Squeezing of the cavity vacuum by charged particles

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It is shown that the electromagnetic vacuum in a cavity is squeezed if one or several charges are present. For a bright electron beam typical of present high-intensity free-electron lasers, the effect is not necessarily small.

INTRODUCTION

With the recent advances in superconducting microwave cavities it has become possible to build and monitor a cavity with extremely high-Q factors ($Q \simeq 10^9$ and higher).[1] In such a cavity a single-quantum mode of the electromagnetic field can be excited and sustained for a long time. If such an empty cavity is left alone, with no external or internal perturbers, the state of the electromagnetic field is the cavity vacuum defined simply by $a\,|\,0\rangle = 0$, where a is the annihilation operator for the single-mode cavity excitation. If atoms are injected into such a cavity, subtle quantum effects like the so-called collapses and revivals[2] can occur as a result of resonant interaction of the atom with the cavity field. Due to modern Rydberg state spectroscopy the cavity field and its dynamics can be monitored experimentally in a very accurate way.

In this paper we present a completely different single-mode cavity effect, leading to squeezed or reduced quantum fluctuations of the cavity electromagnetic field. We show that if instead of a resonant atom a pointlike charged particle is injected into an empty cavity, the cavity vacuum is modified by the interaction of the charge with the zero mode of the empty cavity vacuum field. This modified vacuum that we shall call the charged vacuum and denote by $|\,\tilde{0}\rangle$ has many interesting properties. In a framework of a very simple quantum electrodynamical model we show that this charged vacuum is a squeezed empty cavity vacuum $|\,0\rangle$. Such states have attracted considerable experimental and theoretical attention in the last few years[3] due to new possibilities of measurements with a background level of noise below the quantum-mechanical fluctuations.

DYNAMICS OF A CHARGED PARTICLE IN A CAVITY

We consider a single charged particle interacting with one electromagnetic mode of frequency ω, confined to a very-high-Q cavity of volume V. The Hamiltonian of such a system has the form

$$H = \hbar\omega a^\dagger a + \frac{1}{2m}(\mathbf{p} - e\,\mathbf{A})^2 \qquad (1)$$

where in Coulomb gauge the operator of the electromagnetic field vector potential is

$$\mathbf{A} = \boldsymbol{\epsilon}\left[\frac{2\pi\hbar c^2}{\omega V}\right]^{1/2}(a + a^\dagger)\ . \qquad (2)$$

In this expression $\boldsymbol{\epsilon}$ denotes the polarization of the field and we have neglected the spatial dependence of the cavity mode. This amounts to a dipole approximation for the electromagnetic field, which in general may not be very well justified. For the solubility of our model we perform this approximation nevertheless, and comment on its applicability at the end of the paper. The dynamics of this simple quantum electrodynamical model can be solved exactly. This is due to the fact that the Heisenberg equations of motion generated by Eq. (1) are linear. Thus they can be solved like c-number equations using, for example, the Laplace transform method (see e.g., Ref. 4). After some simple but tedious algebra the solution for the field variables has the following form:

$$a(t) = f_1(t)a(0) + f_2(t)a^\dagger(0) + f_3(t)\ , \qquad (3a)$$

$$a^\dagger(t) = [f_1(t)]^* a^\dagger(0) + [f_2(t)]^* a(0) + [f_3(t)]^*\ , \qquad (3b)$$

where the c-number complex functions $f_i(t)$ are given by the following expressions:

$$f_1(t) = e^{-i\Omega t}\cosh^2 r - e^{i\Omega t}\sinh^2 r\ , \qquad (4a)$$

$$f_2(t) = -i\sin(\Omega t)\sinh(2r)\ , \qquad (4b)$$

$$f_3(t) = -\alpha(e^{-i\Omega t} - 1)\cosh r + \alpha(e^{i\Omega t} - 1)\sinh r\ . \qquad (4c)$$

