such aspects of the dynamical behavior, which have been of paramount importance in the field of nonequilibrium thermodynamics, has been marked by the establishment of stability theories involving the kind of Lyapunov functional as was developed by Glansdorff and Prigogine.[17] As shown below, by constructing an H functional playing the role of a Lyapunov functional, an approach with use of an H theorem is available to explore not only a global but also a local stability analysis for the present nonlinear Fokker-Planck equations. An H theorem[8] for systems described by finite dimensional linear Fokker-Planck equations was studied by several authors[18,19] to ensure the approach of an arbitrary initial distribution function to a unique stationary solution for $t\to\infty$, under the condition that a matrix of diffusion coefficients be positive definite and the drift coefficients having no singularities do not allow explosive solutions. It should be noted that due to this uniqueness of the asymptotically approached solution one cannot expect any bifurcations of solutions to occur in finite dimensional Fokker-Planck dynamical systems with the above-mentioned condition.

As an H functional for the present infinite system described by the nonlinear Fokker-Planck equation (2.7), we take the following expression:[15]

$$H(p(\cdot))=\int_{\mathbf{R}} p(x)\ln\frac{p(x)}{q(x)}dx , \tag{3.1}$$

$$q(x)=\exp\left[D^{-1}\left[\Phi(x)+x_0x-\frac{1}{2\epsilon}x_0^2\right]\right], \quad (3.2)$$

$$x_0=\epsilon\int_R xp(x)dx , \quad (3.3)$$

where $p(x)$, is an arbitrary probability density chosen so as to make $H(p(\cdot))$ well defined and to satisfy the normalization condition

$$\int_R p(x)dx=1 . \quad (3.4)$$

The H functional defined above differs from that developed by Green[18] and Schlögl[25,26] for finite dimensional Fokker-Planck dynamics and a general analysis of thermodynamic nonequilibrium states, respectively. The usual H functional[8,18,19,26] takes the form of Kullback information[27] or information gain, in which, when $p(x)$ assumes a solution of a linear Fokker-Planck equation, $q(x)$ corresponding to Eq. (3.2) is also given by another either time-dependent or stationary solution of the equation. In our construction of the H functional, even when $p(x)$ assumes a solution $p(t,x)$ of the nonlinear Fokker-Planck equation (2.7), $q(t,x)$ does not satisfy Eq. (2.7), unless $p(t,x)$ is equal to its stationary solution, because the expression for $q(t,x)$ contains $p(t,x)$ through $x_0(t)$ $[q(t,x)=q(x;x_0(t))]$. In the present case, however, $q(x)$ is easily seen, by comparing the expressions for $q(x)$ and $p_{st}(x)$, to satisfy a seemingly stationarity condition

$$D\frac{\partial^2 q(x)}{\partial x^2}-\frac{\partial}{\partial x}\left[\frac{d\Phi(x)}{dx}+x_0\right]q(x)=0 , \quad (3.5)$$

irrespective of whether the x_0 defined in Eq. (3.3) is time dependent or not. Moreover, when $p(x)$ assumes a time-dependent solution $p(t,x)$ of Eq. (2.7), it is easy to see

$$\int_R \frac{p(t,x)\dot{q}(t,x)}{q(t,x)}dx=0 . \quad (3.6)$$

When we define entropy S to be[25,26]

$$S=-\int_R p\ln p\, dx , \quad (3.7)$$

and energy U consisting of the potential and interaction parts as

$$U=-\int_R \Phi(x)p(x)dx-\frac{\epsilon}{2}\left[\int_R xp(x)dx\right]^2 , \quad (3.8)$$

the functional $H(p(\cdot))$ given by Eq. (3.1) can be rewritten to be identified as being proportional to a generalized Helmholtz free energy $\mathcal{F}$ per particle of a nonequilibrium state of the present nonlinear stochastic system under the condition that D be viewed as temperature:

$$H(p(\cdot))=\int_R p\ln p\, dx-\int_R pD^{-1}\left[\Phi(x)+\frac{1}{2\epsilon}x_0^2\right]dx$$
$$=D^{-1}(U-DS)=\mathcal{F}/D . \quad (3.9)$$

If the system is in equilibrium, the generalized Helmholtz free energy $\mathcal{F}$ turns out to take the form quite analogous to the usual expression for the free energy of thermodynamic systems:

$$\mathcal{F}=-D\ln\int_R q\,dx$$
$$=-D\ln\int_R \exp\left[D^{-1}\left[\Phi(x)+x_0x-\frac{1}{2\epsilon}x_0^2\right]\right]dx , \quad (3.10)$$

with x_0 determined by Eq. (2.9). This is easily observed by substituting the stationary distribution function Eq. (2.8) into the expression (3.1) for the H functional.

A. *H* theorem

The functional $H(p(\cdot))$ satisfies the following two important properties, which is usually called an H theorem.[8]

(1) $H(p(\cdot))$ is bounded from below:

$$H(p(\cdot))>\text{const} . \quad (3.11)$$

(2) When p is substituted for $H(p(\cdot))$ by a solution $p(t,x)$ of the nonlinear Fokker-Planck equation (2.7) satisfying the normalization condition (3.4), $H(p(t,\cdot))$ is a monotonically decreasing function of time:

$$\frac{d}{dt}H(p(t,\cdot))=-\int_R Dp(t,x)\left[\frac{\partial}{\partial x}\ln\frac{p(t,x)}{q(t,x)}\right]^2 dx\leqq 0 . \quad (3.12)$$

Since the H theorem provides us with very important and useful information for the understanding of the dynamical behavior of an irreversible process towards equilibrium, we present the details of the proof of the theorem.

Proof of (1)

Letting $q_1(x)$ and $q_2(x)$ be any probability densities and noting the inequality,

$$x-1\geq\ln x \quad (x>0) , \quad (3.13)$$

we obtain

$$q_2-q_1\geqq q_1\ln(q_2/q_1) . \quad (3.14)$$

Integration of the above inequality yields the so-called relative entropy inequality

$$\int_R q_1\ln(q_1/q_2)dx\geqq 0 . \quad (3.15)$$

Defining $Z(x_0)$ to be a partition function associated with the function $q(x)$ in Eq. (3.2),

$$Z(x_0)=\int_R q(x)dx$$
$$=\int_R \exp\left[D^{-1}\left[\Phi(x)+x_0x-\frac{1}{2\epsilon}x_0^2\right]\right]dx , \quad (3.16)$$

and substituting $q_1=p$ and $q_2=q/Z$ into inequality (3.15), we obtain

$$\int_R p\ln(p/q)dx\geqq -\ln Z(x_0) . \quad (3.17)$$

To prove the boundedness from below of $-\ln Z$, we first consider the case where $\Phi(x)$ takes the form derived from a ferromagnetic potential $\tilde{\Phi}_{FM}(x)$

$$\Phi(x)=\Phi_{FM}(x)\equiv-\tilde{\Phi}_{FM}(x)-\frac{\epsilon}{2}x^2 =ax^2-bx^4 \quad (b>0) . \tag{3.18}$$

Noting that $\ln Z$ can be written as

$$\ln Z(x_0)=\frac{-1}{2D\epsilon}x_0^2+\ln\int_R e^{D^{-1}[\Phi(x)+x_0x]}dx , \tag{3.19}$$

we differentiate $\ln Z$ with respect to x_0 to see the behavior of the function. We have

$$D\frac{d}{dx_0}\ln Z(x_0) =\frac{-x_0}{\epsilon}+D\frac{\partial}{\partial x_0}\ln\int_R e^{D^{-1}[\Phi_{FM}(x)+x_0x]}dx . \tag{3.20}$$

To see the sign of $d\ln Z/dx_0$, it is convenient to resort to a graphical representation of each term on the right-hand side (rhs) of Eq. (3.20) viewed as a function of x_0. The second term of the rhs, which we denote by $\hat{x}(x_0)$, expresses the average of x over the probability distribution $\propto\exp\{D^{-1}[\Phi_{FM}(x)+x_0x]\}$ and hence satisfies the following properties:

$$\text{(i)}\ \frac{d\hat{x}(x_0)}{dx_0}>0,\quad \text{(ii)}\ \frac{d^2}{dx_0^2}\hat{x}(x_0)\leqq 0 \quad (x_0\geqq 0) ,$$

$$\text{(iii)}\ -\hat{x}(-x_0)=\hat{x}(x_0),\quad \text{(iv)}\ \hat{x}(x_0)\leqq Kx_0^{1/3} \quad (\text{for sufficiently large } x_0) . \tag{3.21}$$

(i) and (iii) can be easily checked and (ii) follows from the GHS inequality.[12,28] A proof of (iv) is given in Appendix C. From these properties of $\hat{x}(x_0)$ it follows that there are only two cases with regard to the way in which the curve representing the second term on the rhs intersects the line for the negative of the first term on the rhs of Eq. (3.20) ($d/dx_0\ln Z=0$, see Fig. 1). The first case, which is characterized by $d\hat{x}(x_0)/dx_0|_{x_0=0}\leqq 1/\epsilon$, yields only one intersection point at $x_0=0$, giving rise to a disordered state in a ferromagnetic model system. The sign of $d\ln Z/dx_0$ changes from a positive one ($x_0<0$) to a negative one ($x_0>0$) at $x_0=0$, where accordingly $\ln Z$ attains its maximum value. The second case, which is characterized by $0<1/\epsilon<d\hat{x}(x_0)/dx_0|_{x_0=0}$ and is responsible for the occurrence of an ordered state, allows three intersection points including $x_0=0$. Hence, $\ln Z$ attains its maximum at the two points with $x_0\neq 0$. Thus we confirm the existence of the maximum value of $\ln Z$ for both cases:

$$\ln Z(x_0)\leqq\max_{x_0}\{\ln Z(x_0)\} , \tag{3.22}$$

the opposite sign of which gives a lower bound for $\int_R p\ln(p/q)dx$:

$$\int_R p\ln(p/q)dx\geqq-\ln Z(x_0)\geqq-\max_{x_0}\{\ln Z(x_0)\} . \tag{3.23}$$

