Percolational and fractal property of random sequential packing patterns in square cellular structures

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The problem for random packing or filling has been treated in many fields. In this paper we form random sequential packing patterns, filling metallic squares with integer length a on insulator substrates divided into square unit cells. We investigate the percolational and fractal property of the packing patterns, and clarify that the maximum critical percolation length a_c (=3 units length) of the packed squares is presented for the insulator-to-metal transition to take place on the random sequential packing textures. When $a > a_c$, no insulator-to-metal transition occurs even at the saturation coverage where no more squares can be filled without any overlap. However even in such patterns with $a > a_c$, the large percolation clusters possess fractal properties, and the fractal dimensions equal 1.94.

I. INTRODUCTION

It was reported that lattice percolation clusters[1-7] and patterns of deposited metal films[8-10] have fractal properties. In our previous paper[11] we presented that random textures shaped on square cellular structures or square meshes have fractal properties. The fractal dimension D_f of the large percolation clusters is equal to 1.9 for all the published cases.

In this paper we shape random sequential packing patterns on square cellular structures and investigate the percolational and fractal property of the packing patterns. Although the problem for random sequential packing or filling has been treated in many fields, the percolational behavior and fractal property of the patterns formed by random sequential packing are little known. We form random sequential packing patterns, filling orientated metallic squares with integer length a at random one by one without any overlap, but permitting any contact among the filled squares, on insulator substrates divided into unit square cells or meshes. Then the sides of the packed squares are just put on the cell boundaries of the square substrate. When $a=1$ the packing (saturation or jamming coverage) fraction p is equal to unity, and the insulator-to-metal transition takes place at a certain metallic volume (area) fraction p in the course to the saturation coverage. As a increases, the packing fraction decreases, but still the packing pattern is metallic at its saturation coverage. When the length a further increases and the probability of contact among the filled squares is very small, the pattern becomes an insulator even at the jamming coverage. It is interesting to find this maximum critical percolation length a_c for the insulator-to-metal transition and to investigate the fractal property of metallic clusters shaped on packing textures.

In Sec. II we present the percolational behavior of random sequential packing patterns shaped on square cellular structures or square meshes. In Sec. III we study the fractal property of the clusters on the patterns.

II. PERCOLATIONAL BEHAVIOR OF RANDOM SEQUENTIAL PACKING CLUSTERS

We form random sequential packing textures on square insulator substrates divided with $N \times N$ unit cells, where N is the length of the substrates. When a is unity and it is assumed that the percolational flow is transmitted only among nearest-neighbor cells, the critical volume (area) fraction v_c for the metal-insulator transition is equal to the critical percolation probability of the site percolation problem on square lattice. Many articles have been published on the critical percolation probability, and many references are given by Ziff and Sapoval.[12] They proposed a very efficient method of computer simulation to obtain critical percolation probabilities, and presented a very accurate value,

$$v_c = 0.592\,745\ . \tag{1}$$

Our computer simulation method is very simple, and we take as v_c the volume fraction when the largest percolation cluster formed on a substrate with 200×200 unit cells begins to reach one side of the substrate from the opposite side in the course of packing metallic squares. To examine the accuracy of our method, at first we apply the method to the case of $a=1$, and obtain the average value of 20 trials

$$v_c = 0.595\ , \tag{2}$$

which agrees well with (1).

For $a=2$ and 3 we obtain by the same method

$$v_c = 0.601\ , \tag{3}$$

and

$$v_c = 0.621\ , \tag{4}$$

respectively. However, for $a=4$ no insulator-to-metal transition or global percolation cluster takes place even at the saturation coverage, where the coverage fraction is

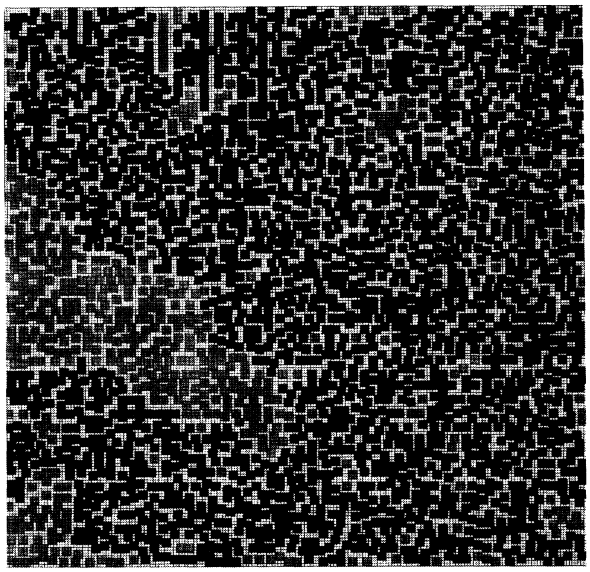

FIG. 1. An example of random sequential packing pattern of $a=3$ at the saturation coverage, generated on a square cellular structure with 200×200 unit cells. The largest cluster, which travels from one side to the opposite side, is represented by black. Other clusters or islands are indicated by fine meshes.

