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Impulsive motion of particles and polarization response of a plasma in a magnetic field

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A percussion on particles moving in a uniform magnetic field induces a distribution of charges in a plasma. The accrued charge tensor $q_{ik}(\mathbf{x},t)$, which is the electric charge that has passed from $t=0$ to t through the unit area normal to x_i due to a percussion of unit strength exerted in the x_k direction at $\mathbf{x}=0$ and $t=0$, is computed. The tensor q_{ik} is equivalent to the space-time polarization response of plasma electrodynamics. In this derivation, neither the Vlasov equation nor Fourier-Laplace transforms are employed. As an application, *a priori* bounds (i.e., independent of the distribution functions and of Laplace transforms) for the growth in time of plane-wave modes of a magnetized plasma are obtained by an operational method. The connection of the impulsion with statistical theoretical concepts is also noted. The fluctuation-dissipation theorem is given in a classical-physics version and it is found that the correlator of microscopic currents for noninteracting particles $\langle j_i j_k \rangle^0_{\mathbf{x},t}$ of a nonequilibrium plasma is related to q_{ik} through $(\partial/\partial t)q_{ik}=(\delta/\delta E)\langle j_i j_k\rangle^0_{\mathbf{x},t}$.

I. INTRODUCTION

The polarization field $\mathbf{P}(\mathbf{x},t)$ and the electric field $\mathbf{E}(\mathbf{x},t)$ in a plasma are connected in the linear approximation by $\mathbf{P}=\hat{\chi}\cdot\mathbf{E}$, where $\hat{\chi}$ is the susceptibility integral tensor operator[1,2]

$$\hat{\chi}_{ij}E_j=\int_0^\infty d\tau\int d^3\xi\, h_{ij}(\xi,\tau)E_j(\mathbf{x}-\xi,t-\tau)\ . \qquad (1)$$

This equation expresses the most general linear, causal, nonlocal relationship between two quantities $\mathbf{E}$ and $\mathbf{P}$ in any time invariant and spatially uniform system. Here $h_{ij}(\mathbf{x},t)$ is the polarization response tensor whose $\mathbf{k},\omega$ Fourier-Laplace transform $\chi_{ij}(\mathbf{k},\omega)=\mathcal{F}_{\mathbf{k}}\mathcal{L}_\omega h_{ij}(\mathbf{x},t)$ (with $\omega=ip$) is the complex susceptibility tensor. While the tensor $\chi_{ij}(\mathbf{k},\omega)$ derived from Vlasov-Maxwell equations[3] for collisionless plasmas is well known,[4,5] the spatiotemporal counterpart $h_{ij}(\mathbf{x},t)$ has seldom been examined, as most questions have been studied by decomposition into plane waves, while the literature for solutions with other geometrical dependences is scarce. We have noted that a direct derivation of $\mathbf{h}$ by physical arguments in $\mathbf{x},t$ can be obtained considering the charges induced by the impulsive motion of particles which suffer a percussion concentrated at $\mathbf{x}=0$ and $t=0$.[6] The method is also useful for the study of electrostatic modes in certain nonhomogeneous plasmas.[7] The peculiarity of this method is that it does not rely on Fourier-Laplace transforms and does not use the plasma kinetic equation. It brings forth in a clear way the basic elements of plasma polarization mechanisms. Given the central position of χ in plasma electrodynamics and other topics of plasma theory, like the foundations of kinetic equations and fluctuations, it seems convenient to have more than one independent approach to compute this tensor available, in this case through the knowledge of $\mathbf{h}$. Moreover, the response tensor $\mathbf{h}$ is important also in certain nonlinear investigations. As an example, we may quote the interesting functional relationship which has been found between the Hamiltonian of ponderomotive forces and $\mathbf{h}$.[8]

The purpose of this paper is to apply the percussion method to obtain $\mathbf{h}$ for a plasma in a uniform magnetic field. The knowledge of $\mathbf{h}(\mathbf{x},t)$ permits the setting up of the integrodifferential equation for any three-dimensional electrodynamical problem

$$c^2\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}+4\pi(\partial/\partial t)\hat{\sigma}\cdot\mathbf{E}+(\partial^2/\partial t^2)\mathbf{E}=-4\pi(\partial/\partial t)\mathbf{J}_e\ , \qquad (2)$$

where $\hat{\sigma}=\partial\hat{\chi}/\partial t$ is the conductivity integral operator and $\mathbf{J}_e(\mathbf{x},t)$ represents possible external currents, or the related integral equation of the quasi-electrostatic approximation

$$(\mathbf{I}+4\pi\hat{\chi})\cdot\mathbf{E}=-4\pi\int_0^t\mathbf{J}_e(\mathbf{x},t')dt'\ . \qquad (3)$$

Further on, we analyze the case $\mathbf{E}(\mathbf{x},t)=\mathbf{E}(t)\exp(i\mathbf{k}\cdot\mathbf{x})$ and derive *a priori* bounds for the growth in time of $\mathbf{E}(t)$. By *a priori* we mean bounds valid for all possible distribution functions of the plasma. Since these bounds are derived from properties of response functions obtained without the use of Laplace transforms, they may also be called pre-Laplace bounds. It is worth noting that, in the absence of these bounds, one cannot in principle discard the possibility of existence of solutions that grow too rapidly in time to be Laplace transformable. Backus noted the problem and solved it for the case of Langmuir modes, in the absence of a magnetic field, by deriving the *a priori* growth bound[9] $\exp(\omega_p t)$ (ω_p is the plasma frequency). No instability can grow faster than this and, in addition, it follows that all solutions fall within the domain of the Laplace method. We extend here this result by giving bounds for several branches of the spectrum of plasma modes in a magnetic field.

There is still another method which may give $h_{ik}(\mathbf{x},t)$ working in the spatiotemporal representation only. This is provided, in the theory of statistical fluctuations, by the fluctuation-dissipation theorem.[10] In thermal equilibrium the space-time correlation function of microscopic currents

$$\langle j_i j_k \rangle^0_{\mathbf{x},t} \equiv \langle j_i(\mathbf{x}_1,t_1) j_k(\mathbf{x}_2,t_2) \rangle$$

(with $\mathbf{x}=\mathbf{x}_1-\mathbf{x}_2$, $t=t_1-t_2$) for noninteracting particles is related to the conductivity response tensor $\sigma_{ik}(\mathbf{x},t) = (\partial/\partial t) h_{ik}(\mathbf{x},t)$ by[11]

$$\langle j_i j_k \rangle^0_{\mathbf{x},t} = T_e \sigma_{ik}(\mathbf{x},t) \;,$$
$$j_i = \sum_a e_a v_i(t) \delta(\mathbf{x}-\mathbf{x}_a(t)) \;. \tag{4}$$

Here the sum runs over all the particles a of a given volume, the bracket indicates the statistical average, and T_e is the temperature (in energy units). Therefore, we shall also consider the relationship between the percussion and the fluctuation approaches in obtaining $\sigma_{ik}(\mathbf{x},t)$ and their equivalence. Since the percussion method may be applied to nonequilibrium distribution functions, we shall extend the fluctuation-dissipation formula (4) to cover these cases within the limits of the correlationless approximation. Finally, we shall explicitly show the equivalence of $\mathbf{h}(\mathbf{x},t)$ derived by the percussion method and the standard complex susceptibility $\boldsymbol{\chi}(\mathbf{k},\omega)$.