The three parameters Ω, r, and α that define these functions have the following explicit relation to the coupling constants of the Hamiltonian [Eq. (1)]:

$$r = \tfrac{1}{4}\ln(1+\sigma) = \tfrac{1}{2}\ln\frac{\Omega}{\omega}\ , \qquad (5a)$$

$$\Omega = \omega(1+\sigma)^{1/2}\ , \qquad (5b)$$

$$\alpha = \boldsymbol{\epsilon}\cdot\mathbf{p}\,\frac{e}{m}\left[\frac{2\pi}{\hbar\Omega V}\right]^{1/2}\frac{1}{\Omega}\ , \qquad (5c)$$

where $\mathbf{p}$ denotes the eigenvalue of the canonical momentum operator $\hat{\mathbf{p}}$ (which is a constant of motion) and

$$\sigma=\frac{4\pi e^2}{m\omega^2 V}=\frac{r_0\lambda^2}{\pi V} . \tag{6}$$

In the last version of Eq. (6) r_0 denotes the classical electron radius and $\lambda=2\pi c/\omega$ the wavelength of the (unperturbed) cavity mode. The explicit form of the time evolution of the electromagnetic field variables gives the full dynamics of the charged particle described by the Hamiltonian (1).

SQUEEZED FLUCTUATIONS

For many of the systems in nonlinear optics which exhibit squeezing, the quantity of interest is rotating-wave squeezing[3] where one and the same of the two quadrature amplitudes of the electric field, i.e., the coefficient of either $\cos(\omega t)$ or $\sin(\omega t)$, has reduced fluctuations independent of time. This is not so in the present case where, owing to the time dependence of $a(t)$ as exhibited in Eqs. (3) and (4), the rotating-wave squeezing will oscillate between one and the other of these amplitudes. We therefore calculate laboratory-frame squeezing, i.e., we compute the fluctuations of the two Hermitian parts of the time-dependent annihilation operator $a(t)=a_1(t)+ia_2(t)$. We impose boundary conditions such that the charge enters at $t=0$ the cavity which at this time is in its vacuum state defined by $a(0)|0\rangle=0$. In this cavity vacuum state we have $[\Delta a_1(0)]^2=[\Delta a_2(0)]^2=\frac{1}{4}$. As a result of the interaction with the injected charge the vacuum of the cavity is changed and the fluctuations in these components are given by the expressions

$$\Delta a_1^2(t)=\tfrac{1}{4}[\cos^2(\Omega t)+e^{-4r}\sin^2(\Omega t)]\le \tfrac{1}{4} , \tag{7a}$$

$$\Delta a_2^2(t)=\tfrac{1}{4}[\cos^2(\Omega t)+e^{4r}\sin^2(\Omega t)]\ge \tfrac{1}{4} , \tag{7b}$$

where $r\ge 0$. From these equations it is clear that during the time evolution the fluctuations of $\Delta a_1^2(t)$ can be reduced below the quantum-mechanical empty-cavity noise. For example, for $\Omega t=\pi/2$ we have

$$\Delta a_1^2\left[t=\frac{\pi}{2\Omega}\right]=\tfrac{1}{4}e^{-4r},\quad \Delta a_2^2\left[t=\frac{\pi}{2\Omega}\right]=\tfrac{1}{4}e^{4r} , \tag{8}$$

i.e., an e^{-4r} reduction of the empty-cavity noise is possible. Note that it is never possible to reduce the a_2 component below the vacuum noise. This is related to the choice of the phase made in Eq. (2) and will be discussed below. As a consequence of the interaction between the charge and the cavity mode the initial complete symmetry between the two variables a_1 and a_2 does not hold any more for finite t. We also note that

$$\Delta a_1^2(t)\Delta a_2^2(t)=\tfrac{1}{16}\{1+\tfrac{1}{2}\sin^2(\Omega t)[\cosh(4r)-1]\} . \tag{9}$$

This implies that [except for $\sin(\Omega t)=0$] the state that develops out of the initial vacuum is no longer a minimum uncertainty state.

If the time scales involved in the measurement process are longer than $1/\Omega$, the maximum squeezing exhibited in Eq. (8) cannot be observed. If a time average over the fast oscillations given by Ωt is performed we obtain

$$\overline{\Delta a_1^2}=\tfrac{1}{8}(1+e^{-4r}),\quad \overline{\Delta a_2^2}=\tfrac{1}{8}(1+e^{4r}) , \tag{10}$$

i.e., the noise in the a_1 quadrature is effectively squeezed by at most a factor of 2.