0.646.[13] We illustrate in Figs. 1 and 2 an example of pattern at the saturation coverage for $a=3$ and 4, respectively. In Fig. 1 we represented by black the largest percolation cluster and by fine meshes we represented other clusters. We can see that the largest cluster does not localize and travels from one side to the opposite side. In Fig. 2 we indicate by black the largest and second largest cluster, and recognize that the two greatest clusters are not connected and localize. We examine this behavior using 20 samples. Hence, when we express by a_c the maximum critical percolation length of the filled square, it is clear that

$$a_c=3\ . \tag{5}$$

TABLE I. Packing and critical volume (area) fraction.

Length of packed square a	1	2	3	4
Packing fraction P	1.000	0.749	0.681	0.646
Critical volume (area)				
Fraction v_c	0.595	0.601	0.621	
Standard deviation of v_c	0.012	0.007	0.006	

In Table I we list the packing fraction[13] and the critical volume fraction v_c against the length a of the filled squares. The values of v_c are the averages of 20 samples.

As shown in Table I, the packing fraction (0.646) for $a=4$ is greater than v_c (0.595, 0.601) for $a=1$ and 2. Therefore, the insulator-to-metal transition occurs on a packing pattern for $a=4$ at the saturation coverage when the packed squares with $a=4$ are divided into squares with $a=1$ or 2, and rearranged.

III. FRACTAL PROPERTY OF RANDOM SEQUENTIAL PACKING PATTERNS

The fractal dimension D_f is defined in many forms.[1] A relation for D_f of the two-dimensional problem is given by

$$P=cA^{D_f/2}\ , \tag{6}$$

where P is the perimeter of fractal objects, c is the proportional contact, and A is the area. Lovejoy[14] used (6) to estimate the fractal dimension of cloud and rain area boundaries, and their D_f's are equal to 1.35. We applied (6) in

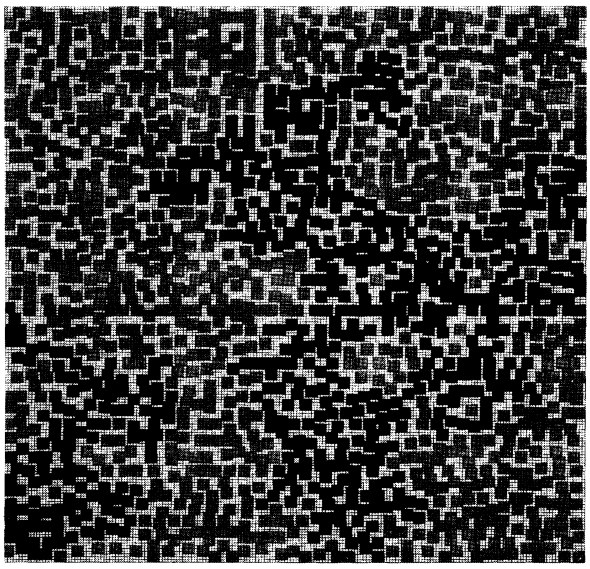

FIG. 2. An example of random sequential packing pattern of $a=4$ at the saturation coverage. The two largest clusters, which are separate, are represented by black.

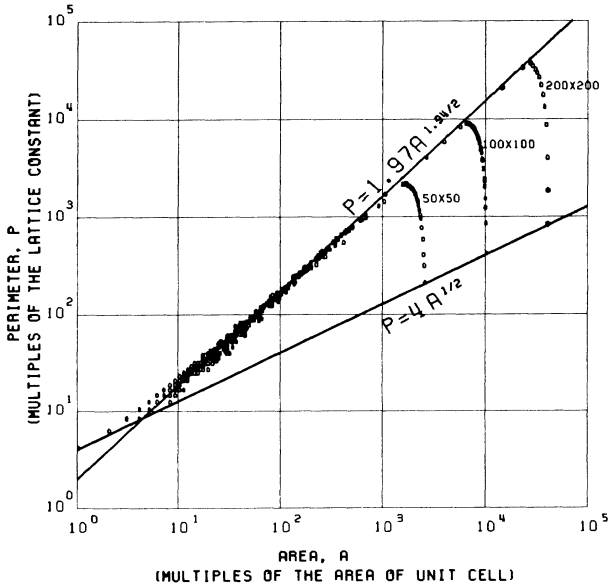

FIG. 3. Scatter plot of the perimeter P vs area A for the clusters of $a=1$.

