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Transport theory of a random planar waveguide with a fixed scatterer: Mode theory

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Based on the unified theory of random medium and random boundaries introduced in a previous paper [Furutsu, J. Opt. Soc. Am. **A2**, 913 (1985)], an exact version of the (normal) mode theory of both the coherent wave and the mutual coherence function is given without assuming a particular model. A mode equation for the second-order Green function is obtained from the governing (Bethe-Salpeter) equation based on a Maclaurin expansion at the set of poles of the first-order (renormalized) Green function, and is eventually given in the form of an equation of radiative transfer. The expansion would not be possible with the "bare" Green function. The overall unitarity of the Bethe-Salpeter equation is investigated in particular detail, and the involved optical relations are shown, not only of the entire system but also of the medium and each of the boundaries (of intrinsically dispersive property) separately. An exact theory of a fixed scatterer embedded in the waveguide is given with several expressions of the solution, including that with the conventional form in scattering theory of a coherent wave, in terms of an effective cross section having negative values in the shadow direction and its neighborhood. A detailed structure of the power equations, constructed by both coherent and incoherent waves in a complex way, is shown in terms of two optical relations for the scatterer's two basic quantities that change the original Bethe-Salpeter equation. Whenever possible, the equations are so written in a general form that they hold true for a wide class of random systems with a fixed scatterer. Specific examples are given.

I. INTRODUCTION

Transport equations have been investigated using coupled power equations, not only for an unbounded space with a few interfaces of different media,[1-6] but also for waveguides with either random walls[7] or a random medium.[8] The coupled power equations are a simplified version of the transport equations that neglect all interference effects. A basic equation to derive the transport equations is an equation for the mutual coherence function, and its exact version is given by an integral equation of the form of Bethe-Salpeter (BS) equation, independent of the characteristics of the random system involved.[9,10] A unified theory of random media and random boundaries was recently given, typically for three random layers with two rough boundaries of interface, based on an unperturbative theory of the BS equation for the second-order Green function, in such a way that the medium and the boundaries are involved on exactly the same footing.[11] Here the solution is given by several expressions in terms of scattering matrices defined for the medium and each of the boundaries separately, by introducing addition formulas for two kinds of scattering matrices with coherent and incoherent characteristics, respectively. Each expression of the solution contains both the coherent and the incoherent waves at the right place, and provides an exact version of the boundary-value solution that could be obtained by solving a transport equation, subjected to the conventional boundary condition which so far has been heuristically given and is really not valid when used in the case of two or more boundaries with separations of small distance. A solution was also obtained for the case of a fixed scatterer embedded in a random medium, and was given by an expression with the same form as in the conventional scattering theory of a coherent wave, in terms of an effective scattering matrix of the scatterer, as affected by the medium fluctuation.

These results can be applied to a random planar waveguide by regarding the two outside spaces of the middle layer as definite media constructing the random walls. In this paper, basic equations of the second-order Green function are first summarized in a compact form, with particular emphasis on the detailed structure of power equations in terms of optical relations ensuring local power conservation at every point (Sec. II). An exact version of the conventional mode theory (for the coherent wave) is then investigated (Sec. III), and is followed by a mode transformation of the BS equation; an exact equation of radiative transfer is derived therefrom as an alternative means of obtaining the Green function, together with the mode expressions of physical quantities and optical relations (Sec. IV). Finally, in Sec. V (and Appendix E), a detailed theory of a fixed scatterer embedded in the waveguide is developed, including related optical relations, in a general form applicable to a wide class of random systems with fixed scatterer whenever possible. As an illustration, specific expressions of statistical parameters are obtained to the ladder approximation in detail for the random boundaries (Sec. VI).

II. PRELIMINARIES AND BASIC EQUATIONS

We employ the following notations: The space coordinate vector is denoted by $\hat{\mathbf{x}}=(x_1,x_2,x_3)=(\boldsymbol{\rho},z)$ in terms of the two-dimensional coordinate vector $\boldsymbol{\rho}=(x_1,x_2)$ and $z=x_3$. The scalar product of two space vectors $\hat{\mathbf{a}}=(\mathbf{a},a_z)$ and $\hat{\mathbf{b}}=(\mathbf{b},b_z)$ is denoted by $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=\mathbf{a}\cdot\mathbf{b}+a_zb_z$ with $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2$. A planar waveguide is considered, and the two rough boundaries are assumed to be described by $z=-\zeta_1(\boldsymbol{\rho})$ and $z=d_2+\zeta_2(\boldsymbol{\rho})$, where $\zeta_j(\boldsymbol{\rho})>0$, $j=1,2$, are surface displacements from two reference boundary planes, say S_1 and S_2, chosen at $z=0$ and d_2, respectively (Fig. 1). A scalar wave function of the form $\psi(\hat{\mathbf{x}})e^{i\omega t}$ (where $\omega>0$ and t is time) is considered, with the wave equation

$$[L-q(\hat{\mathbf{x}})]\psi(\hat{\mathbf{x}})=j(\hat{\mathbf{x}})\ , \tag{2.1a}$$

$$L=-\left[\frac{\partial}{\partial\hat{\mathbf{x}}}\right]^2-k_0^2,\quad k_0=\omega/c\ . \tag{2.1b}$$

The medium is assumed to be nondissipative for the time being, and $q(\hat{\mathbf{x}})=q^*(\hat{\mathbf{x}})$ is the random part; $j(\hat{\mathbf{x}})$ provides a source term.

The boundary conditions on the rough boundaries can be transferred onto the reference boundary planes S_1 and S_2, and can be given by an equivalent boundary equation of the form[12,13]

$$-\partial_n\psi=B^{(j)}\psi\Big|_{S_j},\quad j=1,2\ . \tag{2.2}$$

$\partial_n=\hat{\mathbf{n}}\cdot\partial/\partial\hat{\mathbf{x}}=\pm\partial/\partial z$, where $\hat{\mathbf{n}}$ is the unit vector (inward normally directed) of S_j, and $B^{(j)}$ is a $\boldsymbol{\rho}$ operator depending on both $\boldsymbol{\rho}$ and $\partial/\partial\boldsymbol{\rho}$ and will be referred to as the surface impedance. If we write $B^{(j)}=B_0+b^{(j)}$ with an impedance B_0 when $\zeta_j(\boldsymbol{\rho})=0$ (which is presently assumed to be a numerical constant, including 0 and ∞), an exact $B^{(j)}$ is obtained [Eq. (75) of Ref. 12]; to the first order in ζ_j,

$$b^{(j)}=\frac{\partial}{\partial\boldsymbol{\rho}}\cdot\left[\zeta_j(\boldsymbol{\rho})\frac{\partial}{\partial\boldsymbol{\rho}}\right]+(k_0^2+B_0^2)\zeta_j(\boldsymbol{\rho})\ , \tag{2.3}$$

which is exactly a Hermitian operator [Eq. (2.9a)] when B_0 is real. Generally, any $\boldsymbol{\rho}$ operator, $B(\boldsymbol{\rho},\partial/\partial\boldsymbol{\rho})$, can be regarded as a $\boldsymbol{\rho}$ matrix having the matrix elements $B(\boldsymbol{\rho}\mid\boldsymbol{\rho}')$, defined according to

$$B(\boldsymbol{\rho}\mid\boldsymbol{\rho}')=(2\pi)^{-2}\int d\boldsymbol{\lambda}\,B\exp[-i\boldsymbol{\lambda}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')]\ , \tag{2.4}$$

where B affects only $\boldsymbol{\rho}$ and not $\boldsymbol{\rho}'$. An explicit expression of Eq. (2.2) can be written as

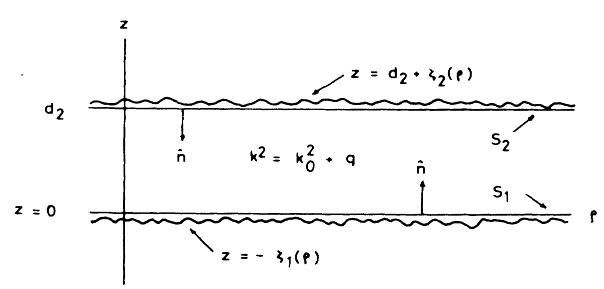

FIG. 1. Geometry and notations of the waveguide for Eqs (2.1) and (2.2). The medium is assumed to be homogeneous ($q=0$) in each boundary space between S_j and the real boundary.

$$-\partial_n\psi_j(\boldsymbol{\rho})=\int_{S_j}d\boldsymbol{\rho}'B^{(j)}(\boldsymbol{\rho}\mid\boldsymbol{\rho}')\psi_j(\boldsymbol{\rho}')\ , \tag{2.5}$$

in terms of the notation $\psi_j(\boldsymbol{\rho})$ for $\psi(\hat{\mathbf{x}})$ bounded on S_j.

Within the waveguide, the power-flux vector $\hat{\mathbf{W}}=(\mathbf{W},W_z)$ of the wave may be given by

$$\hat{\mathbf{W}}(\hat{\mathbf{x}})=\psi^*(\hat{\mathbf{x}})\hat{\boldsymbol{\alpha}}\psi(\hat{\mathbf{x}})\ , \tag{2.6a}$$

with an operator vector $\hat{\boldsymbol{\alpha}}=(\boldsymbol{\alpha},\alpha_z)$, defined by

$$\hat{\boldsymbol{\alpha}}=(2i)^{-1}\left[\frac{\overleftarrow{\partial}}{\partial\hat{\mathbf{x}}}-\frac{\overrightarrow{\partial}}{\partial\hat{\mathbf{x}}}\right]\ , \tag{2.6b}$$

where the arrows $\leftarrow$ and $\rightarrow$ mean the operation on the left- and right-hand sides, respectively. The power equation is

$$\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot\hat{\mathbf{W}}=(2i)^{-1}(\psi^*j-\psi j^*)\ . \tag{2.7}$$

Here, using the boundary Eq. (2.5), the power component normal to S_j, $W_j^{(n)}(\boldsymbol{\rho})$, can be given by

$$\begin{aligned}W_j^{(n)}(\boldsymbol{\rho})&=(2i)^{-1}\psi^*(\overleftarrow{\partial}_n-\overrightarrow{\partial}_n)\psi\Big|_{S_j}\\&=\frac{1}{2i}\int d\boldsymbol{\rho}'[\psi_j^*(\boldsymbol{\rho})B^{(j)}(\boldsymbol{\rho}\mid\boldsymbol{\rho}')\psi_j(\boldsymbol{\rho}')\\&\qquad-\psi_j(\boldsymbol{\rho})B^{(j)*}(\boldsymbol{\rho}\mid\boldsymbol{\rho}')\psi_j^*(\boldsymbol{\rho}')]\ ,\end{aligned} \tag{2.8}$$

in terms of the surface impedance $B^{(j)}$. Hence, when the boundary is perfectly nondissipative, the condition that the total vertical power component integrated over S_j be zero leads to a constraint that

$$B^{(j)\dagger}(\boldsymbol{\rho}\mid\boldsymbol{\rho}')\equiv B^{(j)*}(\boldsymbol{\rho}'\mid\boldsymbol{\rho})=B^{(j)}(\boldsymbol{\rho}\mid\boldsymbol{\rho}')\ , \tag{2.9a}$$

i.e., that the $\boldsymbol{\rho}$ matrix $B^{(j)}$ must be Hermitian. Hereafter, the dagger will be attached to mean the Hermitian conjugation.

The Hermitian condition for $B^{(j)}$ enables us to introduce a relation of the form [cf. Eq. (242) in Ref. 12]

$$\begin{aligned}\frac{\partial}{\partial\boldsymbol{\rho}}\cdot\mathbf{s}^{(j)}(\boldsymbol{\rho}\mid\boldsymbol{\rho}_1;\boldsymbol{\rho}_2)=\frac{1}{2i}[&B^{(j)*}(\boldsymbol{\rho}\mid\boldsymbol{\rho}_1)\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_2)\\&-B^{(j)}(\boldsymbol{\rho}\mid\boldsymbol{\rho}_2)\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_1)]\ ,\end{aligned} \tag{2.9b}$$

which reproduces the condition (2.9a) by the $\boldsymbol{\rho}$ integration, because there is no contribution from the left-hand-side space divergence. Hence the right-hand side of Eq. (2.8) can be written as a two-dimensional space divergence:

$$\begin{aligned}W_j^{(n)}(\boldsymbol{\rho})=-\frac{\partial}{\partial\boldsymbol{\rho}}\cdot\int d\boldsymbol{\rho}_1d\boldsymbol{\rho}_2\mathbf{s}^{(j)}(\boldsymbol{\rho}\mid\boldsymbol{\rho}_1;\boldsymbol{\rho}_2)&\\\times\psi_j^*(\boldsymbol{\rho}_1)\psi_j(\boldsymbol{\rho}_2)\ .&\end{aligned} \tag{2.10}$$

This is really the case of expression (2.3) for $b^{(j)}$, and the substitution into Eq. (2.8) directly leads to the divergence expression

$$W_j^{(n)}(\boldsymbol{\rho})=-\frac{\partial}{\partial\boldsymbol{\rho}}\cdot[\zeta_j(\boldsymbol{\rho})(\psi_j^*\boldsymbol{\alpha}\psi_j)(\boldsymbol{\rho})]\ , \tag{2.11}$$

where α is the ρ operator defined by Eq. (2.6b).

The power equation (2.7) does not hold on S_1 and S_2, although it does everywhere inside the waveguide, but the power-flux vector can be redefined such that a new power equation holds everywhere including S_1 and S_2; that is, if the vertical component W_3 is understood to change to zero after crossing S_1 and S_2, the term $\partial W_3/\partial z$ in power equation (2.7) yields an additional term $\delta_j(z)W_j^{(n)}(\rho)$ with the factor

$$\delta_j(z)=\delta(z-d_j),\quad d_1=0 \tag{2.12}$$

owing to the discontinuity at S_j. Therefore, the continuity of power flux is ensured everywhere including S_1+S_2 by subtracting this additional term from Eq. (2.7) yielding, with Eq. (2.10), a new power equation, given by

$$\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot\hat{\mathbf{W}}(\hat{\mathbf{x}})+\frac{\partial}{\partial\rho}\cdot\sum_{j=1}^{2}\int d\rho_1 d\rho_2 \mathbf{s}^{(j)}(\hat{\mathbf{x}}\mid\rho_1;\rho_2)\,\psi_j^*(\rho_1)\psi_j(\rho_2)$$
$$=(2i)^{-1}[\psi^* j(\hat{\mathbf{x}})-\psi j^*(\hat{\mathbf{x}})]\ , \tag{2.13}$$

in terms of the notation

$$s^{(j)}(\hat{\mathbf{x}}\mid\rho_1;\rho_2)=\delta_j(z)\mathbf{s}^{(j)}(\rho\mid\rho_1;\rho_2)\ . \tag{2.14a}$$

The equation shows that the continuity of power flux can be ensured only with an additional flux, given by $\mathbf{s}^{(j)}\psi^*\psi(\hat{\mathbf{x}})$, meaning a contribution from a surface wave.