To deal with cases with more general potentials $\Phi(x)$, we assume that $\Phi(x)$ can be expressed as a sum of Φ_{FM} corresponding to a ferromagnetic potential and Φ_1 satisfying the condition

$$\int_R e^{2\Phi_1(x)/D}dx<\infty , \tag{3.24}$$

so that

$$\Phi(x)=\Phi_{FM}(x)+\Phi_1(x) . \tag{3.25}$$

Hence, using the Schwaltz inequality, we obtain

$$\int_R\exp\{D^{-1}[\Phi(x)+x_0x]\}dx=\int_R\exp\{D^{-1}[\Phi_{FM}(x)+x_0x+\Phi_1(x)]\}dx \leqq\left[\int_R\exp\{2D^{-1}[\Phi_{FM}(x)+x_0x]\}dx\right]^{1/2}\left[\int_R\exp[2D^{-1}\Phi_1(x)]dx\right]^{1/2} . \tag{3.26}$$

From Eq. (3.19) and inequality (3.24), we have

$$\ln Z(x_0)\leqq\frac{-1}{2D\epsilon}x_0^2+\tfrac{1}{2}\ln\int_R\exp\{2D^{-1}[\Phi_{FM}(x)+x_0x]\}dx +\tfrac{1}{2}\ln\int_R\exp[2D^{-1}\Phi_1(x)]dx . \tag{3.27}$$

Since, as has been proved, for the ferromagnetic potentials

$$\frac{-1}{2D\epsilon}x_0^2+\tfrac{1}{2}\ln\int_R\exp\{2D^{-1}[\Phi_{FM}(x)+x_0x]\}dx\leqq c , \tag{3.28}$$

it follows that

$$\ln Z(x_0)\leqq c+\tfrac{1}{2}\ln\int_R\exp[2D^{-1}\Phi_1(x)]dx\equiv c' . \tag{3.29}$$

Thus we proved that H has a lower bound $-c'$:

$$H(p(\cdot))=\int_R p\ln(p/q)dx\geqq-\ln Z(x_0)\geqq-c' . \tag{3.30}$$

Proof of (2)

Letting $p(t,x)$ be a solution of the nonlinear Fokker-Planck equation (2.7) and substituting it into Eq. (3.1), we take the derivative of $H(p(t,\cdot))$ with respect to time t. With use of Eq. (2.7) we obtain

$$\begin{aligned}
\frac{dH(p(t,\cdot))}{dt} &= \int_{\mathbb{R}} \dot{p}\ln\frac{p}{q}dx + \int_{\mathbb{R}} p\left[\frac{\dot{p}}{p}-\frac{\dot{q}}{q}\right]dx \\
&= \int_{\mathbb{R}}\left[D\frac{\partial^2 p}{\partial x^2}-\frac{\partial}{\partial x}\left[\frac{d\Phi}{dx}+x_0(t)\right]p\right]\ln\frac{p}{q}dx - \int_{\mathbb{R}}\frac{p\dot{q}}{q}dx \\
&= \int_{\mathbb{R}} p\left[D\frac{\partial^2}{\partial x^2}+\left[\frac{d\Phi}{dx}+x_0(t)\right]\frac{\partial}{\partial x}\right]\ln\frac{p}{q}dx - \int_{\mathbb{R}}\frac{p\dot{q}}{q}dx \\
&= \int_{\mathbb{R}} p\left[D\frac{\partial}{\partial x}+\frac{d\Phi}{dx}+x_0(t)\right]\frac{\partial/\partial x\,(p/q)}{p/q}dx - \int_{\mathbb{R}}\frac{p\dot{q}}{q}dx \\
&= \int_{\mathbb{R}} p\left[\frac{1}{p/q}\left[D\frac{\partial}{\partial x}+\frac{d\Phi}{dx}+x_0(t)\right]\frac{\partial}{\partial x}\left[\frac{p}{q}\right]-\frac{D[\partial/\partial x(p/q)]^2}{(p/q)^2}\right]dx - \int_{\mathbb{R}}\frac{p\dot{q}}{q}dx \\
&= \int_{\mathbb{R}} q\left[D\frac{\partial^2}{\partial x^2}+\left[\frac{d\Phi}{dx}+x_0(t)\right]\frac{\partial}{\partial x}\right]\frac{p}{q}dx - \int_{\mathbb{R}} pD\left[\frac{\partial}{\partial x}\ln\frac{p}{q}\right]^2 dx - \int_{\mathbb{R}}\frac{p\dot{q}}{q}dx \\
&= \int_{\mathbb{R}}\frac{p}{q}\left[D\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\left[\frac{d\Phi}{dx}+x_0(t)\right]\right]q\,dx - \int_{\mathbb{R}} pD\left[\frac{\partial}{\partial x}\ln\frac{p}{q}\right]^2 dx - \int_{\mathbb{R}}\frac{p\dot{q}}{q}dx \; .
\end{aligned} \tag{3.31}$$

Here dots represent time derivatives and we used the normalization condition for p in deriving the second line and integrations by parts in the third and the last lines. Since from Eqs. (3.5) and (3.6) the first and third terms of Eq. (3.31) vanish, we finally obtain

$$\frac{dH}{dt} = -\int_{\mathbb{R}} pD\left[\frac{\partial}{\partial x}\ln\frac{p}{q}\right]^2 dx \leqq 0 \; . \tag{3.32}$$

The H theorem states that the functional $H(t)$ representing the free energy of the system continues monotonically to decrease in time but that it cannot decrease indefinitely due to the property (1). Thus we find that dH/dt must vanish for large times and from Eq. (3.32) it follows that

$$\frac{\partial}{\partial x}\ln\left[\frac{p}{q}\right]=0 \quad (t\to\infty) \tag{3.33a}$$

and hence

$$\frac{p}{q}=\text{const (independent of } x) \quad (t\to\infty) \; . \tag{3.33b}$$

This implies that a probability distribution function as a solution of the nonlinear Fokker-Planck equation converges to a function proportional to $q(t,x)=q(x;x_0(t))$ as $t\to\infty$. We note here that $x_0(t)$ in $q(t,x)$ must satisfy Eq. (3.3) with x_0 and $p(x)$ being replaced by $x_0(t)$ and const$\cdot q(x;x_0(t))$, respectively, in the limit $t\to\infty$:

$$x_0(t)=\frac{\epsilon\int_{\mathbb{R}} xq(x;x_0(t))dx}{\int_{\mathbb{R}} q(x;x_0(t))dx} \quad (t\to\infty) \; . \tag{3.34}$$

Then, by noting the x dependence of the expression for $q(x;x_0(t))$ and comparing Eqs. (2.9) and (3.34), $x_0(t)$ in the $t\to\infty$ limit is seen to have to assume one of the static values x_0 given by Eq. (2.9). There is no possibility that $x_0(t)$ in $q(x;x_0(t))$ admits an oscillatory or wandering motion among x_0's given by Eq. (2.9), because, as is easily checked, the nonlinear Fokker-Planck equation (2.7) yields no solutions of the form

$$q(x;x_0(t))\Big/\int_{\mathbb{R}} q(x;x_0(t))dx \; ,$$

with $x_0(t)$ being other than the time-independent constant values characterized by Eq. (2.9). Hence, the large time limit $q(\infty,x)$ should coincide with one of the stationary solutions given by Eq. (2.8) except for a normalization constant. Thus it follows that a time-dependent solution to the nonlinear Fokker-Planck equation (2.7) should always approach for large times one of the stationary probability densities given by Eq. (2.8) with x_0 satisfying the self-consistent equation (2.9). In view of this convergence property, we may be able to state that the present H theorem implies "pre-global stability" of the set of the Gibbs-type stationary solutions Eq. (2.8). Here the term "global stability" is used in the sense that there is no other attractor than a set of fixed-point-type attractors corresponding to the stationary solutions given by Eqs. (2.8) and (2.9) and that any time-dependent solutions of Eq. (2.7) lying far from equilibrium must be attracted by either one of those stationary solutions without any possibility of exhibiting runaway behavior or limit cycle type oscillations. Accordingly, if the system admits only one stationary distribution, as for example in the case of paramagnetic state ($D>D_c$) of the ferromagnetic model or in systems with no phase transitions, global stability in the usual sense of the stationary distribution follows from the H theorem. Namely, a uniquely determined stationary distribution function of the form Eq. (2.8) is always globally stable. In such cases, it is no longer necessary to conduct a local stability analysis presented below. Paraphrasing the above situation in the case of the uniquely determined stationary state, in the language of nonlinear stability analysis, the functional $H(p(t,\cdot))$ constructed here plays the role of a Lyapunov functional of the nonlinear Fokker-Planck equation (2.7) appearing in the so-called "second method of Lyapunov"[29] applied for stability analysis of dynamical systems.