II. IMPULSIVE MOTION

Delta-like electric fields $\mathscr{E}_i^{(k)} = \mathscr{E}^{(k)} \delta_{ik} \delta(\mathbf{x}) \delta(t)$, $k=1,2,3$, generate impulsive motion[12] of charges that pass through $\mathbf{x}=0$ at $t=0$. Particles with charge e and mass m suffer a velocity jump $\Delta v_i^{(k)} = (e/m) \mathscr{E}^{(k)} \delta_{ik} \delta(\mathbf{x})$ without changes in their position. The strength of the percussions is denoted by $\mathscr{E}^{(k)}$. We assume that the plasma is neutral and focus the attention on one species of particles only. The velocity distribution function at the origin, at $t=0^+$, is $\tilde{f}(\mathbf{v}_0) = f_0(\mathbf{v}_0 - \Delta\mathbf{v}^{(k)})$. Here f_0 is normalized to one and n_0 is the particle density. Thus the corresponding charge density per unit of velocity space is

$$\rho^{(k)}(\mathbf{x}_0,\mathbf{v}_0,0) = e n_0 f_0(\mathbf{v}_0) + e n_0 [f_0(\mathbf{v}_0 - \Delta\mathbf{v}^{(k)}) - f_0(\mathbf{v}_0)] \;. \tag{5}$$

The first term does not contribute to the net charge since, in the total balance, it is neutralized by similar additions from all the other species. Considering the first order in $\mathscr{E}^{(k)}$, we have

$$\rho^{(k)}(\mathbf{x}_0,\mathbf{v}_0,0) = -e n_0 \, \Delta v_i^{(k)} (\partial/\partial v_{0i}) f_0 \;,$$

where $\mathbf{x}_0, \mathbf{v}_0$ denote the initial position and velocity of a particle which is at $\mathbf{x}=\mathbf{x}_0 + \int_0^t \mathbf{v}(t')dt'$, with velocity $\mathbf{v}(t)$ at the present time. We ignore here, for simplicity, the contribution to $\Delta\mathbf{v}^{(k)}$ of the magnetic field induced by the impulsive electric field $\mathscr{B}^{(k)} = -c \int_0^t rot\, \mathscr{E}^{(k)} dt'$. This restricts the validity of the treatment to the quasielectrostatic approximation of plasma electrodynamics when $f_0 = f_0(v_\perp, v_\parallel)$ is a nonisotropic function. Otherwise, the results may be approximate to a degree indicated by the inequality $v/c \ll 1$, where v is a typical particle velocity. However, when $f_0 = f_0(v^2)$, isotropic, no limitation ensues, because the additional change of velocity due to the magnetic field is $\delta\mathbf{v}^{(k)} = (e/mc) \int_0^t \mathbf{v} \times \mathscr{B}^{(k)} dt'$ and therefore $\delta v_i^{(k)} (\partial/\partial v_i) f_0 = 2 \delta v_i^{(k)} v_i f_0' = 0$. Impulsive motions with induced magnetic field included have been studied for a plasma without external magnetic field.[13] In the present case we have

$$\rho^{(k)}(\mathbf{x},\mathbf{v},t) = -(e^2 n_0/m) \mathscr{E}^{(k)} (\partial f_0/\partial v_{0k}) \times \delta\left[\mathbf{x} - \int_0^t \mathbf{v}(t')dt'\right] \;, \tag{6}$$

for the charge density at time t, per unit of velocity space, generated by the percussion.

We define now the *accrued charge tensor* $q_{ik}(\mathbf{x},t)$ as the electric charge that has passed, from $t=0$ up to the present time t, at the position $\mathbf{x}$, through the unit area normal to x_i, generated by a percussion of unit strength exerted in the x_k direction at $\mathbf{x}=0$ and $t=0$. This tensor is given by $\mathscr{E}^{(k)} q_{ik}(\mathbf{x},t) = \int_0^t J_i^{(k)}(\mathbf{x},t')dt'$, where $J_i^{(k)}$ are the current densities generated by the δ-function-like fields $\mathscr{E}_i^{(k)}$. We can see then from Eq. (1) that $q_{ik}(\mathbf{x},t)$ is equivalent to $h_{ik}(\mathbf{x},t)$, the polarization response tensor. The crux of the percussion method is that the accrued charge tensor can be computed directly by simple geometrical and kinematical considerations.

III. THE ACCRUED CHARGE TENSOR

We take the z axis in the direction of the external magnetic field $\mathbf{B}_0$ and evaluate the elements of the accrued charge tensor at a point P with coordinates $\mathbf{x}(P) = (l,0,z)$. Then, a rotation around the z axis gives the values of q_{ik} for arbitrary $\mathbf{x}$. First, we consider the effect of the percussions $\mathscr{E}^{(1)}$ and $\mathscr{E}^{(2)}$ along x and y, respectively, which modify the position of the gyration centers of the particles. Referring to Fig. 1, where the projection of the motion on the plane x,y is represented, we see that the orbits which pass through $\mathbf{x}=0$ and $\mathbf{x}(P)$ have their gyrocenters C on the plane $x=l/2$. The impulsive field $\mathscr{E}^{(1)}$ produces changes $\Delta v_\perp^1 = -(e/m)\sin\theta$, $\Delta v_n^1 = (e/m)\cos\theta$ on the tangential and normal velocity components, respectively, of a particle with gyrocenter C at $y=R\sin\theta$. Here we have taken $\mathscr{E}^{(1)}=1$ and $R=v_\perp/\Omega$ is the Larmor radius of the particle ($\Omega = eB_0/mc$). After the percussion,

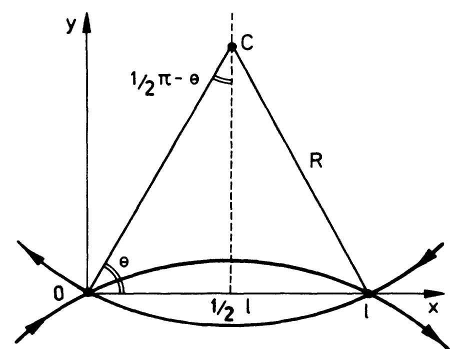

FIG. 1. Projection on the (x,y) plane of the orbits of particles that pass through the points $\mathbf{x}=0$ and $\mathbf{x}(P)=(l,0,z)$.

the radius R is modified by $\Delta R = \Delta v_\perp/\Omega$ and the line $0C$ rotates at an angle $\epsilon = -\Delta v_n/v_\perp$. Hence, to first order in the impulsive field, we have $(R+\Delta R)\cos(\theta+\epsilon) \approx R\cos\theta = l/2$. Therefore, the impulsive motion does not modify the plane $x=l/2$ of the gyrocenters. The aforementioned statements hold for θ in the interval $(-\pi/2,\pi/2)$, where negative values of θ correspond to centers C located at negative values of y in Fig. 1. The velocity component $v_\perp$ varies accordingly from a minimum value $v_m = l\Omega/2$, when C is on the x axis $(\theta=0)$, to $v_\perp=\infty$ when C is at infinity $(\theta=\pm\pi/2)$. A particle which passes through $\mathbf{x}=0$ at $t=0$ arrives at $\mathbf{x}(P)=(l,0,z)$ at time t only when the following relationship holds between $v_\parallel$ and $v_\perp$. Let s be the integer number of half turns of the particle in the magnetic field from $t=0$ to time t. Then, $\Omega t = \pi + 2\theta + s\pi$ for $\theta<0$ and $\Omega t = \pi+2\theta+(s-1)\pi$ for $\theta>0$. Since $v_\parallel t = z$,

$$\begin{aligned} &v_\perp(t) = v_m/|\sin(\Omega t/2)| \ , \\ &v_\parallel = u_s(v_\perp) \equiv (z\Omega/2)/[\arcsin(v_m/v_\perp)+s\pi/2],\ s \text{ even} \\ &\qquad \equiv (z\Omega/2)/[\pi - \arcsin(v_m/v_\perp) \\ &\qquad\qquad + (s-1)\pi/2],\ s \text{ odd} \ . \end{aligned} \tag{7}$$

Similar arguments for $\mathcal{E}^{(2)}$, which produces the velocity jump $\Delta v_\perp^2 = (e/m)\cos\theta$, $\Delta v_n^2 = (e/m)\sin\theta$, show that the gyrocenters lying on the plane $x = l/2-(e/m)/\Omega$ are shifted to the plane $x=l/2$ by the impulsive motion with strength $\mathcal{E}^{(2)}=1$.