So far the medium has been assumed to be nondispersive, and given in the form $k^2(\hat{\mathbf{x}})=k_0^2+q(\hat{\mathbf{x}})$. However, it is now straightforward to rewrite the equations to meet with the more general case of a dispersive anisotropic medium, in which k^2 is an $\hat{\mathbf{x}}$ operator with the matrix elements $k^2(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}')$:

$$k^2\psi(\hat{\mathbf{x}})=\int d\hat{\mathbf{x}}'k^2(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}')\psi(\hat{\mathbf{x}}')\ . \tag{2.14b}$$

The same relation as Eq. (2.9b) holds true also for the medium with the replacement of $B^{(j)}\to k^2$, i.e.,

$$\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot\hat{\mathbf{s}}^{(q)}(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)$$
$$=\frac{1}{2i}[k^{*2}(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}_1)\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_2)-k^2(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}_2)\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_1)]\ , \tag{2.14c}$$

where $\hat{\mathbf{s}}^{(q)}=(\mathbf{s}^{(q)},s_z^{(q)})$ with the vertical component $s_z^{(q)}$. This results in changing the power Eq. (2.13) by an additional divergence term of $\hat{\mathbf{s}}^{(q)}$ and adjusts the power flux to meet with the dispersive effect (Sec. IV).

The wave equations (2.1) and the boundary equation (2.2) can be unified to be written by one equation of the form

$$(L-q-B^{(1)}-B^{(2)})\psi(\hat{\mathbf{x}})=j(\hat{\mathbf{x}})\ . \tag{2.15}$$

Here both $B^{(j)}$ and q are now regarded as $\hat{\mathbf{x}}$ coordinate matrices, having the elements

$$B^{(j)}(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}')=\delta_j(z)B^{(j)}(\rho\mid\rho')\delta_j(z') \tag{2.16}$$

and $q(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}')=q(\hat{\mathbf{x}})\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}')$; the solution $\psi(\hat{\mathbf{x}})$ is subject to a new boundary condition that

$$\partial_n\psi=0 \tag{2.17}$$

in the regions outside of S_1 and S_2. The proof is given by integrating both sides of Eq. (2.15) with respect to z over an infinitesimal range enclosing S_j, i.e., $d_j+0\geq z\geq d_j-0$; hence, the boundary Eq. (2.2) is reproduced.

In the same way, $\psi^*(\hat{\mathbf{x}})$ is the solution of

$$\psi^*(\overleftarrow{L}-q-B^{(1)}-B^{(2)})=j^*\ , \tag{2.18}$$

when the medium and the boundaries are both nondissipative [Eq. (2.9a)]. It may be remarked that the power equation (2.13) is reproduced directly by Eqs. (2.15) and (2.18) with the relation (2.9b).

A deterministic Green function of the new wave Eq. (2.15), $g(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}')$, is defined by

$$(L-q-B^{(1)}-B^{(2)})g(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}')=\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}')\ , \tag{2.19a}$$

or in matrix form by

$$(L-v)g=1,\quad v=q+B^{(1)}+B^{(2)}\ , \tag{2.19b}$$

wherein v may be regarded as an effective medium representing both the medium and the boundaries on an equal basis.

By virtue of the boundary condition, it holds the symmetry

$$g^T(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}')\equiv g(\hat{\mathbf{x}}'\mid\hat{\mathbf{x}})=g(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}')\ , \tag{2.20a}$$

i.e., $g^T=g$, as may be directly shown by applying the Green theorem over the whole space enclosed by the real boundaries (with the impedance of a constant B_0). It follows from Eq. (2.19b) that $v^T=v$; hence, also $B^{(j)T}=B^{(j)}$, i.e.,

$$B^{(j)}(\rho'\mid\rho)=B^{(j)}(\rho\mid\rho')=B^{(j)*}(\rho'\mid\rho)\ , \tag{2.20b}$$

where the last equality is from Eq. (2.9a) and is valid only when the boundaries are nondissipative.

A. Statistical Green functions

Equation (2.19b) has the same form as that equation in an inhomogeneous random medium v and this enables the statistical Green functions also to be obtained in exactly the same form without making any distinction between the medium and the boundaries. To obtain the Green function of first order, $G=\langle g\rangle$, we first introduce an effective matrix M of v, defined according to

$$MG=\langle vg\rangle\ , \tag{2.21a}$$

which can be divided into three equations of $M^{(q)}$, $M^{(1)}$, and $M^{(2)}$, defined in the same fashion by

$$M^{(q)}G=\langle qg\rangle,\quad M^{(j)}G=\langle B^{(j)}g\rangle,\quad j=1,2\ . \tag{2.21b}$$

By averaging Eq. (2.19b), we obtain

$$(L-M)G=1,\quad M=M^{(q)}+M^{(1)}+M^{(2)}\ . \tag{2.22}$$

Since $G^T=G$ from Eq. (2.20a), Eq. (2.22) shows that $M^T=M$ or

$$M^{(q)T}+M^{(1)T}+M^{(2)T}=M^{(q)}+M^{(1)}+M^{(2)} , \qquad (2.23)$$

although this is generally not true for each term, i.e., $M^{(q)T}\neq M^{(q)}$ and $M^{(j)T}\neq M^{(j)}$, when taking into account the high-order terms.[11]

To the first order of q and ζ_j [Eq. (2.3)],

$$M^{(q)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')=\langle qGq \rangle(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') = \langle q(\hat{\mathbf{x}})q(\hat{\mathbf{x}}')\rangle G(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') , \qquad (2.24a)$$

$$M^{(j)}(\boldsymbol{\rho} \mid \boldsymbol{\rho}')=\langle b^{(j)}Gb^{(j)}\rangle(\boldsymbol{\rho} \mid \boldsymbol{\rho}') , \quad \langle q \rangle = \langle b^{(j)}\rangle = 0 , \qquad (2.24b)$$

where, in the last equation, G is to be regarded as a $\boldsymbol{\rho}$ matrix bounded on the plane S_j. Also it may be remarked that some of the higher-order terms of $M^{(q)}$ explicitly dependent on the boundary characteristic [as contrasted to the implicit dependence through G in Eq. (2.24a)] are appreciable in neighborhood of the boundaries, within a separation of the order of the medium correlation distance. The same is also for the $M^{(j)}$'s, whose higher-order terms also have an explicit dependence on the medium. In Sec. III, however, related statistical equations are formulated quite generally independent of the detailed structure of the medium and the boundaries.

Also for the Green function of second order, defined by

$$I(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2 \mid \hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')=\langle g^*(\hat{\mathbf{x}}_1 \mid \hat{\mathbf{x}}_1')g(\hat{\mathbf{x}}_2 \mid \hat{\mathbf{x}}_2')\rangle , \qquad (2.25)$$

or in matrix form by

$$I(1;2)=\langle g^*(1)g(2)\rangle \qquad (2.26)$$

(here and also hereafter, the subscript 1 is attached to the coordinates of quantities of the complex-conjugate wave function, and the subscript 2 is attached to those of quantities of the original wave function), the medium and the boundaries can be treated on exactly the same footing, by introduction of a quantity Δv, defined by

$$\Delta v = v - M = \Delta q + \Delta b^{(1)} + \Delta b^{(2)} , \qquad (2.27a)$$

where

$$\Delta q = q - M^{(q)}, \quad \Delta b^{(j)} = B^{(j)} - M^{(j)} . \qquad (2.27b)$$

We obtain an expression of the deterministic g, as

$$g = G(1+\Delta v g), \quad \langle \Delta v g \rangle = 0 , \qquad (2.28)$$

and, with the complex-conjugate expression for g^*, the substitution into the right-hand side of Eq. (2.26) yields a BS equation for $I(1;2)$ of the form

$$I(1;2)=G^*(1)G(2)[1+K(1;2)I(1;2)] . \qquad (2.29)$$

The factor $K(1;2)$ is defined according to[10]

$$K(1;2)I(1;2)=\langle \Delta v^*(1)\Delta v(2)g^*(1)g(2)\rangle , \qquad (2.30)$$

and can be divided into three major parts $K^{(q)}(1;2)$ and $K^{(j)}(1;2)$, $j=1,2$, that are independently contributed from the medium and the boundaries, respectively, on neglect of other cross terms depending on the both quantities.[11] Hence,

$$K(1;2) \simeq K^{(q)}(1;2)+K^{(1)}(1;2)+K^{(2)}(1;2) . \qquad (2.31)$$

To the perturbative approximation of Eqs. (2.24),

$$K^{(q)}(1;2)=\langle q^*(1)q(2)\rangle , \qquad (2.32a)$$

$$K^{(j)}(1;2)=\langle b^{(j)*}(1)b^{(j)}(2)\rangle, \quad j=1,2 \qquad (2.32b)$$

where $q^*=q$ and $b^{(j)*}=b^{(j)}$ in the nondissipative case [Eq. (2.20b)]. When the medium and/or the boundaries are composed of discrete scatterers and/or embossed surfaces, on the other hand, $M^{(q)}, K^{(q)}$, and/or $M^{(j)}, K^{(j)}$ are given in terms of the scattering amplitudes of each scatterer and/or embossed surface, averaged over all possible size, shape, orientation, etc., and of the densities per unit volume and/or per unit area.[10,12]

Here it may be remarked that Eqs. (2.24) for $M^{(q)}$ and $M^{(j)}$ involve the still unknown G; however, together with Eq. (2.22) for G, they provide a set of equations which determine M self-consistently. Also with this M, $K^{(q)}$ and $K^{(j)}$ given by (2.32) to the ladder approximation strictly fulfill the optical relation (2.48) necessary for power conservation of the entire system, by virtue of the relation

$$\delta(1;2)G(2)K(1;2)=\delta(1;2)M(2) , \qquad (2.32c)$$

as it is shown by using relation

$$\delta(1;2)A^*(1)B(2)=\delta(1;2)A^\dagger(2)B(2) .$$

B. Incoherent scattering matrix and physical quantities

The solution of the BS equation (2.29) can be written, with a coherent propagator $U^{(C)}(1;2)$, defined by

$$U^{(C)}(1;2)=G^*(1)G(2) , \qquad (2.33)$$

in the form

$$I(1;2)=U^{(C)}(1;2)+U^{(C)}(1;2)S(1;2)U^{(C)}(1;2) . \qquad (2.34)$$

The second term gives the incoherent part in terms of an incoherent scattering matrix $S(1;2)$, defined by

$$S(1;2)=K(1;2)[1+U^{(C)}(1;2)S(1;2)] \qquad (2.35)$$

(similar to the Lippmann-Schwinger equation for a coherent wave), and given in matrix form by

$$S=(1-KU^{(C)})^{-1}K = K+KU^{(C)}K+KU^{(C)}KU^{(C)}K+\cdots . \qquad (2.36)$$

Basic relations involved in K, $U^{(C)}$, S, and I are

$$KI=SU^{(C)}, \quad IK=U^{(C)}S , \qquad (2.37)$$

$$S=K+KIK , \qquad (2.38)$$

where, in the last equation, S is the same function of K and I as I in Eq. (2.34) is of $U^{(C)}$ and S.

The power fluxes of the coherent and the incoherent wave, $\hat{\mathbf{W}}^{(C)}=[\mathbf{W}^{(C)}, W_z^{(C)}]$ and $\hat{\mathbf{W}}^{(I)}=[\mathbf{W}^{(I)}, W_z^{(I)}]$, respectively, can be written, on using Eq. (2.6b), as

$$\hat{\mathbf{W}}^{(C)}(\hat{\mathbf{x}})=\hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}} \mid 1;2)U^{(C)}(1;2) . \qquad (2.39a)$$

$$\hat{\mathbf{W}}^{(I)}(\hat{\mathbf{x}})=\hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}} \mid 1;2)U^{(C)}SU^{(C)}(1;2) , \qquad (2.39b)$$

in terms of a vector matrix $\hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}} \mid 1;2)$, defined by the matrix elements

$$\hat{\alpha}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)=\delta(\hat{\mathbf{x}}_1-\hat{\mathbf{x}})\hat{\alpha}\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_2) . \tag{2.40}$$

An additional power flux due to the surface waves, $\mathbf{W}^{(S)}$, is given by

$$\mathbf{W}^{(S)}(\hat{\mathbf{x}})=\boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2)I(1;2) , \tag{2.41}$$

in terms of $\boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2)$ to be defined by Eq. (2.46).

C. Optical relations

The matrices M and K involved in the BS equation (2.29) are subject to a relation that ensures power conservation in the scattering at every point (optical relation). Here the relation can be written in a compact form by introduction of a coordinate matrix $\delta(\hat{\mathbf{x}} \mid 1;2)$, defined by the elements $\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_1)\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_2)$, in such a way that the product $\delta(\hat{\mathbf{x}} \mid 1;2)A^*(1)B(2)$ represents

$$\int d\hat{\mathbf{x}}_1 d\hat{\mathbf{x}}_2 \delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_1)\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_2)A^*(\hat{\mathbf{x}}_1 \mid \hat{\mathbf{x}}_1')B(\hat{\mathbf{x}}_2 \mid \hat{\mathbf{x}}_2') = A^*(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}_1')B(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}_2') . \tag{2.42}$$

For example, Eq. (2.9b) can be written simply by

$$\frac{\partial}{\partial \boldsymbol{\rho}}\cdot \mathbf{s}^{(j)}(\hat{\mathbf{x}} \mid 1;2)=\delta(\hat{\mathbf{x}} \mid 1;2)(2i)^{-1}[B^{(j)*}(1)-B^{(j)}(2)] , \tag{2.43}$$

in terms of the matrix $\mathbf{s}^{(j)}(\hat{\mathbf{x}} \mid 1;2)$, defined by the elements (2.14a); therefore, the sum $\mathbf{s}^{(b)}=\mathbf{s}^{(1)}+\mathbf{s}^{(2)}$ is

$$\frac{\partial}{\partial \boldsymbol{\rho}}\cdot \mathbf{s}^{(b)}(\hat{\mathbf{x}} \mid 1;2)=\delta(\hat{\mathbf{x}} \mid 1;2)(2i)^{-1}[v^*(1)-v(2)] , \tag{2.44}$$

in terms of the medium-boundary matrix v [Eq. (2.19b)], wherein the diagonal matrix q makes no contribution. It may be remarked that even when the medium is originally dispersive, having off-diagonal matrix elements, $q(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$, Eq. (2.44) still holds true with an additional divergence term of $\hat{\mathbf{s}}^{(q)}(\hat{\mathbf{x}} \mid 1;2)$ on the left-hand side [Eq. (2.14c)].