B. Stability analysis of bifurcating solutions

Since the self-consistent equation for x_0 yields, in general, more than one solution for such systems capable of exhibiting phase transitions as ones with ferromagnetic potentials, it is very important to determine which of the stationary solutions is approached for large times in the course of the time evolution or, in other words, to consider a problem of the change in stability between the stationary solutions. To this end we employ a method of nonlinear stability analysis for the present nonlinear Fokker-Planck equation. Noting a close similarly between the properties of our H functional and the free energy of thermodynamic systems, we are led to be concerned with analyses of a local structure of the H functional around a stationary solution $p_{\rm st}$ or, more specifically, an analysis of the second-order variation of the H functional. In fact, since the functional $H(p(t,\cdot))$ is a decreasing function of time, the condition for asymptotic stability of a stationary state is given by the positive definiteness of the second-order variation of the H functional evaluated at the stationary state $p=p_{\rm st}$,[17,25,26]

$$\delta^{(2)}H[\delta p,\delta p]>0 \quad (\textit{for}\ \forall\delta p\neq 0)\ . \tag{3.35}$$

It is easily seen from the discussion given below that the above condition indeed ensures local stability of a stationary state. The condition of positive definiteness for $\delta^{(2)}H[\delta p,\delta p]$ implies that

$$H(p(\cdot))\geqq H(p_{\rm st}(\cdot))$$

holds for p in a certain neighborhood Ω of a stationary state $p_{\rm st}$ with the equality sign being valid for $p=p_{\rm st}$ only. Since $H(p(t,\cdot))$ is a monotonically decreasing function of t, the condition for the second-order variation will also imply that any solution of the nonlinear Fokker-Planck equation (2.7) starting in Ω always remains in Ω. Now that the domain of definition for the H functional $H(p(\cdot))$ is restricted to Ω, it is easily seen, by following the same discussion as in the end of Sec. III A that $(d/dt)H(p(t,\cdot))$ is negative definite in Ω; that is, $(d/dt)H(p(t,\cdot))=0$ $(p\in\Omega)$ is satisfied only by $p=p_{\rm st}$. Hence, the application of the H theorem to the case with the restricted region Ω yields that any p in Ω tends into the stationary state $p_{\rm st}$ (local stability of $p_{\rm st}$). In this case the stationary state $p_{\rm st}$ is a local minimum of the H functional and the H functional with its domain of definition being restricted to a neighborhood of the $p_{\rm st}$ turns out to be a Lyapunov functional of the nonlinear Fokker-Planck equation (2.7).

Defining δp as

$$\delta p=p-p_{\rm st}\ , \tag{3.36}$$

$$\int_{\mathbb{R}}\delta p\,dx=0\ , \tag{3.37}$$

we calculate the variations of $H(p(\cdot))$ around $p=p_{\rm st}$:

$$H(p(\cdot))-H(p_{\rm st}(\cdot))=\delta^{(1)}H[\delta p]+\delta^{(2)}H[\delta p,\delta p]+\cdots\ . \tag{3.38}$$

Differentiating $H(p(\cdot))$ with respect to p, we obtain

$$\begin{aligned}\delta H&=\int_{\mathbb{R}}\delta p\ln\frac{p}{q}dx+\int_{\mathbb{R}}p\left[\frac{\delta p}{p}-\frac{\delta q}{q}\right]dx\\&=\int_{\mathbb{R}}\delta p\ln\left[\frac{p}{q}\right]dx-\int_{\mathbb{R}}\frac{p\,\delta q}{q}dx\ ,\end{aligned} \tag{3.39}$$

with

$$\delta q=D^{-1}q\left[\epsilon x\int_{\mathbb{R}}x\,\delta p\,dx-\epsilon\int_{\mathbb{R}}xp\,dx\int_{\mathbb{R}}x\,\delta p\,dx\right]\ . \tag{3.40}$$

Substitution of Eq. (3.40) into Eq. (3.39) yields

$$\delta H=\int_{\mathbb{R}}\delta p\ln\left[\frac{p}{q}\right]dx\ . \tag{3.41}$$

Putting $p=p_{\rm st}$, we see that the first-order variation $\delta^{(1)}H[\delta p]$ vanishes,

$$\delta^{(1)}H[\delta p]=0\ . \tag{3.42}$$

This implies that the stationary distribution functions given by Eq. (2.8) correspond to extreme points of the H functional. The second-order variation can be obtained by differentiating δH with respect to p and using Eqs. (3.40) and (3.41),

$$\begin{aligned}2\delta^{(2)}H&=\int_{\mathbb{R}}\delta p\left[\frac{\delta p}{p}-\frac{\delta q}{q}\right]dx\\&=\int_{\mathbb{R}}\frac{(\delta p)^2}{p}dx-\frac{\epsilon}{D}\left[\int_{\mathbb{R}}x\,\delta p\,dx\right]^2\ .\end{aligned} \tag{3.43}$$

Putting $p=p_{\rm st}$ gives the second-order variation evaluated at the stationary state

$$2\delta^{(2)}H[\delta p,\delta p]=\int_{\mathbb{R}}\frac{(\delta p)^2}{p_{\rm st}}dx-\frac{\epsilon}{D}\left[\int_{\mathbb{R}}x\,\delta p\,dx\right]^2\ . \tag{3.44}$$

When we put $\epsilon=0$ in the above expression, we recover for $\delta^{(2)}H[\delta p,\delta p]$ the form obtained by Schlögl.[25]

To determine the sign of $\delta^{(2)}H[\delta p,\delta p]$, we proceed further to rewrite the expression (3.44). Considering a δp of the form

$$\delta p=\varphi(x)p_{\rm st}(x)^{1/2},\quad \varphi(x)\in L^2(R) \tag{3.45}$$

and letting H_1 be the subspace of $L^2(R)$ spanned by $\{p_{\rm st}^{1/2},xp_{\rm st}^{1/2}\}$ and $H_1^{\perp}$ the orthogonal complement of H_1, we decompose φ into φ_{H_1} and $\varphi_{\perp}$,

$$\begin{aligned}\varphi&=\varphi_{H_1}+\varphi_{\perp}\\&=\alpha p_{\rm st}(x)^{1/2}+\beta xp_{\rm st}(x)^{1/2}+\varphi_{\perp}(x)\end{aligned} \tag{3.46}$$

From the requirement (3.37) for δp, we have

$$\int_{\mathbb{R}} \delta p\, dx = \int_{\mathbb{R}} (\alpha p_{\rm st} + \beta x p_{\rm st} + \varphi_{\perp} p_{\rm st}^{1/2})dx = \alpha + \beta \langle x \rangle_{\rm st} = 0 \; . \tag{3.47}$$

Substituting Eqs. (3.45) and (3.46) into Eq. (3.44) and using Eq. (3.47), we obtain

$$\begin{aligned} 2\delta^{(2)} H[\delta p, \delta p] &= \int_{\mathbb{R}} (\varphi_{H_1}^2 + \varphi_{\perp}^2) dx - \frac{\epsilon}{D} \left[\int_{\mathbb{R}} x p_{\rm st}^{1/2} \varphi_{H_1} dx \right]^2 \\ &= \int_{\mathbb{R}} \varphi_{\perp}^2 dx + \beta^2 \langle x^2 \rangle_{\rm st} - \beta^2 \langle x \rangle_{\rm st}^2 - \frac{\epsilon}{D} [\beta(\langle x^2 \rangle_{\rm st} - \langle x \rangle_{\rm st}^2)]^2 \\ &= \int_{\mathbb{R}} \varphi_{\perp}^2 dx + \beta^2 \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st} \left[1 - \frac{\epsilon}{D} \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st} \right] , \end{aligned} \tag{3.48}$$

which can be rewritten, by noting the relation (2.12), as

$$2\delta^{(2)} H[\delta p, \delta p] = \int_{\mathbb{R}} \varphi_{\perp}^2 dx + \epsilon \beta^2 \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st} \times \left[\frac{\partial (x_0/\epsilon)}{\partial x_0} - \frac{\partial}{\partial x_0} \int_{\mathbb{R}} x p_{\rm st}(x) dx \right] . \tag{3.49}$$

Here, $\langle \; \rangle_{\rm st}$ denotes an average over a stationary distribution $p_{\rm st}$,

$$\langle A(x) \rangle_{\rm st} = \int_{\mathbb{R}} A(x) p_{\rm st}(x) dx \; . \tag{3.50}$$

Now we find that if the term involving the static moments in Eq. (3.48) is positive,

$$1 - \frac{\epsilon}{D} \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st} > 0 \tag{3.51}$$

or, equivalently,

$$\frac{\partial}{\partial x_0} \left[\frac{x_0}{\epsilon} \right] - \frac{\partial}{\partial x_0} \int_{\mathbb{R}} x p_{\rm st}(x) dx > 0 \;\; (for\ \epsilon > 0) \; , \tag{3.52}$$

$\delta^{(2)} H[\delta p, \delta p]$ becomes positive definite to satisfy the stability condition (3.35), and thus the stationary distribution function $p_{\rm st}$ can be approached by any initial distributions lying in a neighborhood of the $p_{\rm st}$ after a long time.

When, on the other hand, the inequality (3.51) has the opposite sign, the stationary solution becomes a saddle point and it can be shown that there exist probability density functions, such that for each deviation δp, $\delta^{(2)} H[\delta p, \delta p]$ assumes a negative value. To see this, it is sufficient to take as a distribution function for variation the Gibbs-type probability density as given by Eq. (2.8) in the neighborhood of the $p_{\rm st}(x)$. Hence we have, in the lowest order in δx_0,

$$\begin{aligned} \delta p &= \delta \left[\frac{\exp\{D^{-1}[\Phi(x) + x_0 x]\}}{\int_{\mathbb{R}} \exp\{D^{-1}[\Phi(x) + x_0 x]\} dx} \right] \\ &= \left[\frac{x - \langle x \rangle_{\rm st}}{D} \right] p_{\rm st} \delta x_0 \; . \end{aligned} \tag{3.53}$$

Substituting this into Eq. (3.44) or (3.48) yields

$$\begin{aligned} &2\delta^{(2)} H[\delta p, \delta p] \\ &\quad = \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st} \left[1 - \frac{\epsilon}{D} \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st} \right] \frac{(\delta x_0)^2}{D^2} \\ &\quad < 0 \; . \end{aligned} \tag{3.54}$$

Thus, it is confirmed that when

$$1 - \frac{\epsilon}{D} \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st} < 0 \; , \tag{3.55}$$

the stationary distribution function turns out to be unstable for deviations that make $\delta^{(2)} H[\delta p, \delta p]$ negative.

At the bifurcation points, where

$$1 - \frac{\epsilon}{D} \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st} = 0 \tag{3.56}$$

holds, one must in general study higher-order variations of the H functional for the determination of the stability of the solution, unless the stationary solution is uniquely determined. As noted previously, a uniquely existing stationary state is always stable.