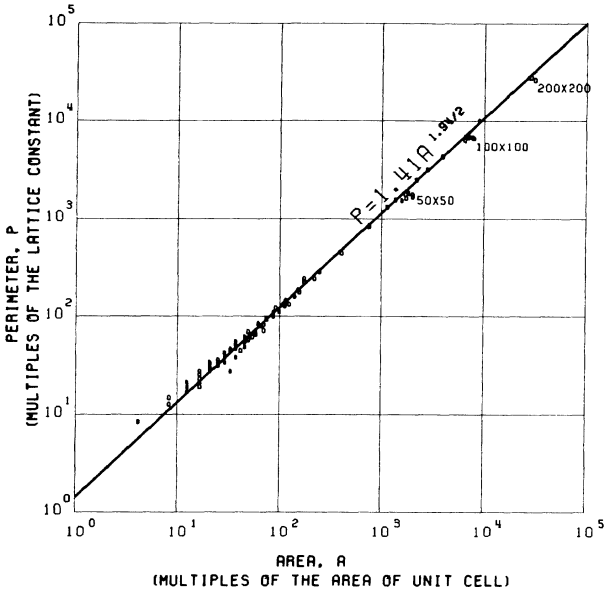

FIG. 4. $a=2$.

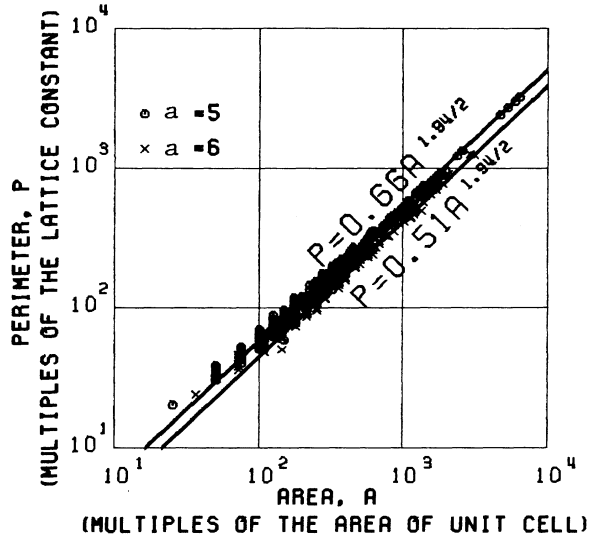

FIG. 6. $a=5$ and 6.

our previous paper[11] with respect to the continuum percolation problem. Here also we use (6).

We plot in Fig. 3 P against A of percolation clusters for $a=1$. Each datum point in the figure corresponds to the area and perimeter of a single cluster, and clusters touching the boundary have also been taken into account. The numbers of unit cells of the substrates are indicated in Fig. 3 as 50×50, 100×100, and 200×200. More than eight randomly generated configurations have been used to obtain Figs. 3–8. As shown in Fig. 3, there are three stages in the relation between P and A. We already expressed these three stages in our previous paper,[11] but restate them again here. In the first stage where the percolation clusters are small and localized, the data points greatly scatter, and we cannot determine a definite D_f for the data. In the second stage, the metallic area fraction is close to the critical fraction v_c and large clusters occur. In this stage we derive D_f and relation (8) graphically as follows:

$$D_f=1.94\ , \tag{7}$$

and

$$P=1.97A^{1.94/2}\ . \tag{8}$$

The fractal dimension (1.94) agrees well with values (1.9) previously published.[1–11]

In the third stage where the metallic volume fraction is

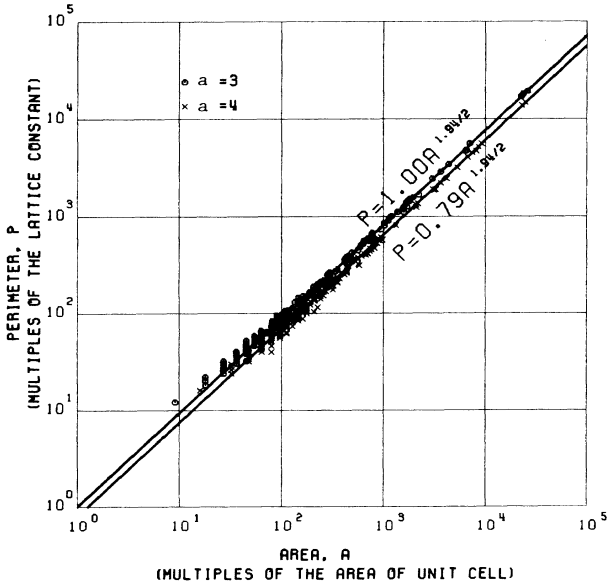

FIG. 5. $a=3$ and 4.