The optical relation is obtained by combining the relation (2.44) with

$$\langle v^*(1)g^*(1)g(2)\rangle=[M^*(1)+G(2)K(1;2)]I(1;2) , \tag{2.45a}$$

$$\langle v(2)g^*(1)g(2)\rangle=[M(2)+G^*(1)K(1;2)]I(1;2) , \tag{2.45b}$$

which directly follow from Eqs. (2.28) and (2.30). Hence in terms of the matrix $\boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2)$, defined according to

$$\boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2)I(1;2)=\langle \mathbf{s}^{(b)}(\hat{\mathbf{x}} \mid 1;2)g^*(1)g(2)\rangle \tag{2.46}$$

[in the same way as $K(1;2)$ has been defined according to Eqs. (2.30)], we find the relation

$$\frac{\partial}{\partial \boldsymbol{\rho}}\cdot \boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2) =(2i)^{-1}\delta(\hat{\mathbf{x}} \mid 1;2) \times\{M^*(1)-M(2)-[G^*(1)-G(2)]K(1;2)\} , \tag{2.47}$$

which demonstrates that the right-hand side can be expressed by the space divergence of a two-dimensional vector

$$\boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2)=(\boldsymbol{\beta}^{(1)}+\boldsymbol{\beta}^{(2)})(\hat{\mathbf{x}} \mid 1;2)$$

that differs from zero only on the reference boundaries S_1 and S_2, meaning a contribution from the surface waves.

The relation (2.47) holds true at every point in the space and on the boundaries, and ensures the BS equation (2.29) to be consistent with the averaged version of the (local) power equation (2.13), and will be referred to as the local optical relation. To ensure merely conservation of the total power, on the other hand, we only need a relation given by the $\hat{\mathbf{x}}$ integration of the local relation, i.e.,

$$\delta(1;2)\{M^*(1)-M(2)-[G^*(1)-G(2)]K(1;2)\}=0 . \tag{2.48}$$

Here no contribution is made by the divergence term and

$$\delta(1;2)= \int d\hat{\mathbf{x}}\, \delta(\hat{\mathbf{x}} \mid 1;2) , \tag{2.49}$$

and has the simple matrix elements $\delta(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2)$.

It may be remarked that a similar optical relation holds true also for each of the medium and the boundaries, independently, as long as that quantity is nondissipative, no matter whether the other quantities are even dissipative. That is, by replacing v in Eqs. (2.45) with each of q, $B^{(1)}$, and $B^{(2)}$, say A, we generally obtain an optical relation of the form[14]

$$\frac{\partial}{\partial \hat{\mathbf{x}}}\cdot \hat{\boldsymbol{\beta}}^{(A)}(\hat{\mathbf{x}} \mid 1;2)=(2i)^{-1}\delta(\hat{\mathbf{x}} \mid 1;2) \times\{M^{(A)*}(1)-M^{(A)}(2) -[G^*(1)-G(2)]K^{(A)}(1;2)\} \tag{2.50}$$

(where $\hat{\boldsymbol{\beta}}^{(q)}(\hat{\mathbf{x}} \mid 1;2)=0$ in the present case). Here $M^{(A)}$ and $K^{(A)}$ are defined according to Eqs. (2.21) and (2.30), respectively, so that to the first order, $K^{(j)}(1;2)$, $j=1,2$, is given by Eq. (2.32b), for example; wherein the first-order Green function G is the same in all the equations. The same is also for the integrated optical relation of each, being given by an equation similar to Eq. (2.48).

To see the detail of power conservation, we first write the power equation of the coherent wave, on using Eqs. (2.39a) and (2.22), as

$$\left[\frac{\partial}{\partial \hat{\mathbf{x}}}\cdot \hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}} \mid 1;2)+\gamma^{(q+b)}(\hat{\mathbf{x}} \mid 1;2)\right]U^{(C)}(1;2) =\delta(\hat{\mathbf{x}} \mid 1;2)(2i)^{-1}[G^*(1)-G(2)] . \tag{2.51}$$

Here

$$\gamma^{(q+b)}(\hat{\mathbf{x}} \mid 1;2)=\delta(\hat{\mathbf{x}} \mid 1;2)(2i)^{-1}[M^*(1)-M(2)] \tag{2.52}$$

$$=\gamma^{(q)}(\hat{\mathbf{x}} \mid 1;2)+ \sum_{j=1}^{2} \gamma^{(j)}(\hat{\mathbf{x}} \mid 1;2) , \tag{2.53}$$

and means a dissipation coefficient of the entire random system in terms of contributions from each of the medium and the boundaries, $\gamma^{(q)}$ and $\gamma^{(j)}$, which are defined by Eq. (2.52) with $M \rightarrow M^{(q)}$ and $M^{(j)}$, respectively. The total power, $\langle \hat{\mathbf{W}}(\hat{\mathbf{x}}) \rangle$, is given according to Eqs. (2.39) and (2.41) by

$$\langle \hat{\mathbf{W}}(\hat{\mathbf{x}}) \rangle = (\hat{\mathbf{W}}^{(C)} + \hat{\mathbf{W}}^{(I)} + \mathbf{W}^{(S)})(\hat{\mathbf{x}}) = (\hat{\boldsymbol{\alpha}} + \boldsymbol{\beta})(\hat{\mathbf{x}} \mid 1;2) I(1;2) , \tag{2.54}$$

and, as will be proved below, satisfies the equation of continuity

$$\frac{\partial}{\partial \hat{\mathbf{x}}} \cdot \langle \hat{\mathbf{W}}(\hat{\mathbf{x}}) \rangle = \delta(\hat{\mathbf{x}} \mid 1;2)(2i)^{-1}[G^*(1) - G(2)] \tag{2.55}$$

which is the same as obtained by averaging the original Eq. (2.13).

A contribution from each kind of the wave to Eq. (2.55) can be seen in detail by rewriting the optical relation (2.47) in terms of $S(1;2)$ as

$$\frac{\partial}{\partial \hat{\mathbf{x}}} \cdot [\hat{\boldsymbol{\alpha}}(\hat{\mathbf{x}} \mid 1;2) U^{(C)} S(1;2) + \boldsymbol{\beta}^{(e)}(\hat{\mathbf{x}} \mid 1;2)] = \gamma^{(q+b)}(\hat{\mathbf{x}} \mid 1;2) \tag{2.56}$$

with

$$\boldsymbol{\beta}^{(e)}(\hat{\mathbf{x}} \mid 1;2) = \boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2)[1 + U^{(C)} S(1;2)] , \tag{2.57}$$

which from Eq. (2.41), gives the power flux of the surface waves by

$$\mathbf{W}^{(S)}(\hat{\mathbf{x}}) = \boldsymbol{\beta}^{(e)}(\hat{\mathbf{x}} \mid 1;2) U^{(C)}(1;2) = \boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2) I(1;2) . \tag{2.58}$$

The proof of Eq. (2.56) is given by use of relation (2.51) as an expression for the factor

$$\delta(\hat{\mathbf{x}} \mid 1;2)(2i)^{-1}[G^*(1) - G(2)]$$

involved in the right-hand side of Eq. (2.47), followed by the multiplication to the right with

$$(1 - U^{(C)} K)^{-1} = 1 + U^{(C)} S . \tag{2.59}$$

Thus by multiplication of Eq. (2.56) to the right with $U^{(C)}(1;2)$, the left-hand side becomes the space divergence of $\hat{\mathbf{W}}^{(I)} + \mathbf{W}^{(S)}$ and the right-hand side means the dissipation of the coherent wave per unit volume; power equation (2.55) is derived therefrom with the aid of (2.51). The continuity of the power flux is ensured everywhere, and on the boundaries,

$$\langle W_j^{(n)}(\boldsymbol{\rho}) \rangle = -\frac{\partial}{\partial \boldsymbol{\rho}} \cdot \boldsymbol{\beta}^{(j)}(\boldsymbol{\rho} \mid 1;2) I(1;2) , \tag{2.60}$$

as it follows from Eqs. (2.10) and (2.46), implying that the power of the surface waves is supplied coherently and not through the incoherent scattering (see also Sec. VI).

III. MODE THEORY OF GREEN FUNCTION FOR THE COHERENT WAVE

To solve Eq. (2.22) for the Green function

$$G(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') = G(z \mid \boldsymbol{\rho} - \boldsymbol{\rho}' \mid z') ,$$

we first observe that $M^{(q)} \simeq M^{(q)T}$ is dependent also on the boundaries [e.g., through G in Eq. (2.24a)] and can be written generally in the form

$$M^{(q)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') = M^{(q)}(z \mid \boldsymbol{\rho} - \boldsymbol{\rho}' \mid z') , \tag{3.1a}$$

wherein taken into account is only the translational invariance. Equation (3.1a) may be approximated more simply by

$$M^{(q)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') \simeq M^{(q)}(\boldsymbol{\rho} - \boldsymbol{\rho}', \mid z - z' \mid) \tag{3.1b}$$

over a range of the points $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$, when separated enough from either of the two boundaries, as compared with the medium correlation distance so that except in the very vicinity of the boundaries, G involved in a short nonvanishing range of $M^{(q)}$ can be approximated by the Green function in a homogeneous random medium of q.

The situation is the same also for the boundary counterpart $M^{(j)} \simeq M^{(j)T}$, $j = 1,2$, which also is dependent on the other boundary S_i, $i \neq j$, as well as the medium; nevertheless, the matrix element can be written in the form

$$M^{(j)}(\boldsymbol{\rho} \mid \boldsymbol{\rho}') = M^{(j)}(\boldsymbol{\rho} - \boldsymbol{\rho}') . \tag{3.2}$$

The solution of Eq. (2.22) for $G(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$ can be written in an eigenfunction series, by introducing a set of eigenfunctions $\phi_a(\hat{\mathbf{x}})$, defined by the eigenvalue equation

$$(L - M^{(q)}) \phi_a(\hat{\mathbf{x}}) = D_a \phi_a(\hat{\mathbf{x}}) \tag{3.3a}$$

of eigenvalue D_a, subjected to the boundary conditions

$$-\partial_n \phi_a = M^{(j)} \phi_a \Big|_{s_j}, \quad j = 1,2 . \tag{3.3b}$$

Here we can set

$$\phi_a(\hat{\mathbf{x}}) = \phi_a(\boldsymbol{\lambda}, z) \exp(-i \boldsymbol{\lambda} \cdot \boldsymbol{\rho}) \tag{3.4}$$

and, to the approximation of using Eq. (3.1b), the use of the Fourier transforms

$$\tilde{M}^{(q)}(\boldsymbol{\lambda}, a^2) = \int d\boldsymbol{\rho} \int_{-\infty}^{\infty} dz \exp[i(\boldsymbol{\lambda} \cdot \boldsymbol{\rho} + az)] \times M^{(q)}(\boldsymbol{\rho}, \mid z \mid) , \tag{3.5a}$$

$$\tilde{M}^{(j)}(\boldsymbol{\lambda}) = \int d\boldsymbol{\rho} \exp(i \boldsymbol{\lambda} \cdot \boldsymbol{\rho}) M^{(j)}(\boldsymbol{\rho}) \tag{3.5b}$$

(which are both even functions of $\boldsymbol{\lambda}$) enables us to represent $M^{(q)}$ and $M^{(j)}$ by operators with the matrix elements

$$M^{(q)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') = \tilde{M}^{(q)}[i\partial/\partial\boldsymbol{\rho}, -(\partial/\partial z)^2] \delta(\hat{\mathbf{x}} - \hat{\mathbf{x}}') , \tag{3.6a}$$

$$M^{(j)}(\boldsymbol{\rho} \mid \boldsymbol{\rho}') = \tilde{M}^{(j)}(i\partial/\partial\boldsymbol{\rho}) \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') . \tag{3.6b}$$

Hence in Eq. (3.4), $\phi_a(\boldsymbol{\lambda}, z)$ can be chosen to be the eigenfunction of $-(\partial/\partial z)^2$, defined by

$$-\left[\frac{\partial}{\partial z}\right]^2 \phi_a(\boldsymbol{\lambda}, z) = a^2 \phi_a(\boldsymbol{\lambda}, z) \tag{3.7a}$$

subject to the boundary conditions

$$-\partial_n \phi_a(\boldsymbol{\lambda}, z) = \tilde{M}^{(j)}(\boldsymbol{\lambda}) \phi_a(\boldsymbol{\lambda}, z) \mid_{z = d_j}, \quad j = 1,2 \tag{3.7b}$$

so that $\phi_a(\hat{\mathbf{x}})$ is simultaneously an eigenfunction of $M^{(q)}$ with the eigenvalue $\tilde{M}^{(q)}(\boldsymbol{\lambda}, a^2)$, in consequence of Eq.

(3.6a), where a is a function of $\boldsymbol{\lambda}$, $a(\boldsymbol{\lambda})$, and $\tilde{M}^{(j)*}(\boldsymbol{\lambda}) \neq \tilde{M}^{(j)}(\boldsymbol{\lambda})$. Hence

$$D_a = \lambda^2 - \kappa_a^2(\boldsymbol{\lambda}) \tag{3.8}$$

with

$$\kappa_a^2(\boldsymbol{\lambda}) = k_0^2 + \tilde{M}^{(q)}(\boldsymbol{\lambda}, a^2) - a^2 \; . \tag{3.9}$$

Thus with the normalization

$$\begin{aligned} &\int_0^{d_2} dz\, \bar{\phi}_a(\boldsymbol{\lambda}, z)\phi_b(\boldsymbol{\lambda}, z) = \delta_{ab} \;, \\ &\sum_a \phi_a(\boldsymbol{\lambda}, z)\bar{\phi}_a(\boldsymbol{\lambda}, z') = \delta(z - z') \end{aligned} \tag{3.10}$$

[so chosen that $\bar{\phi}_a = \phi_a$ in the present case of Eqs. (3.7)], Eq. (2.22) gives the solution

$$G(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') = (2\pi)^{-2} \int d\boldsymbol{\lambda} \sum_a D_a^{-1}(\boldsymbol{\lambda})\phi_a(\boldsymbol{\lambda}, z)\bar{\phi}_a(\boldsymbol{\lambda}, z') \times \exp[-i\boldsymbol{\lambda}\cdot(\boldsymbol{\rho} - \boldsymbol{\rho}')] \; . \tag{3.11}$$

Also in the general case (3.1a) of $M^{(q)}$, the expression (3.11) remains unchanged with the replacement of $\phi_a \to \phi_{a'}$, $\bar{\phi}_a \to \bar{\phi}_{a'}$, and $\kappa_a^2 \to k_0^2 - a'^2(\boldsymbol{\lambda})$, in terms of a new set of eigenfunctions $\phi_{a'}(\boldsymbol{\lambda}, z)$ and $\bar{\phi}_{a'}(\boldsymbol{\lambda}, z) = \phi_{a'}(-\boldsymbol{\lambda}, z)$, defined by the eigenvalue equations

$$-\left[\frac{\partial}{\partial z}\right]^2 \phi_{a'}(\boldsymbol{\lambda}, z) - \int_0^{d_2} dz'' \tilde{M}^{(q)}(z \mid \boldsymbol{\lambda} \mid z'')\phi_{a'}(\boldsymbol{\lambda}, z'') = a'^2\phi_{a'}(\boldsymbol{\lambda}, z) \;, \tag{3.12a}$$

$$-\left[\frac{\partial}{\partial z}\right]^2 \bar{\phi}_{a'}(\boldsymbol{\lambda}, z) - \int_0^{d_2} dz'' \bar{\phi}_{a'}(\boldsymbol{\lambda}, z'')\tilde{M}^{(q)}(z'' \mid \boldsymbol{\lambda} \mid z) = a'^2\bar{\phi}_{a'}(\boldsymbol{\lambda}, z) \;, \tag{3.12b}$$

subject to the boundary conditions (3.7b) and the normalization (3.10), where $\tilde{M}^{(q)}(z \mid \boldsymbol{\lambda} \mid z') = \tilde{M}^{(q)}(z' \mid -\boldsymbol{\lambda} \mid z)$ is the Fourier transform of Eq. (3.1a) with respect to $\boldsymbol{\rho} - \boldsymbol{\rho}'$.