The above results of the nonlinear stability analysis for the bifurcation problem of the nonlinear Fokker-Planck equation (2.7) show that the sign of $1 - (\epsilon/D) \langle (x - \langle x \rangle_{\rm st})^2 \rangle_{\rm st}$ is relevant to the stability condition in such a way as to determine the sign of the second-order variation of the H functional and hence to ensure local stability of the bifurcating solutions. To conclude this section, we note that the stability criterion derived can be seen to provide us with an intuitive and practical recipe for stability analysis by employing a graphical interpretation of the inequality (3.52). Namely, in the graphical representation (Fig. 1) of Eq. (2.9), when the curve for $\int_{\mathbb{R}} x p_{\rm st}(x, x_0) dx$ intersects the straight line x_0/ϵ with the slope of the former being less than that of the latter, the stationary state corresponding to the intersection point is stable. Such a graphical method for the stability analysis coincides with what has been extensively used in studies of thermodynamic phase transitions of mean-field type, in which, however, a rigorous proof of the validity of this method, made from the viewpoint of the theory of time evolutions of thermodynamic systems, seems not to have been given thus far.

IV. LINEAR-RESPONSE THEORY AND FLUCTUATION-DISSIPATION THEOREM

In this and the following sections we deal with another kind of dynamical behavior associated with phase transitions in our coupled-oscillator systems. In systems undergoing phase transitions, exchange in stability takes place at the bifurcation points, as examined in the preceding sections, and accordingly we expect, in general, the behavior of dynamical responses and fluctuations of such macroscopic variables as the order parameter to be subject to some influences due to the occurrence of the instability. A critical slowing down[1,23] exhibited by the order-parameter fluctuations is a well-known and typical dynamical phenomenon characteristic of phase transitions in such thermodynamic systems as ferromagnets. Also, in the case of our stochastic systems, the existence of singularities in the dynamical behavior of the fluctuations as typified by a critical slowing down can be shown from the analysis made within the framework of linear-response theory with use of the Callen-Welton-Kubo fluctuation-dissipation theorem, which has been frequently employed to investigate the critical behavior of fluctuations in systems undergoing thermodynamic phase transitions.

Nowadays it is well known that the fluctuation-dissipation theorem holds in thermodynamic systems and has played a very important role in the statistical mechanics of the irreversible processes.[30] The fluctuation-dissipation theorem states that a linear response to a weak external perturbation of a system in thermal equilibrium is related to equilibrium fluctuations. More specifically, susceptibility tensors or response functions describing dissipation are connected to correlation functions of thermodynamic fluctuations in the equilibrium state. The fluctuation-dissipation theorem for such nonthermodynamic systems as the stochastic ones described by finite dimensional Fokker-Planck equations has been derived by Agarwal.[20] In order to discuss the fluctuation-dissipation theorem for our stochastic systems with infinitely many degrees of freedom and to lay foundations for calculations in Sec. V, we begin, by following the methods developed by Agarwal, with N-dimensional Fokker-Planck dynamics corresponding to the set of stochastic differential equations (2.1) with terms representing external perturbations being taken into account.

We consider the case in which each oscillator described by Eq. (2.1) is subjected to a temporally driven uniform external field $K(t)$, so that the Fokker-Planck equation for the N-particle system reads

$$\begin{aligned}\frac{\partial W(\{x\},t)}{\partial t} &= -\sum_i \frac{\partial}{\partial x_i}\left[-\frac{\partial\tilde{\Phi}(x_i)}{\partial x_i}+\frac{\epsilon}{N}\sum_j (x_j-x_i)+K(t)\right]W(\{x\},t)\\ &\quad+\tfrac{1}{2}\sigma^2\sum_i\frac{\partial^2}{\partial x_i^2}W(\{x\},t)\\ &\equiv L_{\mathrm{FP}}(\{x\},t)W(\{x\},t)\ . \end{aligned} \tag{4.1}$$

We split $L_{\mathrm{FP}}(\{x\},t)$ into unperturbed and time-dependent parts:

$$L_{\mathrm{FP}}(\{x\},t)=L^0_{\mathrm{FP}}(\{x\})+L_{\mathrm{ext}}(\{x\},t) \tag{4.2}$$

with

$$\begin{aligned} L^0_{\mathrm{FP}}(\{x\}) &= -\sum_i \frac{\partial}{\partial x_i}\left[-\frac{\partial\tilde{\Phi}(x_i)}{\partial x_i}+\frac{\epsilon}{N}\sum_j(x_j-x_i)\right] +\tfrac{1}{2}\sigma^2\sum_i\frac{\partial^2}{\partial x_i^2}\ ,\\ L_{\mathrm{ext}}(\{x\},t) &= -K(t)\sum_i\frac{\partial}{\partial x_i}\ . \end{aligned} \tag{4.3}$$

Since we are concerned with a linear deviation from the stationary solution W_{st} due to a small external perturbation L_{ext}, we linearize Eq. (4.1) around W_{st} by putting

$$W=W_{\mathrm{st}}+\delta W(\{x\},t) \tag{4.4}$$

to obtain an equation for δW as

$$\frac{\partial \delta W}{\partial t}=L^0_{\mathrm{FP}}(\{x\})\delta W+L_{\mathrm{ext}}(\{x\},t)W_{\mathrm{st}}\ . \tag{4.5}$$

A solution of this equation is obtained by integration in the form

$$\delta W(\{x\},t)=\int_{-\infty}^{t}\exp[L^0_{\mathrm{FP}}(\{x\})(t-s)]\times L_{\mathrm{ext}}(\{x\},s)W_{\mathrm{st}}\,ds\ . \tag{4.6}$$

The change in the average value of a dynamical variable $A(\{x\})$ is expressed as

$$\begin{aligned}\delta\langle A\rangle_t &= \int A(\{x\})\delta W(\{x\},t)d\{x\}\\ &= -\int_{-\infty}^{t} ds\,K(s)\int A(\{x\})\exp[L^0_{\mathrm{FP}}(\{x\})(t-s)]\times\sum_i\frac{\partial}{\partial x_i}W_{\mathrm{st}}(\{x\})d\{x\}\ . \end{aligned} \tag{4.7}$$

The last line was obtained by substitution of the explicit form of L_{ext}. Using

$$\frac{\partial W_{\mathrm{st}}(\{x\})}{\partial x_i}=\frac{D_i}{D}W_{\mathrm{st}}(\{x\}) \tag{4.8}$$

with

$$\begin{aligned} D_i(\{x\}) &= -\frac{d}{dx_i}\tilde{\Phi}(x_i)+\frac{\epsilon}{N}\sum_j(x_j-x_i)\ ,\\ D &= \tfrac{1}{2}\sigma^2\ , \end{aligned} \tag{4.9}$$

and noting the identity[20]

$$L^0_{\mathrm{FP}}(\{x\})x_iW_{\mathrm{st}}(\{x\})=D_i(\{x\})W_{\mathrm{st}}(\{x\})\ , \tag{4.10}$$

Eq. (4.7) is rewritten as

$$\delta\langle A\rangle_t=\frac{-1}{D}\int_{-\infty}^{t}ds\,K(s)\int A(\{x\})\exp[L_{\rm FP}^{0}(\{x\})(t-s)]L_{\rm FP}^{0}(\{x\})\sum_i x_i W_{\rm st}(\{x\})d\{x\} \tag{4.11}$$

$$=\frac{-1}{D}\int_{-\infty}^{t}ds\,K(s)\frac{d}{dt}\int A(\{x\})\exp[L_{\rm FP}^{0}(\{x\})(t-s)]\left[\sum x_i\right]W_{\rm st}(\{x\})d\{x\} \tag{4.12}$$

$$=\frac{-1}{D}\int_{-\infty}^{t}ds\,K(s)\frac{d}{dt}\int A(\{x\})\exp[L_{\rm FP}^{0}(\{x\})(t-s)]\delta(\{x\}-\{x'\})\left[\sum x_i'\right]W_{\rm st}(\{x'\})d\{x\}d\{x'\} \tag{4.13}$$

$$=\frac{-1}{D}\int_{-\infty}^{t}ds\,K(s)\frac{d}{dt}\int A(\{x\})\left[\sum x_i'\right]W(\{x\},t-s\mid\{x'\},0)W_{\rm st}(\{x'\})d\{x\}d\{x'\} \tag{4.14}$$

$$=\frac{-1}{D}\int_{-\infty}^{t}ds\,K(s)\frac{d}{dt}\left\langle A(\{x(t-s)\})\left[\sum_i x_i(0)\right]\right\rangle_{L_{\rm FP}^{0}} . \tag{4.15}$$

Here the symbol $\langle\ \rangle_{L_{\rm FP}^0}$ in the last line denotes an average with respect to the stationary Markov process characterized by the Fokker-Planck operator $L_{\rm FP}^0$. Since we are concerned with linear responses of the order parameter of the system, which is a macroscopic variable corresponding to the "total magnetization," we now take $A(\{x\})$ to be a sum of the variable x of each oscillator divided by N,

$$A=A_N(\{x\})=\frac{1}{N}\sum_{i=1}^{N}x_i(t) \tag{4.16}$$

and we define its fluctuations,

$$y_N(t)=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[x_i(t)-\langle x_i\rangle]$$
$$=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[x_i(t)-\langle A_N\rangle] . \tag{4.17}$$

Then $\delta\langle A_N\rangle_t$ can be written as

$$\delta\langle A_N\rangle_t=\int_0^\infty K(t-s)\frac{-1}{D}\frac{d}{ds}\langle y_N(s)y_N(0)\rangle_{L_{\rm FP}^0}ds . \tag{4.18}$$

Here we changed the variable for integration in Eq. (4.15). This is the well-known Callen-Welton-Kubo fluctuation-dissipation theorem talking about the relation between a response of the macroscopic variable A_N and the fluctuations y_N.