We compute now the q_{11} component of the accrued charge tensor at $\mathbf{x}(P),t$. For this purpose we define a small area element at $\mathbf{x}(P)$ normal to the x axis, $dS_x = dy\,dz$, which is crossed by impelled particles at the same instant t. Figure 2 shows how dS_x is determined: we consider a small arc of gyrocenters $\widehat{CC'}$ on the circle of radius $R=v_\perp/\Omega$ and center 0. This corresponds to a set of particles at $\mathbf{x}=0$, $t=0$ with tangential velocity in the angle interval $\theta+\pi/2$, $\theta+\pi/2+d\varphi$. These particles pass through the segment $\overline{QQ}'$ (projection of PP' onto the x,y plane) exactly at the same time t. From Fig. 2 we have $dy = l\,d\varphi$ (neglecting terms $O[(d\varphi)^2]$) and therefore $dS_x = lz\,d\varphi dv_\parallel/v_\parallel$. To obtain the element of charge passing across dS_x between t and $t+dt$, we consider the associated velocity interval $v_\perp$, $v_\perp+dv_\perp$ and follow the reasoning that supports Eq. (5). Then,

$$\delta\rho = -en_0[f_0(v_\perp-\Delta v_\perp,u_s)-f_0(v_\perp,u_s)]v_\perp\,d\varphi\,dv_\parallel dv_\perp \ , \tag{8}$$

for centers C with $\theta>0$, since for these centers the charge flows negatively through dS_x. Retaining linear terms in the velocity jumps only and computing the charge accrued per unit area and unit velocity, we get

$$\begin{aligned} &dq_{11}^s/dv_\perp = g_{11}^s \ , \\ &g_{11}^s(l,0,z,v_\perp) \equiv -(e^2n_0/mlz)u_s(v_\perp) \\ &\qquad\qquad \times (v_\perp^2-v_m^2)^{1/2}(\partial f_0/\partial v_\perp)\,|_{v_\parallel=u_s} \ , \end{aligned} \tag{9}$$

where $v_\perp$ is related to t by Eq. (7). The last formula is valid for gyrocenters with $\theta<0$ also, since the sign of $\Delta v_\perp$

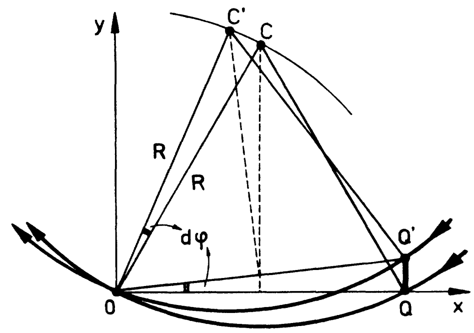

FIG. 2. Projection $\overline{QQ}'$ of the area element dS_x on the (x,y) plane. The two orbits shown correspond to particles that cross dS_x at the same time t, and have tangential velocities with directions $\theta+\pi/2$ and $\theta+\pi/2+d\varphi$ at $t=0$.

and the direction of circulation of charges across dS_x both change in this case

To obtain q_{11} we add all the contributions dq_{11}^s from $t=0$ to t, i.e., we integrate over $v_\perp$. Attention must be paid to the fact that only charges with centers $\theta<0$ pass during the first semiperiod of time $(0,\tau/2)$, where $\tau=2\pi/\Omega$, which corresponds to the $v_\perp$ interval (∞,v_m). For the second semiperiod $(\tau/2,\tau)$, the charges correspond to centers with $\theta>0$, and $v_\perp$ varies through the reverse interval (v_m,∞), and so on. Thus gyrocenters with $\theta<0$ and $\theta>0$ contribute alternatively to the polarization, adding integrals like $\int_{v_m}^\infty g_{11}^n dv_\perp$ successively up to an integer number of semiperiods $s=[2t/\tau]$. There is a final contribution from charges arriving between $s\tau/2$ and t, so that

$$\begin{aligned} q_{11}(l,0,z,t) = &\sum_{n=0}^{s-1}\int_{v_m}^\infty dv_\perp\, g_{11}^n(l,0,z,v_\perp) \\ &+ \int dv_\perp\, g_{11}^s(l,0,z,v_\perp) \ . \end{aligned} \tag{10}$$

The limits of integration for the last term are $[v_m,v_\perp(t)]$ for s odd and $[v_\perp(t),\infty]$ for s even, where $v_\perp(t)$ is defined by Eq. (7). Thus we have obtained the $h_{11}(l,0,z,t)$ component of the polarization response tensor. Note that the spatiotemporal conductivity response $\sigma_{11} = \partial h_{11}/\partial t$ can be obtained from

$$\begin{aligned} &\sigma_{11}(l,0,z,t) = g_{11}^s(l,0,z,v_\perp(t))\,|dv_\perp/dt| \ , \\ &|dv_\perp/dt| = (\Omega^2 l/4)\,|\cos(\Omega t/2)|/\sin^2(\Omega t/2) \ . \end{aligned} \tag{11}$$

The remaining tensor components are computed in a similar way. The velocity jumps for $\mathcal{E}^{(2)}$ and $\mathcal{E}^{(3)}$ are $\Delta v_\perp^2 = (e/m)\cos\theta$ and $\Delta v_\parallel^3 = e/m$, respectively. The elements of area normal to the y and z directions are given by $dS_y = lz(v_\perp^2-v_m^2)^{1/2}d\varphi\,dv_\parallel/v_\parallel v_m$, $dS_z = lz \times (v_\perp^2-v_m^2)^{1/2}d\varphi\,dv_\parallel/v_\parallel^2$. The expression for the tensor $d\mathbf{q}^s/dv_\perp$ is, therefore,

$$d\mathbf{q}^s/dv_\perp=\mathbf{g}^s(l,0,z,v_\perp)=-(\omega_p^2/4\pi l z)(v_\perp^2-v_m^2)^{-1/2}v_\perp u_s((-1)^s(v_\perp^2-v_m^2)^{1/2},-v_m,u_s)^{\mathrm{T}}$$
$$\times((-1)^s(1-v_m^2/v_\perp^2)^{1/2},v_m/v_\perp,1)\mathrm{diag}(\partial f_0/\partial v_\perp,\partial f_0/\partial v_\perp,\partial f_0/\partial v_\parallel)\,|_{v_\parallel=u_s(v_\perp)}\ ,$$

with $u_s(v_\perp)$ given by Eq. (7) and T denoting matrix transposition. The elements of the 3×1 column matrix are the components of the velocity of the induced charge passing through $\mathbf{x}=(l,0,z)$ after completing s half turns. Also, the elements of the 1×3 row matrix are proportional (up to a factor of e/m) to the changes in $v_\perp,v_\parallel$ caused by the percussion $\mathscr{E}^{(k)}=1$, $k=1,2,3$. The velocity $v_m=l\Omega/2$ is the minimum possible value of $v_\perp$ that a particle must have in order to pass through the point $\mathbf{x}=(l,0,z)$. Now, for a generic point $\mathbf{x}=(r\cos\alpha,r\sin\alpha,z)$, we have

$$\mathbf{g}^s(\mathbf{x},v_\perp)=\mathbf{R}(-\alpha)\mathbf{g}^s(r,0,z,v_\perp)\mathbf{R}(\alpha)\ , \tag{12}$$

where $\mathbf{R}(\alpha)$ is the matrix corresponding to a rotation of axes of magnitude α about the z axis. Finally, $\mathbf{h}(\mathbf{x},t)$ and $\boldsymbol{\sigma}(\mathbf{x},t)$ are obtained using formulas similar to Eqs. (10) and (11). Thus, for $t>0$,

$$\mathbf{h}(\mathbf{x},t)=-(\omega_p^2/4\pi r z)\sum_{n=0}^{s}\int dv_\perp(v_\perp^2-v_m^2)^{-1/2}v_\perp u_n\mathbf{R}(-\alpha)((-1)^n(v_\perp^2-v_m^2)^{1/2},-v_m,u_n)^{\mathrm{T}}$$
$$\times((-1)^n(1-v_m^2/v_\perp^2)^{1/2},v_m/v_\perp,1)\mathbf{R}(\alpha)\mathrm{diag}(\partial f_0/\partial v_\perp,\partial f_0/\partial v_\perp,\partial f_0/\partial v_\parallel)\,|_{v_\parallel=u_n}\ , \tag{13}$$

where the limits of integration are those of Eq. (10). Note that the action of $\mathbf{R}(-\alpha)$ on the column matrix gives