A. Normal mode expansion

In the boundary Eq. (3.7b), $\tilde{M}^{(j)}(\boldsymbol{\lambda})$ is an entire function of $\boldsymbol{\lambda}$ as long as $M^{(j)}(\boldsymbol{\rho})$ is a function of finite range that becomes exactly zero for $|\boldsymbol{\rho}|$ of values exceeding some finite value, as realized when the correlation distance of the boundary fluctuation is finite in Eq. (2.24b). By the same reason, $\tilde{M}^{(q)}(\boldsymbol{\lambda}, a^2)$ is an entire function of $\boldsymbol{\lambda}$ and a^2, and therefore, so are κ_a^2 and D_a [see Eq. (3.9)]. This results in the eigenvalue equations (3.7) remaining unchanged for arbitrary contourings in the $\boldsymbol{\lambda}$ complex plane, although each eigenfunction may have branch cuts in its own plane so that the contourings may give rise to an interchange of some eigenfunctions. Hence, the integrands of the series (3.11), for example, have no branch cuts, as the whole, and are analytic everywhere except at the set of poles given by the factors $D_a^{-1}(\boldsymbol{\lambda})$.

If we presume that in Eq. (3.11)

$$\tilde{G}(z \mid \boldsymbol{\lambda} \mid z') \equiv \sum_a D_a^{-1}(\boldsymbol{\lambda})\phi_a(\boldsymbol{\lambda}, z)\bar{\phi}_a(\boldsymbol{\lambda}, z') \tag{3.13}$$

tends to zero as $|\boldsymbol{\lambda}| \to \infty$ [the proof is given in Appendix B to the approximation of Eqs. (2.32)], then the integrand can be expanded at the set of poles in a Maclaurin series. To this end, we first introduce a two-dimensional unit vector $\boldsymbol{\Omega} = (\Omega_1, \Omega_2)$, $\boldsymbol{\Omega}^2 = 1$ to change the Fourier variable $\boldsymbol{\lambda}$ according to $\boldsymbol{\lambda} = \lambda\boldsymbol{\Omega}$ with the element $d\boldsymbol{\lambda} = \lambda d\lambda d\boldsymbol{\Omega}$; then, for given $\boldsymbol{\Omega}$, we denote a complete set of roots of λ of $D_a(\lambda\boldsymbol{\Omega}) = 0$ by $\pm k_a(\pm\boldsymbol{\Omega})$. From Eq. (3.8),

$$k_a = \kappa_a(k_a\boldsymbol{\Omega}), \quad \mathrm{Im}[\kappa_a] < 0 \;, \tag{3.14}$$

and $k_{-a}(\boldsymbol{\Omega}) \equiv -k_a(-\boldsymbol{\Omega})$ is another set of roots to make the set complete.

Also, a normal set of mode wave functions, $\phi_a(\boldsymbol{\Omega}, \hat{\mathbf{x}})$, is defined by

$$\phi_a(\boldsymbol{\Omega}, \hat{\mathbf{x}}) = \phi_a(\boldsymbol{\Omega}, z)\exp(-ik_a\boldsymbol{\Omega}\cdot\boldsymbol{\rho}) \tag{3.15a}$$

with

$$\phi_a(\boldsymbol{\Omega}, z) = \phi_a(\boldsymbol{\lambda}, z)\,|_a \;, \tag{3.15b}$$

where $|_a$ means setting $\boldsymbol{\lambda} = k_a\boldsymbol{\Omega}$. Hence for given $\boldsymbol{\Omega}$, we obtain the Maclaurin expansion of $\tilde{G}(z \mid \boldsymbol{\lambda} \mid z')$ as

$$\tilde{G}(z \mid \boldsymbol{\lambda} \mid z') = \sum_{\pm a} \left[\frac{\partial}{\partial\lambda} D_a(\lambda\boldsymbol{\Omega})\bigg|_a\right]^{-1} \times \phi_a(\boldsymbol{\Omega}, z)\bar{\phi}_a(\boldsymbol{\Omega}, z')[\lambda - k_a(\boldsymbol{\Omega})]^{-1} \; . \tag{3.16}$$

Here

$$\frac{\partial}{\partial\lambda} D_a(\lambda\boldsymbol{\Omega})\bigg|_a = 2k_a(\boldsymbol{\Omega})[1 - \kappa'_a(\boldsymbol{\Omega})] \;, \tag{3.17a}$$

$$\kappa'_a(\boldsymbol{\Omega}) = \frac{\partial}{\partial\lambda}\kappa_a(\lambda\boldsymbol{\Omega})\bigg|_a \; . \tag{3.17b}$$

Obtaining the Green function in the asymptotic region of $|k_a(\boldsymbol{\rho} - \boldsymbol{\rho}')| \gg 1$ is straightforward by substituting Eq. (3.16) into Eq. (3.11) and making the integration with the aid of formula (A3). Hence the result is obtained, in terms of the residue values at the poles $\lambda = +k_a(\boldsymbol{\Omega})$ which are distributed on the lower half-plane of λ, as

$$\begin{aligned} G(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') \simeq \sum_a & 2^{-1}[2\pi k_a(\boldsymbol{\Omega})\,|\boldsymbol{\rho} - \boldsymbol{\rho}'|\,]^{-1/2} \\ & \times [1 - \kappa'_a(\boldsymbol{\Omega})]^{-1}\phi_a(\boldsymbol{\Omega}, z)\bar{\phi}_a(\boldsymbol{\Omega}, z') \\ & \times \exp[-ik_a(\boldsymbol{\Omega})\,|\boldsymbol{\rho} - \boldsymbol{\rho}'| - i\pi/4] \;, \end{aligned} \tag{3.18}$$

where

$$\boldsymbol{\Omega} = (\boldsymbol{\rho} - \boldsymbol{\rho}')/|\boldsymbol{\rho} - \boldsymbol{\rho}'|$$

and

$$|k_a(\boldsymbol{\rho} - \boldsymbol{\rho}')| \gg 1 \; .$$

An exact version of Eq. (3.18) also can be obtained by using formula (A4), which is available only when the system is isotropic in the horizontal direction, however.

B. Miscellaneous relations associated with the normal mode wave functions

To prepare for several relations frequently quoted in the following sections, we first consider the particular case (3.1b) of $M^{(q)}$, and introduce a set of quantities, defined in terms of the notation $|_{a,b}$ meaning $\boldsymbol{\lambda}_1=k_a^*\boldsymbol{\Omega}$ and $\boldsymbol{\lambda}_2=k_b\boldsymbol{\Omega}$ by

$$k_{ab}(\boldsymbol{\Omega})=(k_a^*+k_b)(\boldsymbol{\Omega})/2=k_{ba}^*(\boldsymbol{\Omega})\ , \tag{3.19a}$$

$$\gamma_{ab}(\boldsymbol{\Omega})=(k_a^*-k_b)(\boldsymbol{\Omega})/i=\gamma_{ba}^*(\boldsymbol{\Omega})\ , \tag{3.19b}$$

$$\begin{aligned}\gamma_{ab}^{(q)}(\boldsymbol{\Omega})&=(2ik_{ab})^{-1}\{\tilde{M}^{(q)*}[\boldsymbol{\lambda}_1,a^{*2}(\boldsymbol{\lambda}_1)]\\&\qquad-\tilde{M}^{(q)}[\boldsymbol{\lambda}_2,b^2(\boldsymbol{\lambda}_2)]\}\,|_{a,b}\\&=\gamma_{ba}^{(q)*}(\boldsymbol{\Omega})\ ,\end{aligned} \tag{3.19c}$$

$$\begin{aligned}\gamma_{ab}^{(j)}(\boldsymbol{\Omega})&=(2ik_{ab})^{-1}[\tilde{M}^{(j)*}(\boldsymbol{\lambda}_1)-\tilde{M}^{(j)}(\boldsymbol{\lambda}_2)]\,|_{a,b}\\&=\gamma_{ba}^{(j)*}(\boldsymbol{\Omega})\ ,\end{aligned} \tag{3.19d}$$

which all constitute Hermitian matrices with respect to the Latin subscripts, and also two functions of $\boldsymbol{\Omega}$ and z, defined by

$$N_{ab}(\boldsymbol{\Omega},z)=\phi_a^*(\boldsymbol{\Omega},z)\phi_b(\boldsymbol{\Omega},z)\ , \tag{3.20}$$

$$N_{ab}^{(3)}(\boldsymbol{\Omega},z)=(2ik_{ab})^{-1}\phi_a^*(\boldsymbol{\Omega},z)\left[\frac{\overleftarrow{\partial}}{\partial z}-\frac{\overrightarrow{\partial}}{\partial z}\right]\phi_b(\boldsymbol{\Omega},z)\ . \tag{3.21}$$

Here

$$\Delta_{ab}\equiv\int_0^{d_2}dz\,N_{ab}(\boldsymbol{\Omega},z) \tag{3.22}$$

becomes δ_{ab} when the system is free from the fluctuation so that $M^{(j)}=B_0$ is a constant independent of $\boldsymbol{\rho}$ [Eqs. (3.7)].

Here the function $N_{ab}^{(3)}(\boldsymbol{\Omega},z)$ satisfies the equation

$$\frac{\partial}{\partial z}N_{ab}^{(3)}(\boldsymbol{\Omega},z)=[\gamma_{ab}(\boldsymbol{\Omega})-\gamma_{ab}^{(q)}(\boldsymbol{\Omega})]N_{ab}(\boldsymbol{\Omega},z) \tag{3.23}$$

with the boundary values

$$\mp N_{ab}^{(3)}(\boldsymbol{\Omega},z)=\gamma_{ab}^{(j)}(\boldsymbol{\Omega})N_{ab}(\boldsymbol{\Omega},z)\,|_{z=d_j},\quad j=1,2 \tag{3.24}$$

as follows directly from Eqs. (3.7)–(3.9). The z integration of Eq. (3.23) over the range $d_2\ge z\ge 0$ leads to the relation

$$(\gamma_{ab}-\gamma_{ab}^{(q)})(\boldsymbol{\Omega})\Delta_{ab}=\sum_{j=1}^{2}\gamma_{ab}^{(j)}(\boldsymbol{\Omega})N_{ab}(\boldsymbol{\Omega},d_j)\ . \tag{3.25}$$

Alternatively, the two Eqs. (3.23) and (3.24) can be written by one equation, as [cf. Eq. (2.15)]

$$\frac{\partial}{\partial z}N_{ab}^{(3)}(\boldsymbol{\Omega},z)=[\gamma_{ab}(\boldsymbol{\Omega})-\gamma_{ab}^{(q+b)}(\boldsymbol{\Omega},z)]N_{ab}(\boldsymbol{\Omega},z)\ , \tag{3.26}$$

$$\gamma_{ab}^{(q+b)}(\boldsymbol{\Omega},z)=\gamma_{ab}^{(q)}(\boldsymbol{\Omega})+\sum_{j=1}^{2}\gamma_{ab}^{(j)}(\boldsymbol{\Omega})\delta_j(z)\ , \tag{3.27}$$

with the new boundary condition

$$N_{ab}^{(3)}(\boldsymbol{\Omega},z)=N_{ab}(\boldsymbol{\Omega},z)=0$$

in the regions of $z<0$ and $z>d_2$. Equation (3.25) is directly reproduced by the z integration of Eq. (3.26) over the range $\infty\ge z\ge-\infty$.

Also in the general case (3.1a) of $M^{(q)}$, various previous equations hold true with minor changes. For example, the modified version of Eq. (3.26) can be written in the same form as

$$\frac{\partial}{\partial z}N_{a'b'}^{(3)}(\boldsymbol{\Omega},z)=[\gamma_{a'b'}(\boldsymbol{\Omega})-\gamma_{a'b'}^{(q+b)}(\boldsymbol{\Omega},z)]N_{a'b'}(\boldsymbol{\Omega},z)\ . \tag{3.28}$$

Here $\gamma_{a'b'}^{(q+b)}(\boldsymbol{\Omega},z)$ represents a z matrix with the elements $\gamma_{a'b'}^{(q+b)}(\boldsymbol{\Omega},z\,|\,z_1;z_2)$ having the medium part

$$\begin{aligned}&\gamma_{a'b'}^{(q)}(\boldsymbol{\Omega},z\,|\,z_1;z_2)\\&\quad=(2ik_{a'b'})^{-1}[\tilde{M}^{(q)*}(z\,|\,\boldsymbol{\lambda}_1\,|\,z_1)\delta(z-z_2)\\&\qquad\quad-\tilde{M}^{(q)}(z\,|\,\boldsymbol{\lambda}_2\,|\,z_2)\delta(z-z_1)]\,|_{a',b'}\ ,\end{aligned} \tag{3.29}$$

and $N_{a'b'}(\boldsymbol{\Omega},z)$ represents

$$N_{a'b'}(\boldsymbol{\Omega},z_1,z_2)=\phi_{a'}^*(\boldsymbol{\Omega},z_1)\phi_{b'}(\boldsymbol{\Omega},z_2) \tag{3.30}$$

with two z coordinates z_1 and z_2, so that the convention employed is

$$\begin{aligned}&\gamma_{a'b'}^{(q+b)}(\boldsymbol{\Omega},z)N_{a'b'}(\boldsymbol{\Omega},z)\\&\quad\equiv\int dz_1dz_2\gamma_{a'b'}^{(q+b)}(\boldsymbol{\Omega},z\,|\,z_1;z_2)N_{a'b'}(\boldsymbol{\Omega},z_1,z_2)\ .\end{aligned} \tag{3.31}$$

Hence the z integration of (3.28) yields the relation

$$\gamma_{a'b'}(\boldsymbol{\Omega})\Delta_{a'b'}=\gamma_{a'b'}^{(q)}(\boldsymbol{\Omega})\Delta_{a'b'}+\sum_{j=1}^{2}\gamma_{a'b'}^{(j)}(\boldsymbol{\Omega})N_{a'b'}(\boldsymbol{\Omega},d_j) \tag{3.32}$$

similar to (3.25), where $\gamma_{a'b'}^{(q)}(\boldsymbol{\Omega})$ is defined by the integral

$$\gamma_{a'b'}^{(q)}(\boldsymbol{\Omega})\Delta_{a'b'}=\int dz\,\gamma_{a'b'}^{(q)}(\boldsymbol{\Omega},z)N_{a'b'}(\boldsymbol{\Omega},z) \tag{3.33}$$

with $\Delta_{a'b'}$ from $N_{a'b'}(\boldsymbol{\Omega},z)$. In fact, (3.33) is reduced to $\gamma_{ab}^{(q)}(\boldsymbol{\Omega})\Delta_{ab}$ in (3.25) when $\phi_{a'}^*(\hat{\mathbf{x}})$ and $\phi_{b'}(\hat{\mathbf{x}})$ happen to be eigenfunctions of $M^{(q)*}$ and $M^{(q)}$, respectively [(3.9)].