When N goes to infinity, the thermodynamic limit A_∞ is expected to exist and is expressible in terms of $p(t,x)$ in Eq. (2.5) as

$$A_\infty(\{x\})=\int_{\mathbb{R}}xp(t,x)dx , \tag{4.19}$$

and an assumption of the existence of the thermodynamic limit y_∞ seems to be reasonable. Furthermore, we can expect that even in the thermodynamic limit ($N\to\infty$) the above equation (4.18) holds:

$$\delta\langle A_\infty\rangle_t=\int_0^\infty K(t-s)\frac{-1}{D}\frac{d}{ds}\langle y_\infty(s)y_\infty(0)\rangle ds . \tag{4.20}$$

When we put $K(t)=h\cos(\omega t)$, integration by parts yields

$$\delta\langle A_\infty\rangle_t=\frac{1}{D}\left[\langle y_\infty(0)^2\rangle h\cos(\omega t)+\int_0^\infty\langle y_\infty(s)y_\infty(0)\rangle h\omega\sin[\omega(t-s)]ds\right] , \tag{4.21}$$

from which we can obtain the generalized dynamic susceptibilities. Here we used the fact that the correlation $\langle y_\infty(\infty)y_\infty(0)\rangle$ vanishes by definition of y_∞. More generally, introducing the Fourier transforms of $\delta\langle A_\infty\rangle_t$, $K(t)$, and $(-1/D)(d/dt)\langle y_\infty(t)y_\infty(0)\rangle$,

$$\delta\tilde{A}(\omega)=\int_{-\infty}^{\infty}\delta\langle A_\infty\rangle_t e^{-i\omega t}dt , \tag{4.22}$$

$$\tilde{K}(\omega)=\int_{-\infty}^{\infty}K(t)e^{-i\omega t}dt , \tag{4.23}$$

$$\chi_A(\omega)=\int_0^\infty\frac{-1}{D}\frac{d}{dt}\langle y_\infty(t)y_\infty(0)\rangle e^{-i\omega t}dt , \tag{4.24}$$

the Fourier transform of Eq. (4.20) is written as

$$\delta\tilde{A}(\omega)=\tilde{K}(\omega)\chi_A(\omega) . \tag{4.25}$$

Then $\chi_A(\omega)$ turns out to be the generalized susceptibility representing the ratio of $\delta\tilde{A}(\omega)$ to $\tilde{K}(\omega)$ and is expressed in terms of its real and imaginary parts as

$$\chi_A(\omega)=\chi_A'(\omega)-i\chi_A''(\omega) . \tag{4.26}$$

Explicit expressions for $\chi_A'(\omega)$ and $\chi_A''(\omega)$ are obtained from Eq. (4.21) or (4.24):

$$\chi_A'(\omega)=\frac{1}{D}\left[\langle y_\infty(0)^2\rangle-\omega\int_0^\infty\langle y_\infty(t)y_\infty(0)\rangle\sin\omega t\,dt\right] , \tag{4.27}$$

$$\chi_A''(\omega)=\frac{\omega}{D}\int_0^\infty\langle y_\infty(t)y_\infty(0)\rangle\cos\omega t\,dt . \tag{4.28}$$

It is easily seen that the static susceptibility $\chi_A(0)$ is related with the variance of the fluctuations through

$$\chi_A(0)=\chi_A'(0)=\frac{1}{D}\langle y_\infty(0)^2\rangle . \tag{4.29}$$

We note that the following sum rule is obeyed:

$$\lim_{\omega\to 0}\frac{D\chi_A''(\omega)}{\omega}=\int_0^\infty\langle y_\infty(t)y_\infty(0)\rangle dt . \tag{4.30}$$

The dynamical behavior of the order-parameter fluctuations y_∞ is of primary concern in the present study. The correlation function, which is related to the generalized susceptibilities through Eq. (4.24), can be given in a more explicit form by expressing it in terms of the power spectrum of the fluctuations. The power spectrum is defined as the Fourier transform of the correlation function:

$$\mathcal{S}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\langle y_\infty(t)y_\infty(0)\rangle e^{-i\omega t}dt \ . \tag{4.31}$$

From Eq. (4.24), we can easily obtain an expression for the power spectrum of the order-parameter fluctuations in terms of the generalized dynamical susceptibility:

$$\mathcal{S}(\omega)=\frac{1}{2\pi}\frac{iD[\chi_A(\omega)-\chi_A(-\omega)]}{\omega} =\frac{1}{\pi}\frac{D\chi''_A(\omega)}{\omega} \ . \tag{4.32}$$

Hence the correlation function of the fluctuations is given by the inverse Fourier transform as

$$\langle y_\infty(t)y_\infty(0)\rangle=\int_{-\infty}^{\infty}\frac{D\chi''_A(\omega)}{\pi\omega}e^{i\omega t}d\omega \ . \tag{4.33}$$

When the correlation function of the fluctuations is described by a single-exponential function $\propto e^{-\gamma t}$, the relaxation time is simply given by the inverse of γ. However, in general, a single-exponential type of relaxation cannot be expected for fluctuations in most systems where the correlation function takes the form of a sum of exponential functions [see Eq. (D12)]. For this type of correlation functions, relaxation time can only be defined as an average over distributed relaxation times appearing in the expression of the correlation functions. To be precise, an explicit expression for the relaxation time is generally given by

$$\tau=\frac{\int_0^\infty\langle y_\infty(t)y_\infty(0)\rangle dt}{\langle y_\infty(0)^2\rangle} \ . \tag{4.34}$$

Combining Eqs. (4.29) and (4.30), it turns out that the relaxation time τ of the fluctuations can be expressed in terms of only the generalized susceptibilities:[1]

$$\tau=\lim_{\omega\to 0}\frac{\chi''_A(\omega)}{\omega\chi_A(0)} \ . \tag{4.35}$$

V. DYNAMICAL CRITICAL PHENOMENA

From the exposition in Sec. IV, we see that all that is needed to obtain the dynamical properties of the fluctuations in the system exhibiting stochastic phase transitions is to know the generalized susceptibilities of the system. We now calculate the linear response of the order parameter of a stationary state to an external time-dependent perturbation $F(t)$ on the basis of the nonlinear Fokker-Planck equation

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial x}\left[\frac{d\Phi(x)}{dx}+\epsilon\int_{\mathbf{R}}xp\,dx+F(t)\right]p+D\frac{\partial^2}{\partial x^2}p \ . \tag{5.1}$$

To this end we linearize the equation around the stable stationary distribution p_{st} as given in Eq. (2.8) by putting

$$p=p_{st}+\delta p \ . \tag{5.2}$$

A linear equation thus obtained for δp is written in the form

$$\frac{\partial\delta p}{\partial t}=L_0\delta p-\left[\epsilon\int_{\mathbf{R}}x\,\delta p\,dx+F(t)\right]\frac{\partial}{\partial x}p_{st} \ , \tag{5.3}$$

with

$$L_0=-\frac{\partial}{\partial x}\left[\frac{d\Phi}{dx}+\bar{x}_0\right]+D\frac{\partial^2}{\partial x^2} \ , \tag{5.4}$$

$$\bar{x}_0=\epsilon\int_{\mathbf{R}}xp_{st}dx \ .$$

A formal solution δp is given by

$$\delta p(t,x)=-\int_{-\infty}^{t}e^{L_0(t-s)}\left[\epsilon\int_{\mathbf{R}}x\,\delta p(s,x)dx+F(s)\right]\times\frac{\partial}{\partial x}p_{st}(x)ds \ . \tag{5.5}$$

Thus the deviation of the expectation value of x over the empirical distribution $p(t,x)$ from its stationary value takes the form

$$\begin{aligned}\int_{\mathbf{R}}x\,\delta p\,dx&=-\int_{-\infty}^{t}ds\left[\epsilon\int_{\mathbf{R}}x\,\delta p(s,x)dx+F(s)\right]\times\int_{\mathbf{R}}xe^{L_0(t-s)}\frac{\partial}{\partial x}p_{st}(x)dx\\&=\int_{-\infty}^{t}ds\left[\epsilon\int_{\mathbf{R}}x\,\delta p(s,x)dx+F(s)\right]R(t-s) \ ,\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}R(t)&=\frac{-1}{D}\frac{d}{dt}\langle x(t)x(0)\rangle_{L_0(\bar{x}_0)}\\&=-\frac{1}{D}\frac{d}{dt}\langle Z(t)Z(0)\rangle_{L_0(\bar{x}_0)}\end{aligned} \tag{5.7}$$

with

$$\begin{aligned}Z(t)&=x(t)-\langle x\rangle\\&=x(t)-\int_{\mathbf{R}}xp_{st}dx \ .\end{aligned} \tag{5.8}$$

Here we followed the same procedure as we did in deriving Eq. (4.15) by making use of Eqs. (4.8) and (4.9) with D_i and W_{st} replaced by $d\Phi/dx+\bar{x}_0$ and p_{st}, respectively.