The situation encountered here is very similar to a recent calculation of the squeezing induced by a harmonic oscillator interacting with one mode of the electromagnetic field.[5] In the latter case it was pointed out that it is the fluctuations of the magnetic field that are decreased while those of the electric field are increased. Equations (7a) and (7b) show that this also holds true in the present case.

THE CHARGED VACUUM

There is a simple explanation of the phenomenon calculated in the previous section. In the empty cavity, $|0\rangle$ is the vacuum state of the cavity energy $\hbar\omega a^\dagger a$. When the cavity interacts with the charged particle, it is clear that the state $|0\rangle$ is no longer the ground state of the full Hamiltonian [Eq. (1)]. As a result of this interaction a new vacuum emerges, which we denote by $|\tilde{0}\rangle$ and call a charged vacuum. It is also clear that $\langle\tilde{0}|0\rangle\neq 1$, leading to all kinds of effects including the squeezing property discussed above.

Because of the quadratic form of the interaction Hamiltonian it is easy to diagonalize the energy operator, i.e., to find solutions of the eigenvalue problem[6]

$$H|\phi_{np}\rangle=E_{np}|\phi_{np}\rangle . \tag{11}$$

Simple algebra shows that

$$|\phi_{np}\rangle=\exp\left[\frac{r}{2}[a^2-(a^\dagger)^2]\right]\exp[\alpha(a^\dagger-a)]|p,n\rangle \tag{12}$$

and

$$E_{np}=\frac{p^2}{2m}+\hbar\Omega(n+\tfrac{1}{2})+\frac{\hbar\alpha^2\Omega^2}{\omega}-\frac{\hbar\omega}{2} , \tag{13}$$

where all the relevant parameters are defined by Eqs. (5) and (6) and $|p,n\rangle$ are the empty cavity eigenstates of $\hat{p}^2/2m+\hbar\omega a^\dagger a$. For given p, it is then clear that $|\phi_{0p}\rangle$ is the state with lowest energy. In particular, for $p=0$ we obtain from this relation the charged vacuum

$$|\tilde{0}\rangle=\exp\left[\frac{r}{2}[a^2-(a^\dagger)^2]\right]|0\rangle \tag{14}$$

where we recognize in the $\exp\{r/2[a^2-(a^\dagger)^2]\}$ the well-known squeezing operator. This operator acts on the empty cavity vacuum $|0\rangle$, dressing it with an infinite number of photons. This formula shows that indeed the charged vacuum is the squeezed vacuum with the squeezing parameter given by r.

The expectation value of the photon number with respect to the state $|\phi_{np}\rangle$ is

$$\tilde{N}=\langle\phi_{np}|a^\dagger(t)a(t)|\phi_{np}\rangle$$
$$=\sinh^2 r+n\cosh(2r)+\alpha^2 e^{-2r} , \tag{15}$$

which is independent of time as it should be. For $p=0$

and $n=0$, only the first term survives and specifies the mean photon number in the squeezed vacuum state (14). Since the parameter r is independent of $\hbar$, so is this first term. Consequently, it appears as a quantum-mechanical contribution to the mean-field energy $\hbar\omega\tilde{N}$. Equation (15) for $n=0$ should be compared with the photon number for the case when the charge enters the empty cavity at $t=0$:

$$N(t)=\langle 0|a^\dagger(t)a(t)|0\rangle$$
$$=\sinh^2(2r)\sin^2(\Omega t)$$
$$+4\alpha^2\sin^2\left[\frac{\Omega t}{2}\right]\left[e^{2r}\cos^2\left[\frac{\Omega t}{2}\right]+e^{-2r}\sin^2\left[\frac{\Omega t}{2}\right]\right] . \quad (16)$$

The time average of $N(t)$ is larger than $\tilde{N}$ for $n=0$.