$$((-1)^n(v_\perp^2-v_m^2)^{1/2}\cos\alpha+v_m\sin\alpha,(-1)^n(v_\perp^2-v_m^2)^{1/2}\sin\alpha-v_m\cos\alpha,u_n)^{\mathrm{T}}\ ;$$

the application of $\mathbf{R}(\alpha)$ on the row matrix produces a similar effect. The conductivity response is

$$\boldsymbol{\sigma}(\mathbf{x},t)=-[\omega_p^2/4\pi t T^2(t)]\mathbf{R}(-\alpha)((r\Omega/2)\cot(\Omega t/2),r\Omega/2,z/t)^{\mathrm{T}}((-1)^N\cos(\Omega t/2),(-1)^N\sin(\Omega t/2),1)\mathbf{R}(\alpha)$$
$$\times\mathrm{diag}(\partial f_0/\partial v_\perp,\partial f_0/\partial v_\perp,\partial f_0/\partial v_\parallel)\,|_{v_\parallel=z/t,\,v_\perp=r/T(t)}\ , \tag{14}$$

with $T(t)=|\sin(\Omega t/2)/(\Omega/2)|$ and $N(t)=[2\pi t/\Omega]$, the integer number of turns of a particle during time t. The constant ω_p is the plasma frequency of the species considered. The previous calculations are valid for $e>0$, $B_0>0$. In the general case, the column matrix in $\mathbf{h}(\mathbf{x},t)$ is $((-1)^n(v_\perp^2-v_m^2)^{1/2},-(\mathrm{sgn}\Omega)v_m,u_n)^{\mathrm{T}}$ and the row matrix $((-1)^n(1-v_m^2/v_\perp^2)^{1/2},(\mathrm{sgn}\Omega)v_m/v_\perp,1)\mathrm{sgn}e$, with sgn denoting the sign function, and the equations are changed accordingly. The space-time representation of the conductivity, Eq. (14), coincides for a Maxwellian distribution function, with the formula derived by Shafranov[11] for the particular case of thermal equilibrium via Eq. (4).

IV. THE TIME EVOLUTION EQUATIONS FOR PLANE WAVES

We consider here the special question of plane wave modes of the form $\mathbf{E}(\mathbf{x},t)=\mathbf{E}(t)\exp[i(k_\perp x+k_\parallel z)]$. Then,

$$\begin{aligned}4\pi\mathbf{j}(\mathbf{x},t)&=\int_0^\infty d\tau\int d^3\xi\,4\pi\boldsymbol{\sigma}(\boldsymbol{\xi},\tau)\mathbf{E}(\mathbf{x}-\boldsymbol{\xi},t-\tau)\\&=\Bigg\{-\omega_p^2\int_0^\infty d\tau[1/\tau T^2(\tau)]\int_0^\infty d\rho\,\rho\int_{-\infty}^{\infty}d\zeta\exp(-ik_\parallel\zeta)\\&\qquad\times\int_0^{2\pi}d\varphi\,\mathbf{M}(\boldsymbol{\xi},\tau)\mathrm{diag}(\partial f_0/\partial v_\perp,\partial f_0/\partial v_\perp,\partial f_0/\partial v_\parallel)\,|_{v_\parallel=\zeta/\tau,\,v_\perp=\rho/T(\tau)}\\&\qquad\times\mathbf{E}(t-\tau)\sum_{n=-\infty}^{\infty}J_n(k_\perp\rho)(-i)^n\exp(-in\varphi)\Bigg\}\exp[i(k_\perp x+k_\parallel z)]\\&=\int_0^\infty d\tau\,4\pi\tilde{\boldsymbol{\sigma}}(\mathbf{k},\tau)\mathbf{E}(t-\tau)\exp[i(k_\perp x+k_\parallel z)]\ ,\end{aligned}$$

with

$$\mathbf{M}(\mathbf{x},t)=\mathbf{R}(-\alpha)((r\Omega/2)\cot(\Omega t/2),r\Omega/2,z/t)^{\mathrm{T}}((-1)^N\cos(\Omega t/2),(-1)^N\sin(\Omega t/2),1)\mathbf{R}(\alpha)\ ,$$

where a well-known series for $\exp(-ik_\perp\rho\cos\varphi)$ has been introduced to facilitate the averaging over φ. Completing the integration over $\boldsymbol{\xi}=(\rho,\varphi,\zeta)$ variables, and after some algebra, we obtain the conductivity response for $\mathbf{k}$ modes:

$$4\pi\tilde{\boldsymbol{\sigma}}(\mathbf{k},t)=\omega_p^2[\mathrm{diag}(\mathbf{R},0)+\mathrm{diag}(\mathbf{R}+\mathbf{S},0)(\tfrac{1}{2})k_\perp(\partial/\partial k_\perp)+(-1)^{N(t)}\mathbf{W}+\mathrm{diag}(0,0,1)(\partial/\partial k_\parallel)k_\parallel]F(k_\perp T,k_\parallel,t)\ , \tag{15}$$

where $\mathbf{S}$ is the 2×2 matrix defined by $\mathbf{S}=\mathrm{diag}(1,-1)$, $\mathbf{R}$ is the 2×2 rotation matrix with elements $R_{11}=R_{22}=\cos(\Omega t)$, $R_{12}=-R_{21}=\sin(\Omega t)$, $N(t)$ is the integer number of turns of a particle during time t $\{N(t)=[\Omega t/2\pi]\}$, and $\mathbf{W}$ is a 3×3 matrix whose elements are zero except for the third column and row where

$$(W_{13}, W_{23}, W_{33})^{T} = (\cos(\Omega t/2), -\sin(\Omega t/2), 0)^{T}(t/T)(\partial/\partial k_{\perp}) ,$$

$$(W_{31}, W_{32}, W_{33}) = (\cos(\Omega t/2), \sin(\Omega t/2), 0)(T/t)(\partial/\partial k_{\parallel}) .$$

Finally, we have set

$$F(k_{\perp}T, k_{\parallel}t) = 2\pi \int_0^{\infty} dv_{\perp}\, v_{\perp} J_0(k_{\perp}Tv_{\perp}) \int_{-\infty}^{\infty} dv_{\parallel} \exp(-ik_{\parallel}v_{\parallel}t) f_0(v_{\perp}, v_{\parallel}) , \tag{16}$$

for the Fourier transform in $v_{\parallel}$, with argument $k_{\parallel}t$, and 2π times the Fourier-Bessel transform of zero order in $v_{\perp}$ with argument $k_{\perp}T$ of the distribution function. In the case of a nonisotropic function of Maxwellian type

$$f_0 = [1/(2\pi)^{3/2} c_{\perp}^2 c_{\parallel}] \exp\{(-\tfrac{1}{2})[(v_{\perp}/c_{\perp})^2 + (v_{\parallel}/c_{\parallel})^2]\} ,$$

with $c_{\perp,\parallel}^2 = T_{\perp,\parallel}/m$, we have

$$F = \exp\{(-\tfrac{1}{2})[(k_{\perp}c_{\perp}T)^2 + (k_{\parallel}c_{\parallel}t)^2]\} .$$

A monoenergetic beam like $f_0 = (1/2\pi v_{\perp})\delta(v_{\perp} - u_0)\delta(v_{\parallel} - v_0)$ gives, instead, $F = J_0(k_{\perp}u_0T)\exp(-ik_{\parallel}v_0t)$. In the absence of a magnetic field, the response function for longitudinal modes depends on the Fourier transform of the distribution function in velocity space.[6] Thus Eq. (16) extends this property to magnetized plasmas. The presence of a Fourier-Bessel transform of zero order is a consequence of the rotational symmetry in the plane perpendicular to the magnetic field. The infinite series of harmonics of Ω with Bessel functions of all orders, which is typical of standard formulas for $\sigma(\mathbf{k},\omega)$, does not appear in the time response $\tilde{\sigma}(\mathbf{k},t)$. However, this response contains the periodic argument $T(t)$ instead. When f_0 is isotropic, $f_0 = f_0(v^2)$, then $v_{\parallel}(\partial f_0/\partial v_{\perp}) = v_{\perp}(\partial f_0/\partial v_{\parallel})$, so that $\tilde{\sigma}_{13} = \tilde{\sigma}_{31}$ and $\tilde{\sigma}_{23} = -\tilde{\sigma}_{32}$.