It may be remarked that, in connection with Eq. (3.28), the vertical power flux is not given by $k_{a'b'}N_{a'b'}^{(3)}(\boldsymbol{\Omega},z)$ and should be corrected by an additional term from $\gamma_{a'b'}^{(q+b)}(\boldsymbol{\Omega},z)$ on the right-hand side;[15] the situation is the same also for Eq. (3.26).

IV. MODE THEORY OF THE SECOND-ORDER GREEN FUNCTION

We begin with the integral equation (2.35) for $S(1;2)$ by substituting the eigenfunction expansions of both $G^*(1)$ and $G(2)$ according to Eq. (3.11). The resulting equation becomes written in terms of an eigenfunction transform of $K(1;2)$, defined by

$$\tilde{K}_{ab;cd}(\boldsymbol{\lambda}_1;\boldsymbol{\lambda}_2\,|\,\boldsymbol{\lambda}_1';\boldsymbol{\lambda}_2')=\int d\hat{\mathbf{x}}_1 d\hat{\mathbf{x}}_2\int d\hat{\mathbf{x}}_1' d\hat{\mathbf{x}}_2'\bar{\phi}_a^*(z_1)\bar{\phi}_b(z_2)\exp[i(-\boldsymbol{\lambda}_1\cdot\boldsymbol{\rho}_1+\boldsymbol{\lambda}_2\cdot\boldsymbol{\rho}_2)]K(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2\,|\,\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')$$
$$\times\phi_c^*(z_1')\phi_d(z_2')\exp[i(\boldsymbol{\lambda}_1'\cdot\boldsymbol{\rho}_1'-\boldsymbol{\lambda}_2'\cdot\boldsymbol{\rho}_2')]\ , \tag{4.1}$$

and a similar transform of $S(1;2)$, say $\tilde{S}_{ab;cd}(\boldsymbol{\lambda}_1;\boldsymbol{\lambda}_2\,|\,\boldsymbol{\lambda}_1';\boldsymbol{\lambda}_2')$. In view of the translational invariance of the system in the horizontal direction, it is convenient to introduce relative coordinates $\mathbf{r}$ and $\boldsymbol{\rho}$, defined by

$$\mathbf{r}=\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\quad \boldsymbol{\rho}=(\boldsymbol{\rho}_2+\boldsymbol{\rho}_1)/2,\quad d\mathbf{r}d\boldsymbol{\rho}=d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2\ , \tag{4.2a}$$

and the corresponding variables of Fourier transformation, $\mathbf{u}$ and $\boldsymbol{\lambda}$, by

$$\mathbf{u}=(\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_1)/2,\quad \boldsymbol{\lambda}=\boldsymbol{\lambda}_2-\boldsymbol{\lambda}_1,\quad d\mathbf{u}d\boldsymbol{\lambda}=d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 \tag{4.2b}$$

with the relation

$$-\boldsymbol{\lambda}_1\cdot\boldsymbol{\rho}_1+\boldsymbol{\lambda}_2\cdot\boldsymbol{\rho}_2=\mathbf{u}\cdot\mathbf{r}+\boldsymbol{\lambda}\cdot\boldsymbol{\rho}\ . \tag{4.2c}$$

The matrix elements of $K(1;2)$ can be written in the form

$$K(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2\,|\,\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')=K(\mathbf{r},z_1,z_2\,|\,\boldsymbol{\rho}-\boldsymbol{\rho}'\,|\,\mathbf{r}',z_1',z_2')\ , \tag{4.3}$$

which gives the transform of the form

$$\tilde{K}_{ab;cd}(\boldsymbol{\lambda}_1;\boldsymbol{\lambda}_2\,|\,\boldsymbol{\lambda}_1';\boldsymbol{\lambda}_2')=(2\pi)^2\delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}')\tilde{K}_{ab;cd}(\mathbf{u}\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}')\ . \tag{4.4}$$

The same is also for the transform of $S(1;2)$ say $\tilde{S}_{ab;cd}(\mathbf{u}\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}')$, and Eq. (2.35) provides its integral equation by

$$\tilde{S}_{ef;cd}(\mathbf{u}'\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}'')=\tilde{K}_{ef;cd}(\mathbf{u}'\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}'')$$
$$+\sum_{a,b}(2\pi)^{-2}\int d\mathbf{u}\,\tilde{K}_{ef;ab}(\mathbf{u}'\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u})[D_a^*(\mathbf{u}-\boldsymbol{\lambda}/2)D_b(\mathbf{u}+\boldsymbol{\lambda}/2)]^{-1}\tilde{S}_{ab;cd}(\mathbf{u}\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}'')\ , \tag{4.5}$$

in which the variable $\boldsymbol{\lambda}$ is involved merely as a constant parameter.

To perform the $\mathbf{u}$ integration in Eq. (4.5), we write the integral $F(\boldsymbol{\lambda})$ in the general form

$$F(\boldsymbol{\lambda})=(2\pi)^{-2}\int d\mathbf{u}\sum_{a,b}[D_a^*(\boldsymbol{\lambda}_1)D_b(\boldsymbol{\lambda}_2)]^{-1}f_{ab}(\mathbf{u},\boldsymbol{\lambda})\ . \tag{4.6}$$

In this particular case,

$$f_{ab}(\mathbf{u},\boldsymbol{\lambda})=\tilde{K}_{ef;ab}(\mathbf{u}'\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u})\tilde{S}_{ab;cd}(\mathbf{u}\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}'') \tag{4.7}$$

(where unnecessary variables and subscripts have been suppressed on the left-hand side). To perform the $\mathbf{u}$ integration in Eq. (4.6), we first introduce a two-dimensional unit vector $\boldsymbol{\Omega}=(\Omega_1,\Omega_2)$, $\boldsymbol{\Omega}^2=1$, to change the variable of integration by $\mathbf{u}=u\boldsymbol{\Omega}$ with the element $d\mathbf{u}=u\,du\,d\boldsymbol{\Omega}$, and also the two scalar variables $u_1=\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}_1$ and $u_2=\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}_2$, so that

$$\boldsymbol{\lambda}_1=u\boldsymbol{\Omega}-\boldsymbol{\lambda}/2=u_1\boldsymbol{\Omega}-\boldsymbol{\lambda}_T/2\ , \tag{4.8a}$$

$$\boldsymbol{\lambda}_2=u\boldsymbol{\Omega}+\boldsymbol{\lambda}/2=u_2\boldsymbol{\Omega}+\boldsymbol{\lambda}_T/2\ , \tag{4.8b}$$

where $\boldsymbol{\lambda}_T$ is the component of $\boldsymbol{\lambda}$ orthogonal to $\boldsymbol{\Omega}$, and

$$\begin{aligned}u_1&=u-\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2\ ,\\ u_2&=u+\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2\ ,\\ u&=(u_1+u_2)/2\ .\end{aligned} \tag{4.9}$$

Then we observe that the integrand can be regarded as an analytic function of u_1 and u_2 everywhere except at the two sets of poles given by the factor

$$[D_a^*(u_1\boldsymbol{\Omega}-\boldsymbol{\lambda}_T/2)D_b(u_2\boldsymbol{\Omega}+\boldsymbol{\lambda}_T/2)]^{-1}\ ,$$

as the whole, although each term of the series generally has branch cuts (cf. Sec. III A), and that the integrand tends to zero as $|u_1|,|u_2|\to\infty$ (Appendix B). This enables the integrand (for given $\boldsymbol{\Omega}$) to be expanded in a Maclaurin series at those poles of u_1 and u_2, $\pm k_a^{*(T)}(\pm\boldsymbol{\Omega})$ and $\pm k_b^{(T)}(\pm\boldsymbol{\Omega})$, given by the roots of

$$D_a^*(k_a^{*(T)}\boldsymbol{\Omega}-\boldsymbol{\lambda}_T/2)=0,\quad D_b(k_b^{(T)}\boldsymbol{\Omega}+\boldsymbol{\lambda}_T/2)=0\ . \tag{4.10}$$

$\mathrm{Im}(k_b^{(T)})<0$ and the superscript (T) mean the dependence on $\boldsymbol{\lambda}_T$; as $\boldsymbol{\lambda}\to 0$, the roots are reduced to the $k_a^*(\boldsymbol{\Omega})$ and $k_b(\boldsymbol{\Omega})$'s of Eq. (3.14), respectively. To the first order of $\boldsymbol{\lambda}_T$,

$$k_a^{*(T)}(\boldsymbol{\Omega})=k_a^*(\boldsymbol{\Omega})-(\boldsymbol{\Omega}-\boldsymbol{\Omega}_a^*)\cdot\boldsymbol{\lambda}/2\ , \tag{4.11a}$$

$$k_b^{(T)}(\boldsymbol{\Omega})=k_b(\boldsymbol{\Omega})+(\boldsymbol{\Omega}-\boldsymbol{\Omega}_b)\cdot\boldsymbol{\lambda}/2\ , \tag{4.11b}$$

in terms of the notation

$$\boldsymbol{\Omega}_a=\frac{\partial}{\partial\mathbf{u}}D_a(\mathbf{u})\,|_a\Big/\boldsymbol{\Omega}\cdot\frac{\partial}{\partial\mathbf{u}}D_a(\mathbf{u})\,|_a,\quad \boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_a=1 \tag{4.12}$$

with the mark $|_a$ defined in Eq. (3.15b).

Thus the integral (4.6) is given by a series of the form

$$F(\boldsymbol{\lambda})=\int_{2\pi}d\boldsymbol{\Omega}\frac{1}{2\pi}\int_0^\infty du\sum_{a,b}(u_1-k_a^{*(T)})^{-1}(u_2-k_b^{(T)})^{-1}\times w_{ab}^{(T)}f_{ab}^{(\lambda)}(\boldsymbol{\Omega})\ . \tag{4.13}$$

Here $\sum_{a,b}$ denotes the summation over the complete sets of poles [including $-k_a^{*(T)}(-\boldsymbol{\Omega})$ and $-k_b^{(T)}(-\boldsymbol{\Omega})$, as in Eq. (3.16)], and with Eqs. (3.17),

$$w_{ab}^{(T)} = (8\pi)^{-1} k_{ab}^{(T)} (k_a^{*(T)} k_b^{(T)})^{-1} (1-\kappa_a^{*\prime(T)})^{-1} \times (1-\kappa_b^{\prime(T)})^{-1} , \tag{4.14a}$$

$$f_{ab}^{(\lambda)}(\mathbf{\Omega}) = f_{ab}(k_{ab}^{(T)}\mathbf{\Omega},\boldsymbol{\lambda}) , \tag{4.14b}$$

where[16]

$$k_{ab}^{(T)} = (k_a^{*(T)} + k_b^{(T)})/2 \simeq k_{ab} + (\mathbf{\Omega}_a^* - \mathbf{\Omega}_b)\cdot\boldsymbol{\lambda}/4 , \tag{4.15}$$

and u_1 and u_2 are functions of u through Eq. (4.9). Hence, all the u integrals involved in Eq. (4.13) can be given in terms of an integral $\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda})$, defined by

$$\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda}) = (2\pi)^{-1} \int_0^\infty du\, w_{ab}^{(T)} (u - \mathbf{\Omega}\cdot\boldsymbol{\lambda}/2 - k_a^{*(T)})^{-1} \times (u + \mathbf{\Omega}\cdot\boldsymbol{\lambda}/2 - k_b^{(T)})^{-1} \tag{4.16}$$

(which evaluation is in Appendix C), yielding

$$F(\boldsymbol{\lambda}) = \int d\mathbf{\Omega} \sum_{a,b} \tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda}) f_{ab}^{(\lambda)}(\mathbf{\Omega}) . \tag{4.17}$$

Here

$$\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda}) = w_{ab}^{(e)}(\mathbf{\Omega},\boldsymbol{\lambda})(\gamma_{ab}^{(T)} - i\mathbf{\Omega}\cdot\boldsymbol{\lambda})^{-1} , \tag{4.18}$$

where, using Eqs. (4.11),

$$\gamma_{ab}^{(T)}(\mathbf{\Omega}) = (k_a^{*(T)} - k_b^{(T)})/i = \gamma_{ab}(\mathbf{\Omega}) + i(\mathbf{\Omega} - \mathbf{\Omega}_{ab})\cdot\boldsymbol{\lambda} \tag{4.19}$$

with $\mathbf{\Omega}_{ab} = (\mathbf{\Omega}_a^* + \mathbf{\Omega}_b)/2$; Eq. (4.18) can be written also as

$$\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda}) = w_{ab}^{(e)}(\mathbf{\Omega},\boldsymbol{\lambda})[\gamma_{ab}(\mathbf{\Omega}) - i\mathbf{\Omega}_{ab}\cdot\boldsymbol{\lambda}]^{-1} . \tag{4.20}$$

The summation $\sum_{a,b}$ includes not only the terms of the mode waves a and b propagating in the same direction $\mathbf{\Omega}$, but also interference terms made by the mode waves propagating in the opposite direction (Appendix C).