Of course, the charged vacuum (14) is squeezed. In the state $|\phi_{0p}\rangle$ we have in place of Eqs. (7a) and (7b)

$$\Delta a_1^2(t)=\tfrac{1}{4}e^{-2r},\quad \Delta a_2^2(t)=\tfrac{1}{4}e^{2r} \quad (17)$$

independently of α and t. Comparing with Eq. (8), we notice that the maximal squeezing [at $t=(2n+1)\pi/2\Omega$] induced by a charge entering the cavity overshoots the steady-state squeezing (17) of the charged vacuum. For small r the latter agrees with the time-averaged squeezing (10) induced by the entering charge.

CONCLUSIONS

Before we estimate the amount of squeezing we want to reemphasize the limitations of our model. These derive from the two standard assumptions that we made for the sake of simplicity, viz., the long-wavelength approximation and the restriction to one mode of the electromagnetic field. It is the long-wavelength approximation that rendered the model readily solvable. For by suppressing the mode functions $\exp(ikz)$ it essentially made the electron momentum a c number. As a consequence, all recoil effects are neglected, and the electron's canonical momentum is conserved. If we keep the mode functions $\exp(ikz)$ the model does not seem to allow for an analytic solution anymore. However, it is not difficult to see that the following effects will show up: first, the total momentum of the electron plus the field will be conserved. Second, the squeezing will, as a function of z, continuously change from one quadrature to the other. In general, neither a_1 or a_2 will be squeezed but rather $ae^{i\phi}+a^\dagger e^{-i\phi}$ where the phase ϕ depends on the state of the electron. This is a well-known effect.[7] The single-mode restriction could be removed at the expense of a significant increase of the involved algebra. This is because due to the $\mathbf{A}^2$ term the individual modes are coupled. We should also emphasize that we did not take proper care of the boundary conditions. We assumed that the electromagnetic field was in its ground state up to the time $t=0$. At this time we turned on the electron-field interaction which is equivalent to turning on the electron's charge. A proper treatment would have to consider that the electron enters (and leaves) the cavity while its charge is conserved.

From Eq. (7a) the noise reduction factor for $\cos(\Omega t)=0$ is

$$e^{-4r}=(1+\sigma)^{-1} . \quad (18)$$

For a single electron in a cavity with the volume V the parameter σ defined in Eq. (6) is normally a very small quantity. For the volume of the cavity cannot be much smaller than λ^3 in which case σ is of the order of $r_0/\lambda \ll 1$. We may, however, consider an electron beam with the density $N_e/V=\rho_e$. Then σ is replaced by

$$\sigma=r_0\lambda^2\rho_e/\pi . \quad (19)$$

Current free-electron-laser facilities employ very bright electron beams with currents up to 10 kA.[8] For squeezing at optical wavelengths the value of σ is under these conditions still very small: for a current of 10 kA and a beam area of 1 mm^2 we obtain $\sigma=2\times10^{-7}$ at $\lambda=1$ μm. However, the situation changes drastically for longer wavelengths; for example, under the same conditions but for $\lambda=0.5$ cm we have $\sigma=5$ corresponding to a noise reduction of approximately 80%. It then appears in view of the λ^2 dependence of the parameter σ that this is a typical infrared effect which would have to be considered with due caution. Again, for very long wavelengths the boundary conditions would have to be reconsidered. However, in one respect long wavelengths appear to be more favorable: in order that the long-wavelength approximation apply, the length of the electron pulse must be small compared to the wavelength. For the assumed high current, this seems to be ruled out at optical wavelengths while it appears possible at $\lambda=0.5$ cm.

The squeezing discussed in this paper (and, in a similar situation, in Ref. 5) is related to fluctuations in the magnetic energy density in a cavity due to the presence of a single charge. On the basis of this calculation one is led to expect this type of squeezing to be an ubiquitous feature on the microscopic level which should be present in the vicinity of atoms and molecules quite generally.[9] It is also interesting to speculate that the charged squeezed vacuum in a cavity considered in this paper bears a strong resemblance to the color dielectric vacuum of quantum chromodynamics inside a hadron.[10] We hope to come back to this aspect of the present model in a future publication.

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