In the quasi-electrostatic approximation we have $\mathbf{E} = -\mathrm{grad}\phi$, where $\phi(\mathbf{x},t)$ is the electrostatic potential, so that $\mathbf{E}(t) = -i\mathbf{k}\phi(t)$. Therefore, Eq. (3) may be reduced to a scalar equation

$$\phi(t) + \int_0^{\infty} d\tau\, 4\pi h(\mathbf{k},\tau)\phi(t-\tau) = g(t) , \tag{17}$$

where $h(\mathbf{k},t) = \int_0^t dt'\, \tilde{\sigma}_{ij}(\mathbf{k},t') k_i k_j / k^2$, and g arises from external sources. The external current or charge terms can also be used to account for the initial conditions, i.e., the past memory of the influence of the electric field on the system or, alternatively and equivalently, the information about the initial perturbation of the particle distribution function (this topic is discussed, for instance, in Refs. 1, 3, and 11). The initial conditions ordinarily generate a time-dependent datum. For initial value problems, the integral in Eq. (17) extends to $(0,t)$ only. Right-hand side terms with roles analogous to g will appear in other equations. Using Eq. (15) we obtain for the quasi-electrostatic time response the remarkable formula

$$4\pi h(\mathbf{k},t) = (\omega_p^2/k^2)[k_{\perp}^2(\sin\Omega t/\Omega) + k_{\parallel}^2 t] F(k_{\perp}T, k_{\parallel}t) . \tag{18}$$

Here, and in the rest of this section, a sum over plasma species, electrons and ions, is implied although not written for simplicity. For $k_{\perp} = 0$, the longitudinal response $4\pi h = \omega_p^2 t \tilde{f}(k_{\parallel}t)$ of Ref. 6 is recovered. Here $\tilde{f}$ indicates the Fourier transform of $\bar{f}_0(v_{\parallel})$, the distribution function f_0 averaged over $v_{\perp}$,

$$\bar{f}_0(v_{\parallel}) = 2\pi \int_0^{\infty} dv_{\perp}\, v_{\perp} f_0(v_{\perp}, v_{\parallel}) .$$

If $\Omega = 0$, we obtain $4\pi h = \omega_p^2 t F(k_{\perp}t, k_{\parallel}t)$, instead. When $k_{\parallel} = 0$, we may note that h is a periodic function of time. This points towards the absence of Landau damping for perpendicular waves (Bernstein modes[3]), since the plasma response incessantly returns to the same values due to the periodic argument $k_{\perp}T(t)$ connected with $v_{\perp}$. Note that the behavior of the argument $k_{\parallel}t$ associated to $v_{\parallel}$ is quite different: It shows a time decay of the response inversely correlated with the velocity spread.

Electromagnetic modes propagating along the magnetic field, $k_{\perp} = 0$, are studied introducing the circularly polarized fields $E_{\pm}(t) = E_x(t) \pm iE_y(t)$. From Eq. (2) we get

$$[(d^2/dt^2) + c^2 k_{\parallel}^2] E_{\pm}(t) + (d/dt) \int_0^{\infty} d\tau\, 4\pi [\tilde{\sigma}_{11}(\mathbf{k},\tau) \pm i\tilde{\sigma}_{21}(\mathbf{k},\tau)] E_{\pm}(t-\tau) = g_{1\pm}(t) . \tag{19}$$

We find then, from Eq. (15),

$$4\pi(\tilde{\sigma}_{11} \pm i\tilde{\sigma}_{21}) \equiv 4\pi\tilde{\sigma}_{\pm}(\mathbf{k},t) = \omega_p^2 \exp(\mp i\Omega t) \tilde{f}(k_{\parallel}t) .$$

For slowly varying modes, such that $|(1/E)(d^2E/dt^2)| \ll c^2 k_{\parallel}^2$, we obtain an initial value problem governed by

$$(\omega_p^2 + c^2 k_{\parallel}^2) E_{\pm}(t) + \int_0^t d\tau\, 4\pi\tilde{\sigma}'_{\pm}(t-\tau) E_{\pm}(\tau) = g_{1\pm}(t), \quad t \geq 0 \tag{20}$$

where $\tilde{\sigma}'_{\pm}(t) \equiv (d/dt)\tilde{\sigma}_{\pm}(t)$,

$$4\pi\tilde{\sigma}'_{\pm}(t) = -i\omega_p^2 \int_{-\infty}^{\infty} dv_{\parallel} \exp[-i(k_{\parallel}v_{\parallel} \pm \Omega)t] \times (k_{\parallel}v_{\parallel} \pm \Omega)\bar{f}_0(v_{\parallel}) . \tag{21}$$

For faster processes, consider the response for electrons only and set $E_{\pm}(t) = E_{1\pm}(t)\exp(-i\omega_0 t)$, with $\omega_0^2 = \omega_p^2 + c^2 k_{\parallel}^2$. After some manipulation, Eq. (19) can be transformed exactly into

$$E_{\pm}(t)+4\pi\int_0^t dt'\exp[i\omega_0(t-t')]\int_0^{t'}dt''\exp[-i\omega_0(t'-t'')]\int_0^{t''}d\tau\,\tilde{\sigma}'_{\pm}(t''-\tau)E_{\pm}(\tau)=g_{2\pm}(t)\ . \tag{22}$$

Finally, we also report the case of electromagnetic modes propagating at 90° to the magnetic field, $k_{\parallel}=0$. The corresponding equations are

$$\begin{aligned}&(d^2/dt^2)E_x+4\pi(d/dt)\int_0^t d\tau[\tilde{\sigma}_{11}(\tau)E_x(t-\tau)+\tilde{\sigma}_{12}(\tau)E_y(t-\tau)]=g_x(t)\ ,\\&c^2k_{\perp}^2E_y+(d^2/dt^2)E_y+4\pi(d/dt)\int_0^t d\tau[\tilde{\sigma}_{21}(\tau)E_x(t-\tau)+\tilde{\sigma}_{22}(\tau)E_y(t-\tau)]=g_y(t)\ ,\end{aligned} \tag{23}$$

when $E_z=0$ (the extraordinary mode) and

$$c^2k_{\perp}^2E_z+(d^2/dt^2)E_z+4\pi\int_0^t d\tau\,\tilde{\sigma}'_{33}(\tau)E_z(t-\tau)=g_z(t)\ , \tag{24}$$

when $E_x=E_y=0$ (the ordinary mode), with the following response components:

$$\begin{aligned}&4\pi\tilde{\sigma}_{11}=\omega_p^2[\cos(\Omega t)+\cos^2(\Omega t/2)k_{\perp}(\partial/\partial k_{\perp})]F(k_{\perp}T,0)\ ,\\&4\pi\tilde{\sigma}_{22}=\omega_p^2[\cos(\Omega t)-\sin^2(\Omega t/2)k_{\perp}(\partial/\partial k_{\perp})]F(k_{\perp}T,0)\ ,\\&4\pi\tilde{\sigma}_{33}=\omega_p^2F(k_{\perp}T,0),\quad \tilde{\sigma}'_{33}(t)=(d/dt)\tilde{\sigma}_{33}\ ,\\&4\pi\tilde{\sigma}_{12}=\omega_p^2\sin(\Omega t)[1+(\tfrac{1}{2})k_{\perp}(\partial/\partial k_{\perp})]F(k_{\perp}T,0)\\&\qquad=-4\pi\tilde{\sigma}_{21}\ .\end{aligned} \tag{25}$$

For $f_0=f_0(v^2)$, isotropic, the restrictions $E_z=0$ for the extraordinary mode and $E_x=E_y=0$ for the ordinary mode are no longer necessary since, in this case, $\tilde{\sigma}_{13}=\tilde{\sigma}_{31}=\tilde{\sigma}_{23}=\tilde{\sigma}_{32}=0$.