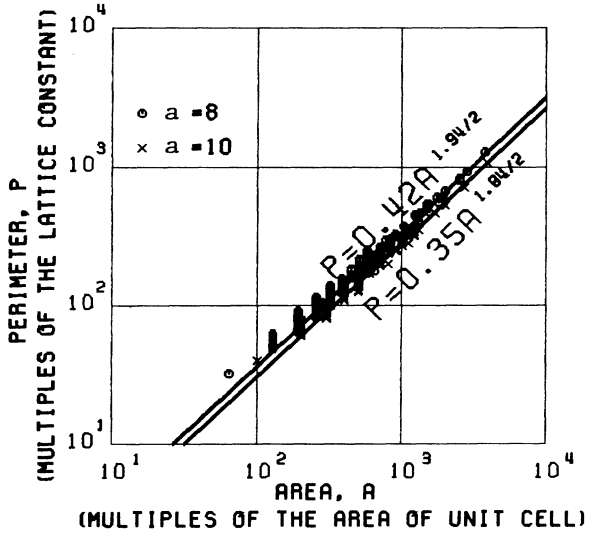

FIG. 7. $a=8$ and 10.

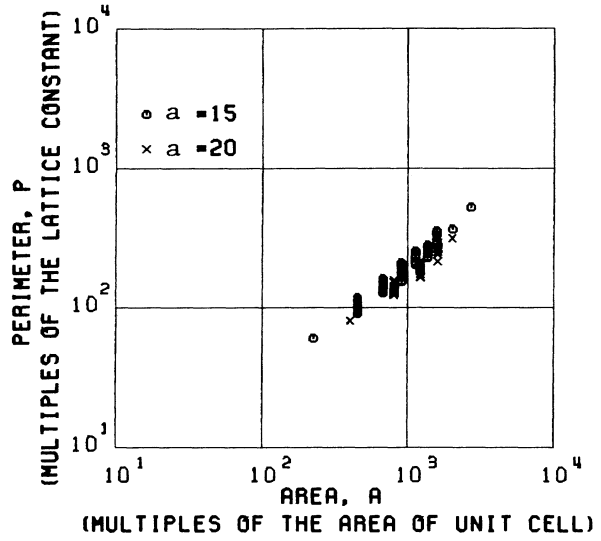

FIG. 8. $a=15$ and 20.

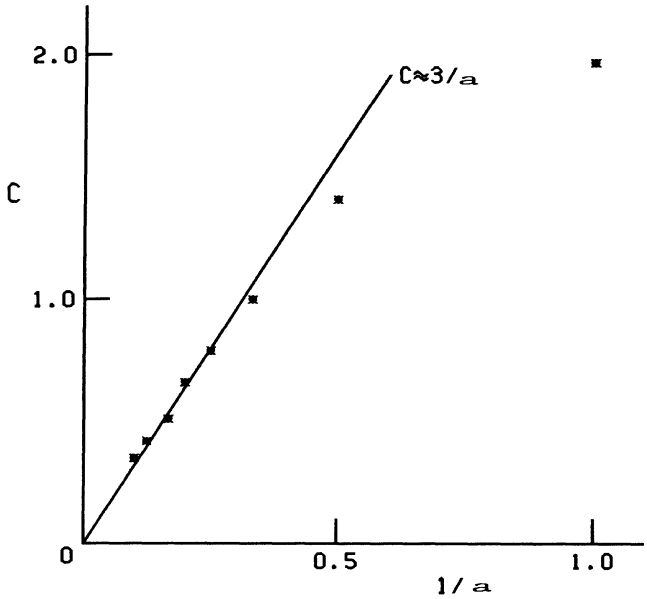

FIG. 9. Relation between c and $1/a$.

beyond v_c, the perimeter of the largest cluster rapidly decreases with an increase of the area, D_f also decreases, and at last it becomes unity when the surfaces of the substrates are covered overall by metal films. These three stages appear independently of the numbers of unit cells in the substrates, though the sizes of the greatest clusters depend on the cell numbers.

In Fig. 4 we plot the same result of $a=2$. As indicated in Fig. 4, D_f in the second stage is the same as $a=1$, but the proportional constant c decreases from 1.97 to 1.41, and the third stage, where the perimeters become small with an increase of the area, little appears.