Putting

$$G(t)=\int_{\mathbf{R}}x\,\delta p(t,x)dx \ , \tag{5.9}$$

Eq. (5.6) is rewritten as

$$G(t)=\int_{-\infty}^{t}[\epsilon G(s)+F(s)]R(t-s)ds \ . \tag{5.10}$$

Since this equation is a linear integral equation for $G(t)$, the solution can be easily found by taking the Fourier transform of Eq. (5.10). Defining the Fourier transforms of $G(t)$, $R(t)$, and $F(t)$

$$\tilde{G}(\omega)=\int_{-\infty}^{\infty}G(t)e^{-i\omega t}dt \ , \tag{5.11}$$

$$\tilde{R}(\omega)=\int_0^\infty R(t)e^{-i\omega t}dt\ , \tag{5.12}$$

$$\tilde{F}(\omega)=\int_{-\infty}^\infty F(t)e^{-i\omega t}dt\ , \tag{5.13}$$

the Fourier transform of Eq. (5.10) reads

$$\tilde{G}(\omega)=[\epsilon\tilde{G}(\omega)+\tilde{F}(\omega)]\tilde{R}(\omega)\ . \tag{5.14}$$

Then $\tilde{G}(\omega)$ can be solved in the form

$$\tilde{G}(\omega)=\frac{\tilde{F}(\omega)\tilde{R}(\omega)}{1-\epsilon\tilde{R}(\omega)}\ . \tag{5.15}$$

From Eqs. (4.25) and (5.15) we obtain the complex susceptibility $\chi(\omega)$ for our system in the following forms:

$$\chi(\omega)=\frac{\tilde{G}(\omega)}{\tilde{F}(\omega)}=\frac{\tilde{R}(\omega)}{1-\epsilon\tilde{R}(\omega)}=\frac{\frac{1}{D}[\langle Z^2\rangle-i\omega\gamma(\omega)]}{1-\frac{\epsilon}{D}[\langle Z^2\rangle-i\omega\gamma(\omega)]}\ . \tag{5.16}$$

Here we used the fact that $\tilde{R}(\omega)$ can be rewritten after an integration by parts as

$$\tilde{R}(\omega)=\frac{1}{D}\langle Z^2\rangle_{\rm st}-\frac{i\omega}{D}\gamma(\omega)\ , \tag{5.17}$$

with

$$\gamma(\omega)=\int_0^\infty e^{-i\omega t}\langle Z(t)Z(0)\rangle_{L_0(\bar{x}_0)}dt\ , \tag{5.18}$$

and $\langle\ \rangle_{\rm st}$ denotes the average over the distribution $p_{\rm st}$. We note that $\gamma(\omega)$ is known to be expressible in terms of the eigenvalues $\{-\lambda_\mu\}$ and eigenfunctions $\{P_\mu\}$ for the Fokker-Planck operator $L_0(\bar{x}_0)$ (Appendix D):

$$\gamma(\omega)=\sum_{\mu=1}^{\infty}\frac{1}{i\omega+\lambda_\mu}\left[\int_{\mathbb{R}}[x-\langle x\rangle_{\rm st}]P_\mu(x)dx\right]^2\ . \tag{5.19}$$

From Eq. (5.16) we can easily obtain an expression for the static susceptibility:

$$\chi(0)=\frac{\frac{1}{D}\langle Z^2\rangle_{\rm st}}{1-\frac{\epsilon}{D}\langle Z^2\rangle_{\rm st}}\ . \tag{5.20}$$

The real and imaginary parts of $\chi(\omega)$ defined in Eq. (4.26) are given as

$$\chi'(\omega)=\frac{\frac{1}{D}\left[\left[1-\frac{\epsilon}{D}[\langle Z^2\rangle_{\rm st}-\omega\gamma_s(\omega)]\right][\langle Z^2\rangle_{\rm st}-\omega\gamma_s(\omega)]-\frac{\epsilon}{D}\omega^2\gamma_c(\omega)^2\right]}{\left[1-\frac{\epsilon}{D}[\langle Z^2\rangle_{\rm st}-\omega\gamma_s(\omega)]\right]^2+\left[\frac{\epsilon}{D}\omega\gamma_c(\omega)\right]^2}\ , \tag{5.21}$$

$$\chi''(\omega)=\frac{\frac{1}{D}\omega\gamma_c(\omega)}{\left[1-\frac{\epsilon}{D}[\langle Z^2\rangle_{\rm st}-\omega\gamma_s(\omega)]\right]^2+\left[\frac{\epsilon}{D}\omega\gamma_c(\omega)\right]^2}\ , \tag{5.22}$$

with

$$\begin{aligned}\gamma_c(\omega)&={\rm Re}\gamma(\omega)=\int_0^\infty \cos\omega t\langle Z(t)Z(0)\rangle_{L_0(\bar{x}_0)}dt\ ,\\ \gamma_s(\omega)&=-{\rm Im}\gamma(\omega)=\int_0^\infty \sin\omega t\langle Z(t)Z(0)\rangle_{L_0(\bar{x}_0)}dt\ .\end{aligned} \tag{5.23}$$

Combining Eqs. (4.32) and (5.22), we find the power spectrum of the order-parameter fluctuations y_∞ as

$$\mathcal{S}(\omega)=\frac{1}{\pi}\frac{\gamma_c(\omega)}{\left[1-\frac{\epsilon}{D}[\langle Z^2\rangle_{\rm st}-\omega\gamma_s(\omega)]\right]^2+\left[\frac{\epsilon}{D}\omega\gamma_c(\omega)\right]^2}\ . \tag{5.24}$$

Then the correlation function of the fluctuations is given by the inverse Fourier transform as

$$\langle y_\infty(t)y_\infty(0)\rangle=\int_{-\infty}^\infty \mathcal{S}(\omega)e^{i\omega t}d\omega=\frac{1}{\pi}\int_{-\infty}^\infty\frac{\gamma_c(\omega)e^{i\omega t}d\omega}{\left[1-\frac{\epsilon}{D}[\langle Z^2\rangle_{\rm st}-\omega\gamma_s(\omega)]\right]^2+\left[\frac{\epsilon}{D}\omega\gamma_c(\omega)\right]^2}\ . \tag{5.25}$$

We note that in the limit $\epsilon\to 0$, $\mathcal{S}(\omega)$ approaches the power spectrum of the fluctuations $\gamma_c(\omega)/\pi$ for a single independent oscillator without any couplings with the others, as expected. Furthermore, since we have from Eq. (5.22)

$$\lim_{\omega\to 0}\frac{D\chi''(\omega)}{\omega}=\frac{\gamma_c(0)}{\left[1-\frac{\epsilon}{D}\langle Z^2\rangle_{\rm st}\right]^2}\ , \tag{5.26}$$

using Eqs. (4.35) and (5.20), we obtain the relaxation time

τ of the fluctuations as

$$\tau=\frac{\gamma_c(0)}{\langle Z^2\rangle_{st}\left[1-\frac{\epsilon}{D}\langle Z^2\rangle_{st}\right]} . \tag{5.27}$$

It is noted that the positivity of $\gamma_c(\omega)$ holds from Eq. (5.19) and that the denominator is also positive, so far as a stable stationary state is concerned.

Since, as shown in Sec. II, bifurcations of solutions to the nonlinear Fokker-Planck equation take place at the point where $1-(\epsilon/D)\langle Z^2\rangle_{st}=0$ holds, it turns out from Eqs. (5.20) and (5.27) that both the static susceptibility $\chi(0)$ and the relaxation time τ diverge as

$$\chi(0)\sim\tau\sim\left[1-\frac{\epsilon}{D}\langle(x-\langle x\rangle_{st})^2\rangle_{st}\right]^{-1} \tag{5.28}$$

on approaching the phase transition point. Here we noted that neither $\gamma_c(0)$ nor $\langle Z^2\rangle_{st}$ exhibits a singular behavior at the phase transition point, since they are associated with the linear Fokker-Planck equation characterized by L_0. The above result confirms, from a general stochastic-theoretical point of view, the widely accepted belief on a critical slowing down for thermodynamic phase transitions of mean-field type[1] that the degree of the divergence of the relaxation time is to the same extent as that of the static susceptibility.

VI. SUMMARY AND DISCUSSIONS

We have investigated the dynamical aspect of nonthermodynamic phase transitions of mean-field type taking place in stochastic systems of infinitely many coupled anharmonic oscillators in a rigorous and systematic way, on the basis of the nonlinear Fokker-Planck equation. Studies have been performed from the two kinds of points of view—an asymptotic approach to equilibrium and a critical anomaly of dynamic responses of the system. We will give the summary and discussions of our results of these two kinds of studies in turn.

Our main results with regard to the first part of the study are the presentations of the H theorem and its applications to the problem of stability analysis, together with a stability criterion expressed in terms of the second-order variation of the H functional of the system. We have confirmed that even for infinite stochastic systems exhibiting phase transitions, an H functional can be constructed to show the validity of an H theorem.[31] It has been shown that the H functional plays an essential role as a Lyapunov functional in conducting a stability analysis of bifurcating stationary states of the system. Although a stability analysis of a nonlinear equation giving rise to bifurcations is usually conducted by means of a linear stability analysis involving a study of the eigenvalues of the linearized equation, we took in the present article an alternative approach, i.e., a method of nonlinear stability analysis making full use of the concept of a Lyapunov functional. The relation between the two approaches for the present nonlinear Fokker-Planck equation will be studied elsewhere. To summarize the role of the H functional in light of the stability analysis involving a Lyapunov functional, the H functional [(3.1)–(3.3)] has been shown to be a Lyapunov functional of the nonlinear Fokker-Planck equation (2.7), ensuring global stability of a stationary solution of the nonlinear Fokker-Planck equation, in the case where that stationary solution is a uniquely existing one. On the other hand, when the nonlinear Fokker-Planck equation admits more than one stationary solution, the H functional has been shown to be entitled to become a Lyapunov functional only in a certain neighborhood of a stationary state making the second-order variation $\delta^{(2)}H[\delta p,\delta p]$ positive definite, which ensures local stability of that stationary state.