V. AN OPERATIONAL METHOD FOR INTEGRAL EQUATIONS

We shall now describe a method to obtain growth bounds for the electric field $\mathbf{E}(t)$ [or the potential $\phi(t)$], based on properties of the convolution product $a*b$ between two functions $a(t)$ and $b(t)$ defined for $t\geq 0$, $(a*b)(t)\equiv\int_0^t d\tau\,a(\tau)b(t-\tau)$. Consider an integral equation of the form

$$E(t)+(E*H)(t)=G(t),\quad t>0 \tag{26}$$

to which several cases of Sec. IV reduce. This equation may represent an initial value problem for $E(t)$. Assume that the following bounds hold on H and G, $|H(t)|\leq\gamma^2t$, $|G(t)|\leq C$. Then, $|E(t)|\leq|G(t)|+\gamma^2t\Theta(t)*|E(t)|$, where Θ is the Heaviside or unit step function. We define for $t\geq 0$,

$$\begin{aligned}a(t)&\equiv|E(t)|-\gamma^2t\Theta(t)*|E(t)|\\&=[\delta(t)-\gamma^2\Theta(t)*\Theta(t)]*|E(t)|\leq|G(t)|\ .\end{aligned}$$

Since $\delta(t)$ is the identity operator $\mathbf{1}$ for a convolution product, we may express the function a in a symbolic operator form as $a=(\mathbf{1}-\gamma^2\Theta^2)|E|$, where Θ^2 indicates the convolution product of Θ with itself. If $(\mathbf{1}-\gamma^2\Theta^2)^{-1}$ denotes the inverse operator of $(1-\gamma^2\Theta^2)$, we may put $|E|=(\mathbf{1}-\gamma^2\Theta^2)^{-1}a$. We now proceed to invert the operator formally as

$$|E|=\left[\sum_{n=0}^{\infty}\gamma^{2n}\Theta^{2n}\right]a=a+\left[\sum_{n=1}^{\infty}\gamma^{2n}\Theta^{2n}\right]a\ ,$$

where Θ^{2n} must be taken as the convolution of Θ^2 n times with itself. Thus $\Theta^{2n}=t^{2n-1}/(2n-1)!$, $n\geq 1$, so that

$$\begin{aligned}|E(t)|&=a(t)+\left[\gamma\sum_{n=1}^{\infty}\gamma^{2n-1}t^{2n-1}\Big/(2n-1)!\right]*a(t)\\&=[\delta(t)+\gamma\sinh(\gamma t)\Theta(t)]*a(t)\ .\end{aligned}$$

If we show that $\delta(t)+\gamma\sinh(\gamma t)$ is the inverse operator, we may then produce a bound for $|E(t)|$ since, using the fact that $a(t)\leq|G(t)|\leq C$, then

$$\begin{aligned}|E(t)|&\leq|G(t)|+\gamma\sinh(\gamma t)\Theta(t)*|G(t)|\\&\leq C+\gamma\sinh(\gamma t)\Theta(t)*C\Theta(t)\\&=C\cosh(\gamma t),\quad t\geq 0\ .\end{aligned} \tag{27}$$

To show that we have the proper inverse operator, note that

$$\begin{aligned}&[\delta(t)+\gamma\sinh(\gamma t)\Theta(t)][\delta(t)-\gamma^2\Theta(t)*\Theta(t)]\\&\quad=\delta(t)+\gamma\sinh(\gamma t)\Theta(t)\\&\qquad-\gamma^2t\Theta(t)-\gamma^3\sinh(\gamma t)\Theta(t)*\Theta(t)*\Theta(t)=\delta(t)\ .\end{aligned}$$

A similar procedure may be used to derive bounds for $E(t)$ when $|H(t)|\leq\alpha$ and $|G(t)|\leq C$. In this case we have $|E(t)|\leq C\exp(\alpha t)$. When $|H(t)|\leq\alpha+\beta^2t$, the bound is more elaborate:

$$|E(t)|\leq C\exp(\alpha t/2)(\cosh\{[(\alpha/2)^2+\beta^2]^{1/2}t\}+[\alpha/(\alpha^2+4\beta^2)^{1/2}]\sinh\{[(\alpha/2)^2+\beta^2]^{1/2}t\})\ .$$

It is possible to replace the bound C on $G(t)$ by

$$M(t)\equiv\sup_{0\leq\tau\leq t}\{|G(\tau)|\}\ .$$

The mathematical background of this operational calculus may be found in Ref. 14. The operator algebra is also useful to solve integral and integro-differential equations and to establish the existence and unicity of solutions via an infinite series of convolutions.

VI. GROWTH BOUNDS FOR PLASMA MODES

With this method, *a priori* bounds for a variety of plasma modes can easily be obtained, a few of which we shall present here.

We consider the quasi-electrostatic approximation first. (i) Let us examine the electron response alone, assuming fixed ions. Since F is such that $|F|\le 1$ for all distribution functions f_0, from Eqs. (16)–(18) we may set the bound $|H|\le\omega_p^2 t\,|F(k_\perp T,k_\parallel t)|\le\omega_p^2 t$. Hence, it follows that no electronic instability can grow faster than $\exp(\omega_p t)$. This generalizes the validity of Backus's result[9] for all possible $\mathbf{k}$ in a magnetized plasma. (ii) For $|\omega_p/\Omega|<1$, a better bound is $|H|\le|\omega_p^2/\Omega|(k_\perp/k)^2+(\omega_p k_\parallel/k)^2 t$, which has an angular dependence. Therefore,

$$|\phi|\le C\exp[|\omega_p^2/2\Omega|(k_\perp/k)^2 t]\times[\cosh(\gamma t)+|\omega_p^2/2\Omega\gamma|(k_\perp/k)^2\sinh(\gamma t)]\ ,$$

with $\gamma=\omega_p[|\omega_p/2\Omega|^2(k_\perp/k)^4+(k_\parallel/k)^2]^{1/2}$. This shows a reduction of the bound due to the magnetic field, since $\gamma<\omega_p$ when $k_\perp\neq 0$. This effect is enhanced as $k_\parallel\to 0$. For $k_\parallel=0$, the bound is $|\phi|\le C\exp(|\omega_p^2/\Omega|t)$ so that in plasma regimes with $|\omega_p/\Omega|\ll 1$, the growth rate of modes perpendicular to $\mathbf{B}_0$ is considerably reduced in comparison with that of modes parallel to $\mathbf{B}_0$. (iii) Next, we consider the ion-acoustic modes by adding the ion response. For this range of the spectrum, however, the electron contribution may ordinarily be replaced, when $k_\parallel\neq 0$, by the term $\phi/(k\lambda_D)^2$ (λ_D: Debye length) in Eq. (17). This corresponds to the assumption of a Boltzmann electron density distribution for thermal equilibrium in the wave potential. The integral equation is again of the type (26) with $H=4\pi h_i/(1+1/k^2\lambda_D^2)$, where h_i is the ion response. By the same arguments used before, we obtain the growth bound $\gamma=kc_s/[1+(k\lambda_D)^2]^{1/2}$ [$c_s=(T_e/m_i)^{1/2}$: ion sound velocity]. This extends to magnetized plasma a bound obtained by Pécseli[15] for ion-acoustic waves in a plasma without magnetic field. The growth bound may be very restrictive for long wavelengths as $\gamma\to 0$ with $k\to 0$. (iv) When $k_\parallel=0$, the electron contribution to Eq. (17) changes and may be approximated now by $\phi(\omega_{pe}/\Omega_e)^2$ if $|(1/E)d^2E/dt^2|\ll\Omega_e^2$. This corresponds to the lower hybrid branch of the spectrum. Thus we have $\gamma=\omega_{pi}^2/\Omega_i[1+(\omega_{pe}/\Omega_e)^2]^{1/2}$. Therefore, again the growth rates are reduced by intense magnetic fields.