A $\boldsymbol{\rho}$ function given by the Fourier inversion of $w_{ab}^{(e)}(\mathbf{\Omega},\boldsymbol{\lambda})$, say $\bar{w}_{ab}^{(e)}(\mathbf{\Omega},\boldsymbol{\rho})$, is a function of a short range of the order of the mode wavelengths, and is appreciable also for the nonpropagative and the interfering mode waves, giving [Eqs. (C10)–(C13)]

$$\int d\boldsymbol{\rho}\, \bar{w}_{ab}^{(e)}(\mathbf{\Omega},\boldsymbol{\rho}) \simeq \begin{cases} w_{ab} & \text{for propagative mode waves} \\ (i\pi)^{-1} \ln|k_b/k_a^*|\, w_{ab} & \\ & \text{for nonpropagative mode waves} \end{cases} . \tag{4.21a}$$

On the other hand, the Fourier inversion of $\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda})$, $U_{ab}(\mathbf{\Omega},\boldsymbol{\rho})$, is given to the approximation of $\mathbf{\Omega}_{ab} \simeq \mathbf{\Omega}$ by

$$U_{ab}(\mathbf{\Omega},\boldsymbol{\rho}) \simeq w_{ab}^{(e)} |\boldsymbol{\rho}|^{-1} \times \exp[-\gamma_{ab}(\mathbf{\Omega})|\boldsymbol{\rho}|]\delta^1(\mathbf{\Omega} - \boldsymbol{\rho}/|\boldsymbol{\rho}|) , \tag{4.21b}$$

where $w_{ab}^{(e)} = w_{ab}^{(e)}(\mathbf{\Omega}, i\partial/\partial\boldsymbol{\rho})$ and, from Eq. (4.20), is a solution of

$$\left[\gamma_{ab} + \mathbf{\Omega}_{ab}\cdot\frac{\partial}{\partial\boldsymbol{\rho}}\right] U_{ab}(\mathbf{\Omega},\boldsymbol{\rho}) = w_{ab}^{(e)}\delta(\boldsymbol{\rho}) . \tag{4.22}$$

The factor $\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda})$ in Eq. (4.17) has a pole at $i\mathbf{\Omega}\cdot\boldsymbol{\lambda} = \gamma_{ab}^{(T)}$, and we often need an expansion of $f_{ab}^{(\lambda)}(\mathbf{\Omega})$ at this pole, particularly when $f_{ab}^{(\lambda)}(\mathbf{\Omega}) = 0$ at the pole [e.g., (D2)]. To this end, we first observe that $f_{ab}^{(\lambda)}(\mathbf{\Omega})$ [(4.14b)] is a function of $u_1 = k_a^{*(\lambda)}$ and $u_2 = k_b^{(\lambda)}$, given from Eq. (4.9), by

$$k_a^{*(\lambda)} \equiv k_{ab}^{(T)} - \mathbf{\Omega}\cdot\boldsymbol{\lambda}/2 = k_a^{*(T)} - i2^{-1}(\gamma_{ab}^{(T)} - i\mathbf{\Omega}\cdot\boldsymbol{\lambda}) , \tag{4.23a}$$

$$k_b^{(\lambda)} \equiv k_{ab}^{(T)} + \mathbf{\Omega}\cdot\boldsymbol{\lambda}/2 = k_b^{(T)} + i2^{-1}(\gamma_{ab}^{(T)} - i\mathbf{\Omega}\cdot\boldsymbol{\lambda}) , \tag{4.23b}$$

and therefore it can be expanded to the first order of $\gamma_{ab}^{(T)} - i\mathbf{\Omega}\cdot\boldsymbol{\lambda}$ by

$$f_{ab}^{(\lambda)}(\mathbf{\Omega}) = f_{ab}(\mathbf{\Omega}) + i2^{-1}(\gamma_{ab}^{(T)} - i\mathbf{\Omega}\cdot\boldsymbol{\lambda}) f_{ab}'(\mathbf{\Omega}) , \tag{4.24}$$

where, on rewriting $f_{ab}(\mathbf{u},\boldsymbol{\lambda})$ as $f_{ab}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$ [Eqs. (4.8)],

$$f_{ab}(\mathbf{\Omega}) = f_{ab}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\,|_{a,b}^{(T)} \equiv f_{ab}(k_a^{*(T)}\mathbf{\Omega} - \boldsymbol{\lambda}_T/2, k_b^{(T)}\mathbf{\Omega} + \boldsymbol{\lambda}_T/2) , \tag{4.25}$$

$$f_{ab}'(\mathbf{\Omega}) = \mathbf{\Omega}\cdot\left[-\frac{\partial}{\partial\boldsymbol{\lambda}_1} + \frac{\partial}{\partial\boldsymbol{\lambda}_2}\right] f_{ab}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\,|_{a,b}^{(T)} \tag{4.26}$$

(where $|_{a,b}^{(T)}$ means to set $u_1 = k_a^{*(T)}$ and $u_2 = k_b^{(T)}$), and are still dependent on the component $\boldsymbol{\lambda}_T$.

Now, applying the formula (4.17) to the present case of Eq. (4.7), Eq. (4.5) is reduced, after setting $\mathbf{u}' = k_{ef}^{(T)}\mathbf{\Omega}'$ and $\mathbf{u}'' = k_{cd}^{(T)}\mathbf{\Omega}''$ to an equation of the form

$$\tilde{S}_{ef;cd}(\mathbf{\Omega}'\,|\,\boldsymbol{\lambda}\,|\,\mathbf{\Omega}'') = K_{ef;cd}^{(\lambda)}(\mathbf{\Omega}'\,|\,\mathbf{\Omega}'') + \sum_{a,b} \int d\mathbf{\Omega}\, K_{ef;ab}^{(\lambda)}(\mathbf{\Omega}'\,|\,\mathbf{\Omega})\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda}) \times \tilde{S}_{ab;cd}(\mathbf{\Omega}\,|\,\boldsymbol{\lambda}\,|\,\mathbf{\Omega}'') . \tag{4.27}$$

Here

$$K_{ef;cd}^{(\lambda)}(\mathbf{\Omega}'\,|\,\mathbf{\Omega}'') = \tilde{K}_{ef;cd}(k_{ef}^{(T)}\mathbf{\Omega}'\,|\,\boldsymbol{\lambda}\,|\,k_{cd}^{(T)}\mathbf{\Omega}'') , \tag{4.28}$$

and $\tilde{S}_{ef;cd}(\mathbf{\Omega}'\,|\,\boldsymbol{\lambda}\,|\,\mathbf{\Omega}'')$ also is defined in exactly the same way. The $\boldsymbol{\lambda}$ dependence of $K_{ef;cd}^{(\lambda)}$ is negligible in many cases, however, because of its slight change as compared with the change due to the factor $\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda})$ [implying that the spatial range of $K(1;2)$ is negligibly small compared with the coherence distances of the mode waves]. With the same approximation, the summation $\sum_{a,b}$ can be restricted to include only the propagative mode waves, say $\sum'_{a,b}$, and further to $\sum'_{a,b}$ $(a=b)$ whenever $|\gamma_{ab}| \gg \gamma_{aa}$, $a \neq b$ [Eq. (C11)].

The Fourier inversion of Eq. (4.27) with respect to $\boldsymbol{\lambda}$ yields an integral equation of $S_{ef;cd}(\mathbf{\Omega}'\,|\,\boldsymbol{\rho}' - \boldsymbol{\rho}''\,|\,\mathbf{\Omega}'')$, as

$$S_{ef;cd}(\mathbf{\Omega}' \mid \boldsymbol{\rho}'-\boldsymbol{\rho}'' \mid \mathbf{\Omega}'')=K_{ef;cd}(\mathbf{\Omega}' \mid \mathbf{\Omega}'')\delta(\rho'-\rho'')+\sum_{a,b}\int d\mathbf{\Omega}\int d\boldsymbol{\rho}\, K_{ef;ab}(\mathbf{\Omega}' \mid \mathbf{\Omega})U_{ab}(\mathbf{\Omega},\boldsymbol{\rho}'-\boldsymbol{\rho})S_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}'' \mid \mathbf{\Omega}'') . \tag{4.29}$$

The superscript (λ) of $K^{(\lambda)}_{ef;ab}(\mathbf{\Omega}' \mid \mathbf{\Omega})$ has been suppressed, implying the neglect of $\boldsymbol{\lambda}$ or a possible replacement of $\boldsymbol{\lambda}\to i\partial/\partial\boldsymbol{\rho}$ in it, and the factor $U_{ab}(\mathbf{\Omega},\boldsymbol{\rho}'-\boldsymbol{\rho})$ may be given by Eq. (4.21b). The solution of Eq. (4.29) is important not only to obtain the mutual coherence function, but also because of being a basic quantity to construct an effective scattering cross section of a fixed scatterer embedded in the random waveguide [e.g., Eq. (5.19)], and may be obtained by a direct iteration method when the scattering volume [where $K(1;2)\neq 0$] is limited, or by Eq. (2.38) in terms of $I(1;2)$.

A. Power fluxes and equation of radiative transfer

The horizontal power flux at $\hat{\mathbf{x}}$ for the coherent wave from a point source at $\hat{\mathbf{x}}'$, $\mathbf{W}^{(C)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$, is given according to Eq. (2.39a) with (3.11) by

$$\mathbf{W}^{(C)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')=\sum_{a,b}(2\pi)^{-2}\int d\boldsymbol{\lambda}\, e^{-i\boldsymbol{\lambda}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}(2\pi)^{-2}\int d\mathbf{u}\,\mathbf{u}\phi_a^*(\boldsymbol{\lambda}_1,z)\phi_b(\boldsymbol{\lambda}_2,z)[D_a^*(\boldsymbol{\lambda}_1)D_b(\boldsymbol{\lambda}_2)]^{-1}\bar{\phi}_a^*(\boldsymbol{\lambda}_1,z')\bar{\phi}_b(\boldsymbol{\lambda}_2,z') , \tag{4.30}$$

which has the form of integral (4.6) and also fulfills the asymptotic condition required for the factor $f_{ab}(\mathbf{u},\boldsymbol{\lambda})$. With formula (4.17) and the followed Fourier inversion with respect to $\boldsymbol{\lambda}$, we obtain

$$W^{(C)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')=\sum_{a,b}\int d\mathbf{\Omega}\, k_{ab}^{(T)}\mathbf{\Omega}N_{ab}^{(T)}(\mathbf{\Omega},z)U_{ab}(\mathbf{\Omega},\boldsymbol{\rho}-\boldsymbol{\rho}')\bar{N}_{ab}^{(T)}(\mathbf{\Omega},z') , \tag{4.31}$$

where $N_{ab}^{(T)}(\mathbf{\Omega},z)$ is the same as $N_{ab}(\mathbf{\Omega},z)$ in Eq. (3.20) except the replacement of $k_a^*\mathbf{\Omega}\to k_a^{*(T)}\mathbf{\Omega}-\boldsymbol{\lambda}_T/2$ and $k_b\mathbf{\Omega}\to k_b^{(T)}\mathbf{\Omega}+\boldsymbol{\lambda}_T/2$ [Eqs. (D5) and (D6)].[16] The same is also for $\bar{N}_{ab}^{(T)}(\mathbf{\Omega},z)$.

In the same way the vertical flux $W_z^{(C)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$ is found to be given by Eq. (4.31) with $\mathbf{\Omega}N_{ab}^{(T)}(\mathbf{\Omega},z)\to N_{ab}^{(3,T)}(\mathbf{\Omega},z)$ [Eq. (3.21)], where $N^{(3,T)}$ is a short form of $N^{(3)(T)}$. The incoherent power flux given by Eq. (2.39b), say $\mathbf{W}^{(I)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$, also can be obtained with the same procedure, and the sum of the two power fluxes is found in the form

$$(\mathbf{W}^{(C)}+\mathbf{W}^{(I)})(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')=\sum_{a,b,c,d}\int d\mathbf{\Omega}\int d\mathbf{\Omega}' k_{ab}^{(T)}\mathbf{\Omega}N_{ab}^{(T)}(\mathbf{\Omega},z)I_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \mathbf{\Omega}')\bar{N}_{cd}^{(T)}(\mathbf{\Omega}',z') . \tag{4.32}$$

Here

$$\begin{aligned}I_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \mathbf{\Omega}')&=U_{ab}(\mathbf{\Omega},\boldsymbol{\rho}-\boldsymbol{\rho}')\delta_{ac}\delta_{bd}\delta(\mathbf{\Omega}-\mathbf{\Omega}')\\&\quad+\int d\boldsymbol{\rho}''d\boldsymbol{\rho}'''U_{ab}(\mathbf{\Omega},\boldsymbol{\rho}-\boldsymbol{\rho}'')S_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho}''-\boldsymbol{\rho}''' \mid \mathbf{\Omega}')U_{cd}(\mathbf{\Omega}',\boldsymbol{\rho}'''-\boldsymbol{\rho}') ,\end{aligned} \tag{4.33}$$

which is just a mode transformed version of Eq. (2.34); an expression corresponding to the BS equation (2.29) also can be obtained, as

$$\tilde{I}_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\lambda} \mid \mathbf{\Omega}')=\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda})\left[\delta_{ac}\delta_{bd}\delta(\mathbf{\Omega}-\mathbf{\Omega}')+\sum_{i,j}\int d\mathbf{\Omega}''K_{ab;ij}^{(\lambda)}(\mathbf{\Omega} \mid \mathbf{\Omega}'')\tilde{I}_{ij;cd}(\mathbf{\Omega}'' \mid \boldsymbol{\lambda} \mid \mathbf{\Omega}')\right] , \tag{4.34}$$

with the aid of Eq. (4.27), in terms of the Fourier transforms.

Equation (4.34) can be rewritten by Eq. (D19) (by multiplying the both sides with $\gamma_{ab}-i\mathbf{\Omega}_{ab}\cdot\boldsymbol{\lambda}$), of which Fourier inversion gives an integro-differential equation for $I_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \mathbf{\Omega}')$ as

$$\begin{aligned}\left[\gamma_{ab}+\mathbf{\Omega}_{ab}\cdot\frac{\partial}{\partial\boldsymbol{\rho}}\right]I_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \mathbf{\Omega}')&=w_{ab}^{(e)}\delta_{ac}\delta_{bd}\delta(\mathbf{\Omega}-\mathbf{\Omega}')\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')\\&\quad+\sum_{i,j}\int d\mathbf{\Omega}''w_{ab}^{(e)}K_{ab;ij}(\mathbf{\Omega} \mid \mathbf{\Omega}'')I_{ij;cd}(\mathbf{\Omega}'' \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \mathbf{\Omega}')\end{aligned} \tag{4.35}$$

similar to the conventional equation of radiative transfer, with a scattering cross section $K_{ab;ij}(\mathbf{\Omega} \mid \mathbf{\Omega}')$ and a term of point source.