The H functional also proves useful in exploring the dynamical behavior of the nonequilibrium process of the system. Since, as is stated in Sec. III, the H functional constructed in the present article is proportional to the generalized Helmholtz free energy of a nonequilibrium state, it follows that, owing to the explicitly expressed time evolution equation for p (nonlinear Fokker-Planck equation), the time change of the free energy of the system is explicitly given and, in principle, amenable to calculation, and furthermore the free energy cannot increase at any moment in the course of the time evolution of the system. The above situation makes a sharp contrast with thermodynamic systems, where the same kind of rule for the free energy (principle of free-energy minimum) holds, because it is difficult, in general, not only to define nonequilibrium free energy for thermodynamic systems but to derive the rule from a first principle such as the Liouville equation rather than by invoking the second law of thermodynamics. In this connection, it is worthwhile to note that an increment or difference of the values of the H functional at two times in the course of the time evolution of the system turns out to be the negative of the so-called entropy production. This can be easily seen by observing the relationship (3.9) between the H functional and the generalized free energy $\mathcal{F}$ and by noting a possible analogy to the thermodynamic relation between the free energy F and the entropy production d_iS:

$$d_iS=dS-\frac{dU}{T}=\frac{-1}{T}dF . \tag{6.1}$$

Accordingly, the H theorem (3.12) implies the positivity of the entropy production[31–34] during an irreversible evolution of the system and we see that the amount of entropy production is calculable for any time interval during the course of the irreversible time evolution of the present infinite system, provided the time-dependent solution p of the nonlinear Fokker-Planck equation is solved. Furthermore, if we confine ourselves to the behavior of the irreversible process near equilibrium (linear region), we can elaborate more about the entropy production.[35] It can be shown that the Prigogine inequality or the law of entropy production rate minimum[31,32] holds near stable stationary states of our system. To be more precise, since, based on Eq. (3.44) and the linearization of Eq. (2.7), we can prove

$$\frac{d^2}{dt^2}\delta^{(2)}H[\delta p,\delta p]$$

$$=2\left\{\int_{\mathrm{R}}\frac{\left[\frac{\partial}{\partial t}\delta p\right]^2}{p_{\mathrm{st}}}dx-\frac{\epsilon}{D}\left[\int_{\mathrm{R}}x\left[\frac{\partial}{\partial t}\delta p\right]dx\right]^2\right\}, \tag{6.2}$$

it turns out from the expressions for the second-order variation $\delta^{(2)}H[\delta p,\delta p]$ Eqs. (3.44) and (3.48) that the stability condition given by the inequality (3.51) implies

$$\frac{d_i^2 S}{dt^2}=-\frac{d^2}{dt^2}\delta^{(2)}H\leqq 0 . \tag{6.3}$$

We now turn to discussing the problems of another type of dynamical behavior, which concerns the critical anomaly of the fluctuations. To investigate the dynamical critical phenomena of the present system, we have calculated dynamical responses of the order parameter to an externally driven field within the framework of linear-response theory. Noting that the fluctuation-dissipation theorem holds even in the thermodynamic limit, we have obtained correlation functions and power spectra of the order-parameter fluctuations in our system. Our main results are the presentation of the expressions for the generalized dynamical susceptibilities together with the spectra of the fluctuations of the system and the confirmation of the occurrence of a critical slowing down for the order-parameter fluctuations in the system, irrespective of the form of the potentials and accordingly of the kind of phase transitions, as long as the system undergoes a phase transition at the point characterized by Eq. (5.28). Critical slowing down has been shown to occur in such a way that the relaxation time of the order-parameter fluctuations diverges with the same power as that of the divergence of the generalized static susceptibility, as a phase transition point is approached. The present result of the qualitative nature of the critical slowing down coincides with that obtained by a well-known mean-field analysis of critical dynamics in stochastic Ising spin systems[23] based on the Glauber model.

As far as we know, rigorous approaches and thorough studies to obtain expressions for the critical divergence of relaxation time and dynamical responses of the order parameter of the infinite system undergoing phase transitions are very few except for the above-mentioned mean-field analysis of the Glauber model for the dynamical critical behavior of the Ising ferromagnet.[23] Kometani and Shimizu[14] studied the phase transitions in the present model system, making use of an approximated theory involving an interrelation between the evolutions of the fluctuations and the macroscopic order parameter, in which a Gaussian approximation for the fluctuations of each oscillator was employed. As stated in their paper, they avoided investigating the dynamical behavior of fluctuations in the neighborhood of the phase-transition points owing to the nature of their approximations. Desai and Zwanzig[10] also investigated the dynamical behavior of the present system for the case of the ferromagnetic potentials on the basis of a self-consistent dynamic mean-field theory, in which they introduced the nonlinear Fokker-Planck equation and derived a hierarchy of equations for the cumulant moments of the time-dependent distribution function. They discussed the linear response of the system and expected the occurrence of a critical slowing down based on a crude analysis. Results of thorough studies including the obtaining of generalized dynamical susceptibilities, spectra of the fluctuations, and relaxation time in explicit forms, however, were not given in their paper.

Although we have a qualitative agreement between the results of the present system and those of the mean-field Ising spin system with respect to the mean-field character of the critical slowing down for the relaxation time of the systems, there are some differences worth noting between the outcomes of the studies for the two systems. Major differences, which come from an arbitrariness in the choice of the potentials in our present approach, are those of the shapes of the relaxation functions together with the extent to which the theoretical considerations can shed light on the mechanisms for the occurrence of the critical slowing down.[36] While the mean-field Glauber model yields a single-exponential type of decay for the relaxation of the order parameter or for the correlation function of the fluctuations, the present system, which can be viewed as a model of continuous spin version of ferromagnets, if for example ferromagnetic potentials are adopted, has been shown to give rise to a multiexponential type of decay[37] for the relaxation, originating from the nonlinearities of the drift terms of the Langevin equation (2.1). The single-exponential type of decay for the fluctuations in the system described by the mean-field Glauber model comes from the fact that the time evolution of the averaged value of magnetization is governed by a one-dimensional ordinary nonlinear differential equation, the linearization of which yields a solution with a single-exponential form. On the other hand, in our treatment the underlying time evolution equation responsible for the bifurcations of the order parameter takes the form of a nonlinear Fokker-Planck equation. Since the linearization of this equation still involves nonlinearity in the variable x in its drift term, the relaxation or correlation function of the order-parameter fluctuations assumes a non-single-exponential type of decay. Since the present approach for treating the critical slowing down allows one a freedom of choice of potentials $\tilde{\Phi}(x)$ for the occurrence of bifurcations involving no broken symmetry, as well as symmetry-breaking instabilities in the system, we are left with the possibility of discussing phenomena of critical slowing down in several kinds of phase transitions.[38] In this sense, expressions for the non-single-exponential type of decay of fluctuations in the present infinite stochastic systems seem to be useful to discuss the dynamical behavior of systems exhibiting various kinds of phase transitions ranging from nonequilibrium to such thermodynamic phase transitions as found in some ferroelectric materials.

Finally, we note the importance of the role played by the fluctuation-dissipation theorem in the study of the critical behavior of fluctuations. Although the nonlinear Fokker-Planck equation (2.7) is available for the description of the fluctuations of each individual oscillator in the system, it provides us with no information on the behavior of the fluctuations of such macroscopic variables as the order parameters of particular concern. We are necessarily led to employ the fluctuation-dissipation theorem to obtain the dynamical behavior of the order-parameter fluctuations. In the present paper, the fluctuation-dissipation theorem extended to the infinite system has been presented by simply taking the $N \to \infty$ limit of an expression derived for a finite-dimensional system of Fokker-Planck dynamics. There, the existence of the $N \to \infty$ limit of the order-parameter fluctuations y_∞ has been assumed. It is shown by Dawson,[12] however, that the behavior of the critical fluctuations just at a phase transition point ($D = D_c$) goes beyond what is claimed by the commonly known central-limit theorem, which admits the $N^{-1/2}$ scaling of the fluctuations as in the definition of y_∞. Accordingly, the present result of the dynamical critical behavior of the fluctuations will not be allowed to apply just at a transition point, though the generalized susceptibilities will still remain meaningful at a stable transition point. In general, more detailed discussions will be needed to justify our procedures in obtaining the fluctuation-dissipation theorem for such infinite systems exhibiting phase transitions as the present case, where the order of taking the limit $N \to \infty$ and $t \to \infty$ becomes crucial and discussions about the central-limit theorem associated with the existence of the order-parameter fluctuations y_∞ are inevitable. Quite recently, in the case where the usual central-limit scaling for y_∞ holds and hence the fluctuations become a Gaussian process, conducting a more rigorous mathematical analysis of the fluctuations in our infinite system, we have been able to prove the fluctuation-dissipation theorem as claimed in the present work, to justify the present approach to obtain the critical slowing down of the order-parameter fluctuations. Details of such mathematical arguments have been omitted in this article and will be reported elsewhere.

APPENDIX A: $D \to \infty$ LIMIT OF EQ. (2.10)

To evaluate the value of $\partial/\partial x_0 \int_{\mathbb{R}} x p_{\rm st} dx$ in the limit $D \to \infty$, we note the identity given by Eq. (2.12). Since we are concerned with the evaluation at $x_0 = 0$ of the second moment of a symmetric potential $\Phi(x)$, it will suffice to prove

$$\frac{1}{D}\langle x^2\rangle_{\rm st} = \frac{1}{D}\frac{\int_{\mathbb{R}} x^2 e^{D^{-1}\Phi(x)}dx}{\int_{\mathbb{R}} e^{D^{-1}\Phi(x)}dx} \to 0 \quad (D \to \infty) . \tag{A1}$$

We assume $\Phi(x)$ to satisfy the following condition: There exist positive M, K_1, K_2, and an integer n (>1) such that

$$-K_2 x^{2n} - M \leqq \Phi(x) \leqq -K_1 x^{2n} + M . \tag{A2}$$

We note that when $\Phi(x)$ is a polynomial, whose term with the highest order is given by $-Kx^{2n}$ ($K > 0$), we can choose M, K_1, and K_2 for $\Phi(x)$ to satisfy the above condition. From the inequality (A2), it follows that

$$\frac{\int_{\mathbb{R}} x^2 e^{D^{-1}\Phi(x)}dx}{\int_{\mathbb{R}} e^{D^{-1}\Phi(x)}dx} \leqq \frac{\int_{\mathbb{R}} x^2 e^{D^{-1}[-K_1 x^{2n}+M]}dx}{\int_{\mathbb{R}} e^{D^{-1}[-K_2 x^{2n}-M]}dx} = \frac{e^{2M/D} D^{1/n}\int_{\mathbb{R}} y^2 e^{-K_1 y^{2n}}dy}{\int_{\mathbb{R}} e^{-K_2 y^{2n}}dy} . \tag{A3}$$

Here we made a change of variable $x = D^{1/2n}y$ for integration. Since

$$0 \leqq \overline{\lim_{D\to\infty}} \frac{\langle x^2\rangle_{\rm st}}{D} \leqq \overline{\lim_{D\to\infty}}\, e^{2M/D} D^{1/n-1} \frac{\int_{\mathbb{R}} y^2 e^{-K_1 y^{2n}}dy}{\int_{\mathbb{R}} e^{-K_2 y^{2n}}dy} = 0 , \tag{A4}$$

we have

$$\lim_{D\to\infty} \frac{\langle x^2\rangle_{\rm st}}{D} = 0 . \tag{A5}$$

APPENDIX B: $D \to 0$ LIMIT OF EQ. (2.10)

To see the behavior of $\langle x^2\rangle_{\rm st}/D$ in the limit $D \to 0$, we consider two cases for types of symmetric potentials.