We pass now to the examination of the electromagnetic modes. (v) Let us first take the parallel propagation case in the low-frequency range where Eqs. (20) and (21) hold. We may write for each species $4\pi|d\tilde\sigma_{\pm e,i}/dt|\le\omega_{pe,i}^2\omega_{1e,i}^{\pm}$, with

$$\omega_{1e,i}^{\pm}\equiv|k_\parallel|\int_{-\infty}^{\infty}dv_\parallel\,|\pm(\Omega_{e,i}/k_\parallel)+v_\parallel|\,\bar f_{0_{e,i}}(v_\parallel)\ .$$

These values clearly depend on the dispersion of the averaged distribution functions $\bar f_{0_{e,i}}$ around the resonant velocities $\pm\Omega_{e,i}/k_\parallel$. Applying the results of Sec. V, we obtain $\gamma=(\omega_{pe}^2\omega_{1e}^{\pm}+\omega_{pi}^2\omega_{1i}^{\pm})/(\omega_{pe}^2+\omega_{pi}^2+c^2k_\parallel^2)$, a limitation which becomes more stringent as $k_\parallel$ increases. This bound covers the range of Alfvén waves, whistlers, and cyclotron resonances. (vi) The high-frequency range of electromagnetic modes parallel to $\mathbf{B}_0$ may be studied using Eq. (22). Consider, for simplicity, the electron contribution only. An approach similar to that of Sec. V gives

$$|E_\pm(t)|\le 4\pi\int_0^t dt'\int_0^{t'}dt''\int_0^{t''}d\tau\,|\tilde\sigma'_\pm(t''-\tau)|\,|E_\pm(\tau)|+|E_\pm(0)|+|E_\pm'^2(0)+\omega_0^2E_\pm^2(0)|^{1/2}t+\int_0^t dt'\int_0^{t'}dt''\,|g_{2\pm}(t'')|\ .$$

In addition, we have the bound $|4\pi\tilde\sigma'_\pm(t)|\le\omega_p^2\omega_1$ [with ω_1 as in (v)]. It can then be proved that $|E(t)|\le(A+Bt+Ct^2)\exp[(\omega_p^2\omega_1)^{1/3}t]$ for all t, with A, B, and C non-negative constants. (vii) As a final example, let us consider the ordinary mode which is ruled by Eq. (24). Here $|4\pi(d/dt)\tilde\sigma_{33}|\le\omega_p^2k_\perp\langle v_\perp\rangle\equiv\omega_p^2\omega_2$, where $\langle v_\perp\rangle=2\pi\int_0^\infty dv_\perp v_\perp^2\bar f_0(v_\perp)$. Note that Eq. (24) is formally identical to Eq. (19). Thus the same treatment used for (vi) holds. Therefore, the growth bound is similar to that of (vi) with ω_1 replaced by ω_2.

VII. THE COMPLEX SUSCEPTIBILITY TENSOR

As an alternative path to the main route,[3] it is possible to evaluate the complex susceptibility tensor $\chi_{ij}(\mathbf{k},\omega)$ from the accrued charge tensor $q_{ij}(\mathbf{x},t)$. We give here this derivation to show the agreement of the wave number-frequential representation with the spatiotemporal description. Starting from Eq. (13), it is convenient to express the integrands of the integrals over velocity space in terms of δ functions introduced *ad hoc*. Next, the δ functions are replaced by the Fourier integral representation, and the passage to the $\mathbf{k},\omega$ description is then immediate. To carry through this calculation, we rewrite Eq. (13) defining $t_n(v_\perp)\equiv z/u_n(v_\perp)$,

$$\phi_n(v_\perp)\equiv\begin{cases}\alpha-\arcsin(v_m/v_\perp), & n\ \text{even}\\ \alpha-\pi+\arcsin(v_m/v_\perp), & n\ \text{odd}\ ,\end{cases}$$

$$\tilde{\mathbf{M}}(v_\perp,v_\parallel,\phi,t)\equiv(v_\perp\cos\phi,-v_\perp\sin\phi,v_\parallel)^{\mathrm T}\times(\cos(\Omega t+\phi),\sin(\Omega t+\phi),1)\ .$$

Thus

$$\begin{aligned}\mathbf{h}(\mathbf{x},t)&=-(\omega_p^2/4\pi r)\sum_{n=0}^{s}\int dv_\perp[1/t_n|\cos(\alpha-\phi_n)|]\tilde{\mathbf{M}}(v_\perp,u_n,\phi_n,t_n)\mathrm{diag}(\partial f_0/\partial v_\perp,\partial f_0/\partial v_\perp,\partial f_0/\partial v_\parallel)|_{v_\parallel=u_n}\\&=-(\omega_p^2/4\pi r)\sum_{n=0}^{s}\int dv_\perp\int_{-\infty}^{\infty}dv_\parallel\,\delta(v_\parallel t_n-z)[1/|\cos(\alpha-\phi_n)|]\tilde{\mathbf{M}}(v_\perp,v_\parallel,\phi_n,t_n)\mathrm{diag}(\partial f_0/\partial v_\perp,\partial f_0/\partial v_\perp,\partial f_0/\partial v_\parallel)\\&=-(\omega_p^2/4\pi)\int_0^\infty dv_\perp v_\perp\int_{-\infty}^{\infty}dv_\parallel\int_0^{2\pi}d\phi\int_0^\infty d\tau\,\Theta(t-\tau)\delta(v_\parallel\tau-z)\delta(A(\tau,\phi))\\&\qquad\times\delta(B(\tau,\phi))\tilde{\mathbf{M}}(v_\perp,v_\parallel,\phi,\tau)\mathrm{diag}(\partial f_0/\partial v_\perp,\partial f_0/\partial v_\perp,\partial f_0/\partial v_\parallel)\ ,\end{aligned}\tag{28}$$

with

$$A(\tau,\phi)\equiv(v_\perp/\Omega)[\sin(\Omega\tau+\phi)-\sin\phi]-r\cos\alpha$$

and

$$B(\tau,\phi)\equiv(v_\perp/\Omega)[\cos(\Omega\tau+\phi)-\cos\phi]+r\sin\alpha\ .$$

We note now that

$$\Theta(t)=(1/2\pi)\int_{-\infty}^{\infty}d\omega(i/\omega)\exp(-i\omega t)$$

and

$$\delta(A(\tau,\phi))=(1/2\pi)\int_{-\infty}^{\infty}dk_x\exp[-ik_x A(\tau,\phi)]\ ,$$

with a similar identity valid for $\delta(B(\tau,\phi))$. It follows that

$$\mathbf{h}(\mathbf{x},t)=(1/2\pi)^4\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\int d^3k\exp(i\mathbf{k}\cdot\mathbf{x})\boldsymbol{\mathcal{X}}(\mathbf{k},\omega)\ ,$$

where

$$\begin{aligned}\boldsymbol{\mathcal{X}}(\mathbf{k},\omega)=(\omega_p^2/4\pi i\omega)\int_0^\infty dv_\perp v_\perp\int_{-\infty}^{\infty}dv_\parallel\int_0^{2\pi}d\phi\int_0^\infty d\tau\exp\{&-ik_x(v_\perp/\Omega)[\sin(\Omega\tau+\phi)-\sin\phi]\\&+ik_y(v_\perp/\Omega)[\cos(\Omega\tau+\phi)-\cos\phi]\\&-i(k_z v_\parallel-\omega)\tau\}\widetilde{\mathbf{M}}(v_\perp,v_\parallel,\phi,\tau)\\&\times\mathrm{diag}(\partial f_0/\partial v_\perp,\partial f_0/\partial v_\perp,\partial f_0/\partial v_\parallel)\end{aligned}\tag{29}$$

is the desired complex susceptibility tensor. This expression coincides with that obtained by solving the linearized Vlasov equation with the Fourier-Laplace transform method in the case of an isotropic f_0 or for quasi-electrostatic modes (see, for instance, Ref. 5, p. 49). The formula for $\boldsymbol{\mathcal{X}}(\mathbf{k},\omega)$ most commonly used in the analysis of plasma dispersion relations follows from Eq. (29) after setting $k_y=0$ and expanding (via the well-known generating function of Bessel functions) the exponentials containing trigonometric functions in a series of harmonics of Ω and Bessel functions of all orders.