B. Power conservation and optical relations

Equation (4.35) is consistent with power conservation at every point in the waveguide and on the boundaries, by virtue of the local optical relation (D17). The z-integrated version of this aspect can be found more simply by multiplying the both sides of Eq. (4.35) [or (D19)] with $\Delta_{ab}^{(T)}k_{ab}^{(e)}$ and making the summation $\sum_{a,b}\int d\mathbf{\Omega}$; hence, with the integrated optical relation (D23), we are led to

$$\frac{\partial}{\partial\boldsymbol{\rho}}\cdot\sum_{a,b}\int d\mathbf{\Omega}\left[\mathbf{\Omega}\Delta_{ab}^{(T)}k_{ab}^{(T)}+\sum_{j=1}^{2}\boldsymbol{\beta}_{ab}^{(j,\lambda)}(\mathbf{\Omega})\right]I_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \mathbf{\Omega}')=\Delta_{cd}^{(T)}k_{cd}^{(e)}w_{cd}^{(e)}(\mathbf{\Omega}',\boldsymbol{\lambda})\delta(\boldsymbol{\rho}-\boldsymbol{\rho}') , \tag{4.36}$$

where $k_{ab}^{(e)}$ is defined by Eq. (D4) and $\Delta_{ab}^{(T)} \sim \Delta_{ab} \sim \delta_{ab}$ [Eq. (D21)]. Here the first term in the large parentheses is obviously a z-integrated version of Eq. (4.32) for $\mathbf{W}^{(C)}+\mathbf{W}^{(I)}$, and the other terms are those from the surface waves, $\mathbf{W}^{(S)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$, given from Eq. (2.41), by

$$\mathbf{W}^{(S)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') = \sum_{j=1}^{2} \sum_{a,b,c,d} \int d\boldsymbol{\Omega} \int d\boldsymbol{\Omega}' \boldsymbol{\beta}_{ab}^{(j,\lambda)}(\boldsymbol{\Omega}) \delta_j(z) I_{ab;cd}(\boldsymbol{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \boldsymbol{\Omega}') \bar{N}_{cd}^{(T)}(\boldsymbol{\Omega}', z') , \tag{4.37}$$

where $\boldsymbol{\beta}_{ab}^{(j,\lambda)}(\boldsymbol{\Omega})$ is given by Eq. (D13) in terms of a mode transform of $\beta^{(j)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2)$.

In the left-hand side of Eq. (4.36) we have, to the approximation of Eq. (D25),

$$\sum_{j=1}^{2} \boldsymbol{\beta}_{ab}^{(j,\lambda)}(\boldsymbol{\Omega}) + \boldsymbol{\Omega} \Delta_{ab}^{(T)} k_{ab}^{(T)} \simeq \boldsymbol{\Omega}_{ab} \Delta_{ab} k_{ab}^{(e)}, \quad \lambda = 0 \tag{4.38}$$

which shows that the total integrated power flux, including those of the surface waves, is the same as given by the z integration of Eq. (4.32) with the replacement of $k_{ab}^{(T)} \rightarrow k_{ab}^{(e)}$ and $\boldsymbol{\Omega} \rightarrow \boldsymbol{\Omega}_{ab}$, i.e.,

$$\int dz\, (\mathbf{W}^{(C)}+\mathbf{W}^{(I)}+\mathbf{W}^{(S)})(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') \simeq \sum_{a,b,c,d} \int d\boldsymbol{\Omega}\, d\boldsymbol{\Omega}' k_{ab}^{(e)} \boldsymbol{\Omega}_{ab} \Delta_{ab} I_{ab;cd}(\boldsymbol{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \boldsymbol{\Omega}') \bar{N}_{cd}(\boldsymbol{\Omega}', z') . \tag{4.39}$$

Since $\Delta_{ab} \sim \delta_{ab}$, the nonpropagative mode waves ($k_{aa}^{(e)} \sim 0$) are actually excluded in the series.

Equation (4.39) holds true also when the medium is intrinsically dispersive [(2.14b)], and indicates that the surface wave terms supplement an overall change caused by the dispersive property of the original system.

C. General case: Sources distributed over separate places

In this section various equations of $I(1;2)$ have been obtained only for a point source, but this constraint can be removed by using another expression, such as

$$I(1;2) = [1 + I(1;2) K(1;2)] U^{(C)}(1;2) , \tag{4.40}$$

i.e., the transposed version of the BS equation (2.29). Hence, by multiplication of Eq. (4.40) to the right with the source factor $j^*(1) j(2)$, the mutual coherence function can be given also by

$$\langle \psi^*(\hat{\mathbf{x}}_1) \psi(\hat{\mathbf{x}}_2) \rangle = \langle \psi^*(\hat{\mathbf{x}}_1) \rangle \langle \psi(\hat{\mathbf{x}}_2) \rangle + \int d\hat{\mathbf{x}}_1' d\hat{\mathbf{x}}_2' \int d\hat{\mathbf{x}}_1'' d\hat{\mathbf{x}}_2'' I(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2 \mid \hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2') K(\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2' \mid \hat{\mathbf{x}}_1'';\hat{\mathbf{x}}_2'') \langle \psi^*(\hat{\mathbf{x}}_1'') \rangle \langle \psi(\hat{\mathbf{x}}_2'') \rangle . \tag{4.41}$$

Here

$$\langle \psi(\hat{\mathbf{x}}) \rangle = \int d\hat{\mathbf{x}}' G(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') j(\hat{\mathbf{x}}') , \tag{4.42}$$

and is given with the exact $G(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$ also containing the nonpropagative mode waves. In the second term giving the incoherent part, the factor $IK(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2 \mid \hat{\mathbf{x}}_1'';\hat{\mathbf{x}}_2'')$ can be given by a mode series of the form

$$IK(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2 \mid \hat{\mathbf{x}}_1'';\hat{\mathbf{x}}_2'') = \sum_{a,b,c,d} \int d\boldsymbol{\rho}' \int d\boldsymbol{\Omega}' N_{ab}^{(T)}(\boldsymbol{\Omega}, z_1, z_2) I_{ab;cd}(\boldsymbol{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \boldsymbol{\Omega}') K_{cd}(\boldsymbol{\Omega}' \mid \boldsymbol{\rho}'-\boldsymbol{\rho}'' \mid \mathbf{r}'', z_1'', z_2'') \tag{4.43}$$

with the same $I_{ab;cd}$ as given by the solution of Eq. (4.35); the factor

$$K_{cd}(\boldsymbol{\Omega}' \mid \boldsymbol{\rho}'-\boldsymbol{\rho}'' \mid \mathbf{r}'', z_1'', z_2'') = K_{cd}^{(\lambda)}(\boldsymbol{\Omega}' \mid \mathbf{r}'', z_1'', z_2'') \delta(\boldsymbol{\rho}'-\boldsymbol{\rho}'') \tag{4.44}$$

is a mode transform of $K(\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2' \mid \hat{\mathbf{x}}_1'';\hat{\mathbf{x}}_2'')$ only with respect to the coordinates $\hat{\mathbf{x}}_1'$ and $\hat{\mathbf{x}}_2'$, according to Eqs. (4.1), (4.3), and (4.28), and $N_{ab}^{(T)}(\boldsymbol{\Omega}, z_1, z_2) = N_{ab}^{(T)}(\boldsymbol{\Omega}, z)$ when $z_1 = z_2 = z$.

Thus with Eq. (4.43), expression (4.41) for the mutual coherence function is valid even when the source is distributed over separate places with large distances, so that the interfering waves can be important to excite the incoherent wave given by the second term.

V. A FIXED SCATTERER EMBEDDED IN THE RANDOM WAVEGUIDE

When a fixed scatterer, described by $q_\alpha(\hat{\mathbf{x}})$ with the center at $\hat{\mathbf{x}} = \hat{\mathbf{x}}_\alpha$, is embedded in the waveguide, the deterministic Green function $g^{(\alpha)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$ is governed by the equation

$$[L - q(\hat{\mathbf{x}}) - q_\alpha(\hat{\mathbf{x}})] g^{(\alpha)}(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') = \delta(\hat{\mathbf{x}} - \hat{\mathbf{x}}') , \tag{5.1}$$

or in matrix form by

$$(L - q - q_\alpha) g^{(\alpha)} = 1 . \tag{5.2}$$

A basic assumption implied here is that $q(\hat{\mathbf{x}})$ and $q_\alpha(\hat{\mathbf{x}})$ can be both nonzero at the same place, with a constraint that $|q_\alpha| \gg |q|$ (or $\gg |M^{(q)}|$, more reasonably; see Appendix E to overcome this aspect). The procedure of obtaining the statistical Green functions with this equation is almost the same as when the medium is unbounded in space,[11] although the specific expressions are not quite the same.

To obtain the first-order Green function $G^{(\alpha)} = \langle g^{(\alpha)} \rangle$, averaging of Eq. (5.2) yields an equation of the form

$$(L - M - q_\alpha') G^{(\alpha)} = 1, \quad q_\alpha' = q_\alpha + \Delta q_\alpha . \tag{5.3}$$

Here M is the effective matrix of the medium plus the boundaries when the scatterer is absent, and is the same as in Eq. (2.22); Δq_α is determined by an effective medium $M_\alpha^{(q)}$, defined by

$$M_\alpha^{(q)} G^{(\alpha)} = \langle q g^{(\alpha)} \rangle, \quad M_\alpha^{(q)} = M^{(q)} + \Delta q_\alpha \ , \tag{5.4}$$

or, more strictly, from Eqs. (2.21) and (2.22), according to

$$M_\alpha G^{(\alpha)} = \langle v g^{(\alpha)} \rangle, \quad M_\alpha = M + \Delta q_\alpha \ , \tag{5.5}$$

which reproduces Eq. (5.4) to the approximation of $M_\alpha^{(j)} \simeq M^{(j)}$, implying no effect of the scatterer on the effective values of the surface impedance $B^{(j)}$. Hence Δq_α can be regarded as an effective change of q_α due to the medium-boundary fluctuation, and to the second order of q [Fig. 2(a)]

$$\Delta q_\alpha = \langle q G T^{(\alpha)} G q \rangle, \quad \langle q \rangle = 0 \tag{5.6}$$

in terms of the scattering matrix $T^{(\alpha)}$ to be defined by Eq. (5.8); a corresponding expression was obtained also for a particulate medium.[11] The change Δq_α will generally be small in actual cases, however.

The solution of Eq. (5.3) can be given in the form

$$G^{(\alpha)} = G + G T^{(\alpha)} G \tag{5.7}$$

in terms of the scattering matrix $T^{(\alpha)}$ of q'_α, defined by

$$\begin{aligned} T^{(\alpha)} &= (1 - q'_\alpha G)^{-1} q'_\alpha \\ &= q'_\alpha + q'_\alpha G q'_\alpha + q'_\alpha G q'_\alpha G q'_\alpha + \cdots \ . \end{aligned} \tag{5.8}$$

Although evaluating $T^{(\alpha)}$ is generally an involved task, it

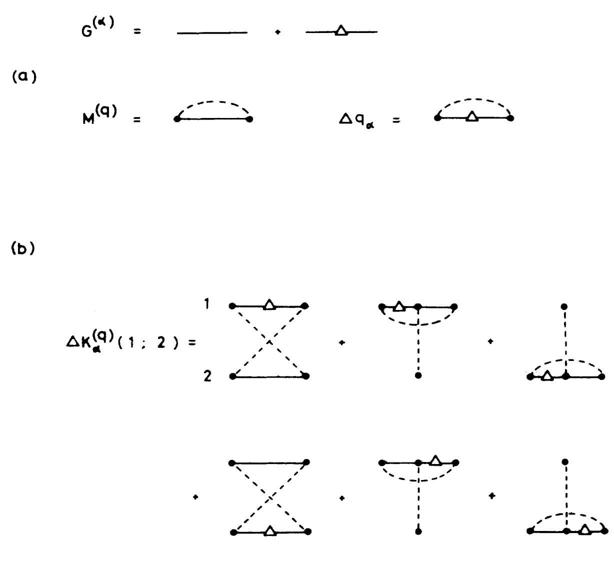

FIG. 2. (a) Schematic diagrams of $G^{(\alpha)}$, $M^{(q)}$, and Δq_α, given by Eqs. (5.7), (2.24a), and (5.6), respectively, are shown. Here G, q, and $T^{(\alpha)}$ are represented by a solid line, filled circle, and triangle, respectively, and are connected in the order of their matrix multiplication; $\langle q \cdots q \rangle$ is represented by a dashed line connecting the q's. (b) Nonvanishing elements of $\Delta K_\alpha(1;2) \simeq \Delta K_\alpha^{(q)}(1;2)$ in Eq. (5.13) are shown to the lowest order, with the same notations as in (a).

is directly connected to a conventional scattering amplitude of the scatterer for the coherent wave and may be a good experimental observable.

The procedure of deriving the BS equation for the second-order Green function in this case, $I_\alpha(1;2)$, is the same as for the previous $I(1;2)$, just with the replacement of $G \to G^{(\alpha)}$ and $M \to M_\alpha$ in Eqs. (2.27)–(2.30) (and with $M^{(q)} \to M_\alpha^{(q)}$ and $M^{(j)} \to M_\alpha^{(j)}$ for each of the medium and the boundaries). The coherent propagator dependent on q_α $U^{(C,\alpha)}(1;2)$, is now defined by

$$U^{(C,\alpha)}(1;2) = G^{(\alpha)*}(1) G^{(\alpha)}(2) \tag{5.9}$$

[cf. (2.33)], and can be written, on using Eq. (5.7), in the form

$$I^{(C,\alpha)} = U^{(C)} + U^{(C)} V^{(\alpha)} U^{(C)} \ . \tag{5.10}$$

Here

$$\begin{aligned} V^{(\alpha)}(1;2) = {} & T^{(\alpha)*}(1) T^{(\alpha)}(2) + T^{(\alpha)*}(1) G^{-1}(2) \\ & + T^{(\alpha)}(2) [G^{-1}(1)]^* \ . \end{aligned} \tag{5.11}$$

Thus the BS equation for $I_\alpha(1;2)$ is obtained, with the procedure of leading to Eq. (2.29), in the same form as

$$I_\alpha(1;2) = U^{(C,\alpha)}(1;2)[1 + K(1;2) I_\alpha(1;2)] \ . \tag{5.12}$$

Strictly speaking, the incoherent factor $K(1;2)$ also should be replaced by $K_\alpha(1;2)$, defined by Eq. (2.30) with $g \to g^{(\alpha)}$, so that

$$K_\alpha(1;2) = K(1;2) + \Delta K_\alpha(1;2) \tag{5.13}$$

with a change ΔK_α caused by the scatterer. A part of ΔK_α contributed by the medium, $\Delta K_\alpha^{(q)}$, is shown in Fig. 2(b) diagrammatically to the first order of $T^{(\alpha)}$, showing that the nonvanishing elements are of the fourth order in q. We shall return to this exact case after Eq. (5.27).

The solution of Eq. (5.12) can be given by

$$I_\alpha = U^{(C,\alpha)} + U^{(C,\alpha)} S_\alpha U^{(C,\alpha)} \tag{5.14}$$

in terms of an incoherent scattering matrix S_α similar to S, defined by [Eq. (2.35)]

$$S_\alpha = K(1 + U^{(C,\alpha)} S_\alpha) \ . \tag{5.15}$$

On using Eq. (5.10), the equation can be rewritten in terms of S as

$$S_\alpha = S(1 + U^{(C)} V^{(\alpha)} U^{(C)} S_\alpha) \tag{5.16}$$

whose formal solution is

$$S_\alpha = (1 - S U^{(C)} V^{(\alpha)} U^{(C)})^{-1} S \ , \tag{5.17}$$

which directly leads to another expression

$$S_\alpha = S + S U^{(C)} V^{(\alpha,K)} U^{(C)} S \tag{5.18}$$

in terms of the notation

$$V^{(\alpha,K)} = (1 - V^{(\alpha)} U^{(C)} S U^{(C)})^{-1} V^{(\alpha)} \ , \tag{5.19}$$

meaning an effective $V^{(\alpha)}$ affected by K.