In Fig. 5 we present the same results of $a=3$ and 4. As shown in Fig. 5, the third stage which appears in $a=1$ and 2 perfectly disappears. Although the percolational behaviors of $a=3$ and 4 are quite different, the difference among the plotted data are unexpectedly small, and the fractal dimensions (1.94) of the large percolation clusters are equal. A small discrepancy appears only in the change of c, and c decreases from 1 to 0.79.

We plot in Figs. 6 and 7 the same data for $a=5,6$ and $a=8,10$, respectively. The fractal dimensions of large clusters are equal to 1.94, although c decreases. When the length of the filled squares becomes increasingly large and equal to 15 or 20, even the second stage where large percolation clusters grow, vanishes, as shown in Fig. 8. The data points fluctuate and D_f has no definite values.

In Fig. 9 we plot the proportional constant c against $1/a$. As shown in Fig. 9 the relation between c and $1/a$ is

$$c \cong 3/a \quad \text{for } a \geqq 2 \ . \tag{9}$$

Therefore, the relation of P and A for the large percolation clusters is empirically presented by

$$P \cong \frac{3}{a} A^{1.94/2} \quad \text{for } a \geqq 2 \ . \tag{10}$$

IV. CONCLUSIONS

In this paper we shaped by computer simulation two-dimensional packing patterns, filling orientated metallic squares with integer length a at random one by one without any overlap, permitting contacts, on the cells of insulator substrates divided into square unit cells. We investigated the percolational and fractal property of the packing textures. The results show the following.

The critical volume or area fractions against a are listed in Table I. When a is more than three, no insulator-to-transition occurs in the packing textures even at their saturation coverages. The large percolation clusters have fractal properties, they do not lose the properties until a becomes considerably greater than three, and the fractal dimensions D_f are equal to 1.94, which agrees well with values previously published. As a increases, only the proportional constant c of Eq. (6) with the same D_f decreases, and c is 1.0 and 0.79 for $a=3$ and 4, respectively. In this case it is reflected not on D_f but on the small difference of c whether the insulator-to-metal transition takes place or not. Equation (10) expresses the relation between the perimeter P and area A of large percolation clusters for $a \geqq 2$.

ACKNOWLEDGMENTS

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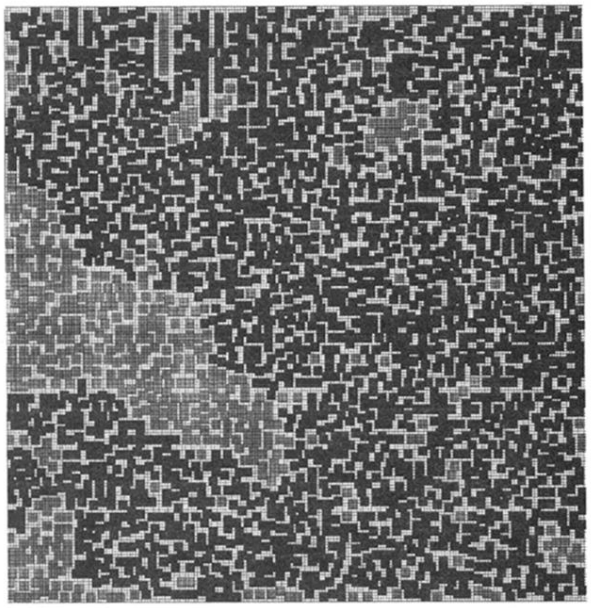

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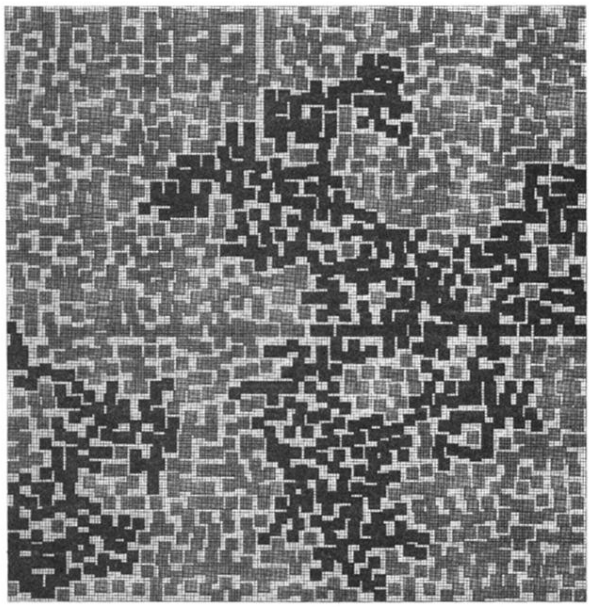

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