VIII. THE CORRELATION TENSOR OF MICROSCOPIC CURRENTS FOR NONINTERACTING PARTICLES

In this section we show the connection between the percussion method and statistical physics concepts. We consider now the total velocity jump $\Delta\mathbf{v}^{(k)}$, due to both $\mathcal{E}^{(k)}$ and the induced magnetic field $\mathcal{B}^{(k)}$, for a particle with velocity $\mathbf{v}_0$ at $\mathbf{x}_0$ for $t=0$. Then, $\Delta v_k^{(k)}=(e/m)\mathcal{E}^{(k)}\delta(\mathbf{x})$, since in the diagonal elements only the electric contribution appears. Furthermore, $\mathbf{v}_0\cdot\Delta\mathbf{v}^{(k)}=(e/m)v_{0k}\delta(\mathbf{x})$, assuming the unit strength for the percussion $\mathcal{E}^{(k)}=1$. From Eq. (5) we obtain, for the complete $\Delta\mathbf{v}^{(k)}$,

$$\rho^{(k)}(\mathbf{x}_0,\mathbf{v}_0,0)=-en_0(\partial f_0/\partial v_{0p})\Delta v_p^{(k)}\ .\tag{30}$$

Assuming the knowledge of the orbits $\mathbf{x}=\mathbf{x}(\mathbf{x}_0,\mathbf{v}_0,t)$, $\mathbf{v}=\mathbf{v}(\mathbf{x}_0,\mathbf{v}_0,t)$, and since $d^3v=d^3v_0$, the current at $\mathbf{x},t$ generated by the (k) percussion can be computed from

$$J_i^{(k)}(\mathbf{x},t)=-en_0\int d^3v_0\, v_i(t)\Delta v_p^{(k)}(\partial f_0/\partial v_{0p})\ .\tag{31}$$

On the other hand, from Eq. (1) and $\sigma_{ik}=(\partial/\partial t)h_{ik}$, it follows that $J_i^{(k)}(\mathbf{x},t)=\sigma_{ik}(\mathbf{x},t)$, i.e., Eq. (31) gives the conductivity response tensor.

The (k) percussion changes the velocity components v_{0p} into $v_{0p}+\Delta v_p^{(k)}$ at $\mathbf{x}_0=0$. The variation of the energy of the particle $E=m|\mathbf{v}_0|^2/2$ is given by $\delta E^{(k)}=mv_{0p}\Delta v_p^{(k)}=ev_{0k}\delta(\mathbf{x}_0)$, since the magnetic part of the percussion does not enter here. Therefore, the variation of the distribution function per unit energy due to the perturbation is

$$\delta f_0/\delta E^{(k)}=(\partial f_0/\partial v_{0p})(\Delta v_p^{(k)}/\delta E^{(k)})$$

and we get

$$(\partial f_0/\partial v_{0p})\Delta v_p^{(k)}=ev_{0k}(\delta f_0/\delta E^{(k)})\delta(\mathbf{x}_0)\ .\tag{32}$$

Replacing Eq. (32) in Eq. (31) we obtain

$$\sigma_{ik}(\mathbf{x},t)=-(\delta/\delta E^{(k)})\left[e^2n_0\int v_i(t)v_{0k}f_0(\mathbf{v}_0)\delta(\mathbf{x}_0)d^3v\right]\ ,\tag{33}$$

where the symbol $(\delta/\delta E^{(k)})$ operates on the distribution function $f_0(\mathbf{v}_0)$ in the sense indicated above.

The quantity within large parentheses in Eq. (33) coincides with the space-time correlation of the microscopic currents for noninteracting particles already introduced in Sec. I, Eq. (4), i.e.,

$$\langle j_i j_k\rangle_{\mathbf{x},t}^0=e^2n_0\int d^3v\, v_i(t)v_{0k}f_0(\mathbf{v}_0)\delta(\mathbf{x}_0)\tag{34}$$

[formula (34) is derived in Ref. 11]. This correlator, which is usually considered for thermal equilibrium, can be extended to nonequilibrium plasmas under the hypothesis of factorization of the distribution function of N particles into a product of N functions of one particle, $f_0(\mathbf{v})$.[10,11] This is the same correlationless approximation which also sustains the Vlasov kinetic equation. From Eqs. (33) and (34), we may state that the impulsive perturbations lead to the relationship

$$\sigma_{ik}(\mathbf{x},t)=-(\delta/\delta E^{(k)})\langle j_i j_k\rangle_{\mathbf{x},t}^0\ .\tag{35}$$

This can be considered as the extension of the fluctuation-dissipation theorem, in a classical-physics form, to a correlationless plasma in the spatiotemporal representation. Equation (35) is, therefore, the nonequilibrium version of Eq. (4).

Equations (30)–(35) include, of course, the cases of the quasi-electrostatic approximation or the isotropic distribution function treated in Sec. III, where

$$(\partial f_0/\partial v_{0p})\Delta v_p^{(k)}=(e/m)(\partial f_0/\partial v_{0k})\delta(\mathbf{x}_0)\ .$$

For a Maxwellian distribution f_0, $\langle j_i j_k\rangle^0_{\mathbf{x},t}$ was computed directly from Eq. (34) in Ref. 11 and σ_{ik} was derived from it through Eq. (4). Equation (14) leads to that same result. Then, the equivalence of σ_{ik} obtained in Sec. III by calculation of the accrued charge and the value obtained from the correlator of microscopic currents can be shown explicitly, in the case of thermal equilibrium, in agreement with the general relationship Eq. (35).

IX. SUMMARY AND CONCLUSIONS

The space-time response kernels σ_{ik}, h_{ik} of the conductivity and susceptibility operators have been derived for a collisionless plasma in a magnetic field, purely from geometrical and kinematical arguments, starting with an impulsive perturbation. If a conceptual alternative is desired, these responses may be taken as the basic elements of the theory for the reason that they are directly related to a physically intuitive description. Besides, in their derivation no restrictions *ab initio* are imposed on the set of functions that the theory may handle. The self-similar dependence on $\mathbf{x}/t$ which characterizes the response functions of plasmas without magnetic field,[6,11,13] is replaced here by the more complicated kinematics of helical orbits, which enters in the distribution function through Eq. (7). The fact that h_{ik} may be obtained without kinetic equations and Fourier-Laplace transforms is not unexpected in view of the result, valid for general electromagnetic perturbations,

$$(\partial/\partial t)h_{ik}(\mathbf{x},t)=-(\delta/\delta E^{(k)})\langle j_i j_k\rangle^0_{\mathbf{x},t}\ ,$$

found in Sec. VIII. Impulsive motion plays in a classical context the same role of the quantum perturbative concepts employed in the derivation of the general fluctuation-dissipation theorem.[10] In the percussion approach, the fundamental formula for σ_{ik} is expressed by Eq. (31). Section III describes a constructive method for performing the calculation explicitly. The information used there: absence of interaction of the particles, knowledge of the unperturbed orbits, and the one-particle distribution function, is fully equivalent to that employed in calculations based on the Vlasov kinetic equation.

The results of Sec. VI provide upper bounds for the growth rate of any possible instability, hydrodynamic or kinetic, of the linear regime in magnetized plasmas. In some cases these bounds are stringent and give useful information in a simple form. It is also worth noting that these bounds are valid for all times. They limit the behavior of all modes, including stable waves, even for the short-time evolution of an initial perturbation. We believe that several plasma electrodynamic problems may be studied directly in space-time using the response functions derived here. A combination of approximation methods (like eigenfunction expansions) and numerical methods may add to the knowledge already obtained through the standard Fourier-Laplace treatment. The compact expression for the time response of quasi-electrostatic modes in a magnetized plasma (Sec. IV) seems particularly suitable for this project, considering also that powerful numerical methods for Volterra integral equations are available.[16]

The impulsion method can be extended to the treatment of nonhomogeneous plasmas where $h_{ik}(\mathbf{x},\mathbf{x}',t)$ maintains a separate dependence on two spatial variables. We found it helpful in deriving Eq. (3) for the electrostatic modes of a system of convergent beams.[7] In particular problems with boundary conditions or initial values for $\mathbf{E}(\mathbf{x},t)$ which show special symmetries, Eq. (2) simplifies, as some coordinates are ignorable. In that case the response functions of the problem are more easily derived considering percussions distributed on surfaces with the appropriate symmetry.

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