Case I

We first deal with the case where $\Phi(x)$ $[= -\tilde{\Phi}(x) - \epsilon x^2/2]$ is given such that its global maximum is attained by more than one point. Let $x_i (i = 1, 2, \ldots, m)$ be such points. For simplicity we assume that around each x_i the potential $\Phi(x)$ can be expanded in a Taylor series as

$$\Phi(x) = A - a_i (x - x_i)^2 + \cdots \tag{B1}$$

with

$$a_i > 0, \quad A = \operatorname*{Max}_{-\infty < x < \infty} \Phi(x) .$$

Using the saddle-point method for integrations involved and taking the limit $D \to 0$ yields

$$\langle x^2\rangle_{\rm st} \to \frac{\left[\exp\frac{A}{D}\right]\sum_{i=1}^m x_i^2 \left[\frac{D\pi}{a_i}\right]^{1/2}}{\left[\exp\frac{A}{D}\right]\sum_{i=1}^m \left[\frac{D\pi}{a_i}\right]^{1/2}} = \frac{\sum_{i=1}^m x_i^2 \left[\frac{1}{a_i}\right]^{1/2}}{\sum_{i=1}^m \left[\frac{1}{a_i}\right]^{1/2}} > 0 . \tag{B2}$$

We thus have in the limit $D \to 0$

$$\frac{\langle x^2\rangle_{\rm st}}{D} \to \infty . \tag{B3}$$

Case II

Here, $\Phi(x)$ is assumed to have a single maximum at $x=0$ and further to be such that the quadratic term in a Taylor expansion of $\Phi(x)$ around $x=0$ is given by

$$\Phi(x)=\left[\frac{\alpha}{2}-\frac{\epsilon}{2}\right]x^2+\cdots \tag{B4}$$

with

$$0<\alpha<\epsilon .$$

Hence using the saddle-point method, we obtain an asymptotic expression for the variance $\langle x^2\rangle_{\rm st}$, when D is sufficiently small

$$\langle x^2\rangle_{\rm st}\sim\frac{D}{\epsilon-\alpha} . \tag{B5}$$

Finally we have

$$\lim_{D\to 0}\frac{\langle x^2\rangle_{\rm st}}{D}=\frac{1}{\epsilon-\alpha}>\frac{1}{\epsilon} . \tag{B6}$$

APPENDIX C: PROOF OF (iv) IN EQ. (3.21)

We examine the asymptotic behavior of $\langle x\rangle_{\rm st}$ as a function of x_0 for the ferromagnetic potentials in the limit $x_0\to\infty$. Let ϵ' be any small positive number satisfying $b>\epsilon'>0$. With ϵ' fixed, it is easily seen that

$$x>x'\equiv\left[\frac{x_0}{b-\epsilon'}\right]^{1/3} \tag{C1}$$

implies

$$\frac{\Phi_{\rm FM}(x)+x_0x}{D}=\frac{ax^2-bx^4+x_0x}{D}<\frac{ax^2-\epsilon'x^4}{D} . \tag{C2}$$

Hence we have

$$\int_{-\infty}^{\infty}e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx\geqq\int_0^{x'}e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx , \tag{C3}$$

$$\begin{aligned}\int_{-\infty}^{\infty}xe^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx&\leqq\int_0^{\infty}xe^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx\\&\leqq\int_0^{x'}x'e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx+\int_{x'}^{\infty}xe^{D^{-1}(ax^2-\epsilon'x^4)}dx\\&\leqq x'\int_0^{x'}e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx+\int_0^{\infty}xe^{D^{-1}(ax^2-\epsilon'x^4)}dx .\end{aligned} \tag{C4}$$

Thus we obtain

$$\begin{aligned}\langle x\rangle_{\rm st}=\frac{\int_{-\infty}^{\infty}xe^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx}{\int_{-\infty}^{\infty}e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx}&\leqq\frac{x'\int_0^{x'}e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx+\int_0^{\infty}xe^{D^{-1}(ax^2-\epsilon'x^4)}dx}{\int_0^{x'}e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx}\\&=x'+\frac{\int_0^{\infty}xe^{D^{-1}(ax^2-\epsilon'x^4)}dx}{\int_0^{x'}e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx} .\end{aligned} \tag{C5}$$

By noting

$$\int_0^{x'}e^{D^{-1}[\Phi_{\rm FM}(x)+x_0x]}dx\geqq e^{x_0/D}\int_1^{x'}e^{D^{-1}\Phi_{\rm FM}(x)}dx \quad (x'>1) , \tag{C6}$$

it follows that for sufficiently large x_0,

$$\langle x\rangle_{\rm st}\leqq x'+e^{-x_0/D}\frac{\int_0^{\infty}xe^{D^{-1}(ax^2-\epsilon'x^4)}dx}{\int_1^{x'}e^{D^{-1}\Phi_{\rm FM}(x)}dx} , \tag{C7}$$

so that

$$\langle x\rangle_{\rm st}\leqq\left[\frac{x_0}{b-\epsilon'}\right]^{1/3}\quad(x_0\to\infty) . \tag{C8}$$

APPENDIX D: PROOF OF EQ. (5.19) (REFS. 3 AND 4)

To prove Eq. (5.19), it will suffice to consider a one-dimensional Fokker-Planck equation with a constant diffusion coefficient,

$$\frac{\partial}{\partial t}P(t,x)=LP(t,x) , \tag{D1}$$

with

$$L=-\frac{\partial}{\partial x}V(x)+D\frac{\partial^2}{\partial x^2} . \tag{D2}$$

Denoting the stationary solution of Eq. (D1) by $e^{-\Psi(x)}$, we transform the Fokker-Planck operator L to L_H:

$$L_H=e^{\Psi/2}Le^{-\Psi/2}=D\frac{\partial^2}{\partial x^2}-V_s(x) \tag{D3}$$

with

$$V_s(x)=\frac{1}{2}\frac{d}{dx}V(x)+\frac{1}{4D}V(x)^2 . \tag{D4}$$

It can be easily checked that L_H is an Hermitian operator. When $V_s(x)\to\infty$ as $|x|\to\infty$, the Schrödinger-type equation

$$L_H\psi_j = -\lambda_j\psi_j \tag{D5}$$

has a point spectrum and its orthonormal eigenfunctions $\{\psi_j\}_{j=0,\ldots}$ constitute a complete base of $L^2(R)$, where as seen from $L_H e^{-\Psi/2}=0$, $\psi_0=e^{-\Psi/2}$ is an eigenfunction with the eigenvalue $\lambda_0=0$. Thus the eigenfunctions for the Fokker-Planck operator L are given by

$$P_j = e^{-\Psi/2}\psi_j = \psi_0\psi_j \tag{D6}$$

with

$$LP_j = -\lambda_j P_j \ . \tag{D7}$$

Since the transition probability density $P(t,x:t',x')$ is given by

$$\begin{aligned} P(t,x:t',x') &= e^{L(t-t')}\delta(x-x') \\ &= e^{L(t-t')}\sum_j \psi_j(x)\psi_j(x') \\ &= e^{L(t-t')}\sum_j \psi_0(x)\psi_j(x)\psi_0(x')^{-1}\psi_j(x') \\ &= \psi_0(x)\psi_0(x')^{-1}\sum_j \psi_j(x)\psi_j(x')e^{-\lambda_j(t-t')} \\ & \qquad (t \geqq t') \ , \end{aligned} \tag{D8}$$

assuming $t \geqq t'$ without loss of generality, the density of the joint probability distribution in the stationary process reads

$$\begin{aligned} P_2(t,x,t',x') &= P(t,x:t',x')e^{-\Psi(x')} \\ &= \psi_0(x)\psi_0(x')\sum_j \psi_j(x)\psi_j(x')e^{-\lambda_j(t-t')} \ . \end{aligned} \tag{D9}$$

A two-time correlation function in the stationary process of a random variable $Z(x)$ is given by

$$\begin{aligned} \langle Z(x(t))Z(x(0))\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} dx\,dx' Z(x)Z(x')P_2(t,x,0,x') \\ &= \sum_{j=0}^{\infty} e^{-\lambda_j t}\left[\int_{\mathbb{R}} Z(x)\psi_0(x)\psi_j(x)dx\right]^2 \\ &= \sum_{j=0}^{\infty} e^{-\lambda_j t}\left[\int_{\mathbb{R}} Z(x)P_j(x)dx\right]^2 \ . \end{aligned} \tag{D10}$$

Substituting

$$Z(x) = x - \langle x\rangle_{st} \ , \tag{D11}$$

we find

$$\begin{aligned} \langle Z_t Z_0\rangle &= \sum_{j=0}^{\infty} e^{-\lambda_j t}\left[\int_{\mathbb{R}} (x-\langle x\rangle_{st})P_j(x)dx\right]^2 \\ &= \sum_{j=1}^{\infty} e^{-\lambda_j t}\left[\int_{\mathbb{R}} (x-\langle x\rangle_{st})P_j(x)dx\right]^2 \ . \end{aligned} \tag{D12}$$

The Fourier transform of the correlation function yields Eq. (5.19).

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