Expression (5.14) for I_α is convenient particularly when the coherent part (the first term) is dominant. There exists another expression:

$$I_\alpha = I + IV^{(\alpha,K)}I \ , \tag{5.20}$$

in which the entire effect of the scatterer is given by the second term, in terms of $V^{(\alpha,K)}$ meaning an effective scattering matrix of the scatterer for the whole incident wave having both the coherent and the incoherent parts. The proof of Eq. (5.20) is given by substituting the expression [cf. Eq. (2.38)]

$$S_\alpha = K + KI_\alpha K \tag{5.21}$$

into the left-hand side of Eq. (5.18), followed by rewriting the right-hand side in terms of I, with the aid of relations (2.38) and (2.37), and finally by dropping the common factors K from the both sides.

Equation (5.20) also can be written in a mode series and when $\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_2 = \hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'_1 = \hat{\mathbf{x}}'_2 = \hat{\mathbf{x}}'$, we obtain [Appendix E]

$$I_\alpha(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}') = \sum_{a,b,c,d} N_{ab}^{(T)}(\mathbf{\Omega},z) I_{ab;cd}^{(\alpha)}(\mathbf{\Omega},\boldsymbol{\rho} \mid \mathbf{\Omega}',\boldsymbol{\rho}') \bar{N}_{cd}^{(T)}(\mathbf{\Omega}',z') , \tag{5.22a}$$

where

$$I_{ab;cd}^{(\alpha)}(\mathbf{\Omega},\boldsymbol{\rho} \mid \mathbf{\Omega}',\boldsymbol{\rho}') = I_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}' \mid \mathbf{\Omega}') + \sum_{i,j,k,l} \int d\mathbf{\Omega}'' d\mathbf{\Omega}''' I_{ab;ij}(\mathbf{\Omega} \mid \boldsymbol{\rho}-\boldsymbol{\rho}_\alpha \mid \mathbf{\Omega}'') V_{ij;kl}^{(\alpha,K)}(\mathbf{\Omega}'' \mid \mathbf{\Omega}''') I_{kl;cd}(\mathbf{\Omega}''' \mid \boldsymbol{\rho}_\alpha - \boldsymbol{\rho}' \mid \mathbf{\Omega}') \ . \tag{5.22b}$$

Here $V_{ij;kl}^{(\alpha,K)}(\mathbf{\Omega}'' \mid \mathbf{\Omega}''')$ is a mode transform of the matrix $V^{(\alpha,K)}$, defined by Eq. (5.19) or by the integral equation

$$V^{(\alpha,K)} = V^{(\alpha)}(1 + U^{(C)} S U^{(C)} V^{(\alpha,K)}) \ , \tag{5.23}$$

whose mode transformed version is

$$V_{ij;kl}^{(\alpha,K)}(\mathbf{\Omega}' \mid \mathbf{\Omega}'') = V_{ij;kl}^{(\alpha)}(\mathbf{\Omega}' \mid \mathbf{\Omega}'') + \sum_{a,b,c,d} \int d\mathbf{\Omega}\, d\mathbf{\Omega}''' V_{ij;ab}^{(\alpha)}(\mathbf{\Omega}' \mid \mathbf{\Omega}) U_{ab}^{(C)} S_{ab;cd} U_{cd}^{(C)}(\mathbf{\Omega} \mid \boldsymbol{\rho}=0 \mid \mathbf{\Omega}''') V_{cd;kl}^{(\alpha,K)}(\mathbf{\Omega}''' \mid \mathbf{\Omega}'') \ . \tag{5.24}$$

Here

$$V_{ij;kl}^{(\alpha)}(\mathbf{\Omega}' \mid \mathbf{\Omega}'') = \sigma_{ij;kl}^{(\alpha)}(\mathbf{\Omega}' \mid \mathbf{\Omega}'') - \gamma_{ij;kl}^{(\alpha)}(\mathbf{\Omega}')\delta(\mathbf{\Omega}' - \mathbf{\Omega}'') \ , \tag{5.25a}$$

where $\sigma_{ij;kl}^{(\alpha)}(\mathbf{\Omega}' \mid \mathbf{\Omega}'')$ and $\gamma_{ij;kl}^{(\alpha)}(\mathbf{\Omega}'')$ are defined by Eqs. (E16) and (E28) in terms of the mode transforms $\tilde{T}_{ik}^{(\alpha)*}$ and $\tilde{T}_{jl}^{(\alpha)}$ of $T^{(\alpha)}$, and mean, respectively, the differential and total cross sections of the scatterer for scattering of the wave having mode index kl and propagating in direction $\mathbf{\Omega}''$ into the wave having index ij and direction $\mathbf{\Omega}'$. $U_{ab}^{(C)} S_{ab;cd} U_{cd}^{(C)}(\mathbf{\Omega} \mid \boldsymbol{\rho} \mid \mathbf{\Omega}')$ is the incoherent part of $I_{ab;cd}(\mathbf{\Omega} \mid \boldsymbol{\rho} \mid \mathbf{\Omega}')$, as given by the second term in Eq. (4.33), and may be obtained by solving radiative transfer Eq. (4.35) or, more directly, Eq. (4.29).

It may be remarked that in Eq. (5.25a), the term of $\gamma_{ij;kl}^{(\alpha)}(\mathbf{\Omega}')$ comes from the interference terms in Eq. (5.11) and makes the cross section $V_{ij;ab}^{(\alpha)}$ negative in the shadow direction, and also that both $V^{(\alpha)}$ and $V^{(\alpha,K)}$ are subject to the same optical relation

$$\sum_{i,j}{}' \int d\mathbf{\Omega}' \Delta_{ij} k_{ij}^{(e)} w_{ij} V_{ij;kl}^{(\alpha)}(\mathbf{\Omega}' \mid \mathbf{\Omega}'') \simeq 0 \ , \tag{5.25b}$$

$$\sum_{i,j}{}' \int d\mathbf{\Omega}' \Delta_{ij} k_{ij}^{(e)} w_{ij} V_{ij;kl}^{(\alpha,K)}(\mathbf{\Omega}' \mid \mathbf{\Omega}'') \simeq 0 \tag{5.25c}$$

($\sum'$ means the summation only over the propagative mode waves), by virtue of (5.24) [Eq. (E31)]. Here the two relations simply mean that the entire scattered power is ultimately equal to what was absorbed by the scatterer itself and make the resultant power transmitted through the second term in Eq. (5.20) zero.

This fact can be directly demonstrated by showing that an effective wave source of the second term, as given by the space divergence of the power flux, is zero everywhere in the space. That is, using power Eq. (4.36) for $I_{ab;cd}$, this space divergence becomes written as

$$\sum_{i,j,k,l} \int d\mathbf{\Omega}'' d\mathbf{\Omega}''' \Delta_{ij}^{(T)} k_{ij}^{(e)} w_{ij}^{(e)} V_{ij;kl}^{(\alpha,K)}(\mathbf{\Omega}'' \mid \mathbf{\Omega}''') \times \delta(\boldsymbol{\rho}-\boldsymbol{\rho}_\alpha) I_{kl;cd}(\mathbf{\Omega}''' \mid \boldsymbol{\rho}_\alpha - \boldsymbol{\rho}' \mid \mathbf{\Omega}') \ , \tag{5.26}$$

which is zero in consequence of the optical relation (5.25c), to the approximation of $\Delta_{ij}^{(T)} \sim \Delta_{ij}$ $(\sim \delta_{ij})$.

It may be remarked that even when the original potential q_α is Hermitian, the effective change Δq_α of q_α is generally not Hermitian, as a result of an incoherent scattering even by the deterministic scatterer. A detailed structure of power conservation for a fixed scatterer embedded in the random system is treated after Eq. (5.37) in a general form.

A. An exact version of the scattering matrix

The expression (5.20) for I_α has been obtained as an exact solution of the BS equation (5.12), which is not quite exact, however, in approximating the incoherent factor $K_\alpha(1;2)$ by $K(1;2)$ [Eq. (5.13)]. Also with the factor $K_\alpha(1;2)$, an exact solution can be effectively formulated, as follows.

With the additional term ΔK_α to K, the new BS equation for $I_\alpha^{(K+\Delta K)}(1;2)$ can be written, in terms of the old solution $I_\alpha(1;2)$, as

$$I_\alpha^{(K+\Delta K)}(1;2) = I_\alpha(1;2)[1 + \Delta K_\alpha(1;2) I_\alpha^{(K+\Delta K)}(1;2)] \ , \tag{5.27}$$

which still has the form of the original equation with the correspondence of $I_\alpha \to U^{(C,\alpha)}$ and $\Delta K_\alpha \to K$. Hence, the solution also can be written in the same form as

$$I_\alpha^{(K+\Delta K)} = I_\alpha + I_\alpha S_\alpha^{(\Delta K,K)} I_\alpha \tag{5.28}$$

similar to Eq. (5.14), in terms of a scattering matrix $S_\alpha^{(\Delta K,K)}$ of ΔK_α, defined by

$$S_\alpha^{(\Delta K,K)} = \Delta K_\alpha (1 + I_\alpha S_\alpha^{(\Delta K,K)}) \; . \tag{5.29}$$

By using expression (5.20) for I_α, Eq. (5.29) can be rewritten as

$$S_\alpha^{(\Delta K,K)} = S^{(\Delta K,K)}(1 + IV^{(\alpha,K)} I S_\alpha^{(\Delta K,K)}) \; , \tag{5.30}$$

in terms of another scattering matrix $S^{(\Delta K,K)}$ of ΔK_α when $V^{(\alpha)}=0$, defined, therefore, by

$$S^{(\Delta K,K)} = \Delta K_\alpha (1 + I S^{(\Delta K,K)}) \; , \tag{5.31}$$

and Eq. (5.30) gives the solution by

$$S_\alpha^{(\Delta K,K)} = (1 - S^{(\Delta K,K)} I V^{(\alpha,K)} I)^{-1} S^{(\Delta K,K)} \; . \tag{5.32}$$

Here it may be convenient to write $S^{(\Delta K,K)}$ of Eq. (5.31) in terms of a scattering matrix $S^{(\Delta K)}$ of ΔK_α, defined by

$$\begin{aligned} S^{(\Delta K)} &= \Delta K_\alpha (1 + U^{(C)} S^{(\Delta K)}) \\ &= (1 - \Delta K_\alpha U^{(C)})^{-1} \Delta K_\alpha \; , \end{aligned} \tag{5.33}$$

which is perfectly free from K and is the same function of ΔK_α as the scattering matrix S is of K [(2.35) and (2.36)]. That is, on substituting expression (2.34) for I into Eq. (5.31), we obtain an expression of $S^{(\Delta K,K)}$ in terms of $S^{(\Delta K)}$ as

$$S^{(\Delta K,K)} = S^{(\Delta K)}(1 + U^{(C)} S U^{(C)} S^{(\Delta K,K)}) \tag{5.34a}$$

$$= (1 - S^{(\Delta K)} U^{(C)} S U^{(C)})^{-1} S^{(\Delta K)} \; , \tag{5.34b}$$

which is the same function of $S^{(\Delta K)}$ as $V^{(\alpha,K)}$ is of $V^{(\alpha)}$ [(5.19)].

Thus using Eq. (5.20), Eq. (5.28) can be written finally in the form

$$I_\alpha^{(K+\Delta K)} = I + I V^{(\alpha,K+\Delta K)} I \; . \tag{5.35}$$

$V^{(\alpha,K+\Delta K)}$ means a resultant scattering matrix of the scatterer, and is given by

$$V^{(\alpha,K+\Delta K)} = V^{(\alpha,K)} + (1 + V^{(\alpha,K)} I) S_\alpha^{(\Delta K,K)} (I V^{(\alpha,K)} + 1) \; , \tag{5.36}$$

in terms of $V^{(\alpha,K)}$ and $S_\alpha^{(\Delta K,K)}$. The second term on the right-hand side gives an entire contribution from ΔK_α, where to the first order of $V^{(\alpha,K)}$,

$$S_\alpha^{(\Delta K,K)} \simeq S^{(\Delta K,K)} + S^{(\Delta K,K)} I V^{(\alpha,K)} I S^{(\Delta K,K)} \; .$$

B. Optical relations and power conservation

Optical relations for the entire system of the random waveguide plus the fixed scatterer can be found in exactly the same way as in the previous section; basic equations remain unchanged simply with the replacement of $M \to M_\alpha$ and $K \to K_\alpha$, as long as the medium and the boundaries are nondissipative, independent of whether the scatterer is contrarily dissipative. For example, the basic relation (2.47) is replaced by

$$\begin{aligned} \frac{\partial}{\partial \rho} \cdot \boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2) = (2i)^{-1} \delta(\hat{\mathbf{x}} \mid 1;2) \\ \times \{ M_\alpha^*(1) - M_\alpha(2) \\ - [G^{(\alpha)*}(1) - G^{(\alpha)}(2)] K_\alpha(1;2) \} \; , \end{aligned} \tag{5.37}$$

in terms of the effective medium-boundary matrix M_α defined by Eq. (5.5) [where, strictly speaking, $\boldsymbol{\beta}(\hat{\mathbf{x}} \mid 1;2)$ also should be written as $\boldsymbol{\beta}_\alpha(\hat{\mathbf{x}} \mid 1;2)$], and, if a dissipation by the scatterer is assumed hereafter with a parameter $\gamma_\alpha^{(a)}$, defined by

$$\begin{aligned} \gamma_\alpha^{(a)}(\hat{\mathbf{x}} \mid 1;2) &= (2i)^{-1} \delta(\hat{\mathbf{x}} \mid 1;2) [q_\alpha^*(1) - q_\alpha(2)] \\ &\equiv (2i)^{-1} (q_\alpha^* - q_\alpha)(\hat{\mathbf{x}} \mid 1;2) \; , \end{aligned} \tag{5.38}$$

then power equation (2.51) for the coherent wave is replaced, in terms of the notation $A(\hat{\mathbf{x}} \mid 1;2) B(1;2) = AB(x \mid 1;2)$, by

$$\begin{aligned} \left[\frac{\partial}{\partial \hat{\mathbf{x}}} \cdot \hat{\alpha} + \gamma_\alpha^{(q+b)} + \gamma_\alpha^{(a)} \right] U^{(C,\alpha)}(\hat{\mathbf{x}} \mid 1;2) \\ = (2i)^{-1} (G^{(\alpha)*} - G^{(\alpha)})(\hat{\mathbf{x}} \mid 1;2) \; . \end{aligned} \tag{5.39}$$

Here

$$\begin{aligned} \gamma_\alpha^{(q+b)}(1;2) &= (2i)^{-1} [M_\alpha^*(1) - M_\alpha(2)] \\ &= (\gamma^{(q+b)} + \Delta\gamma_\alpha)(1;2) \; , \end{aligned} \tag{5.40}$$

with $\gamma^{(q+b)}$ defined by Eq. (2.52) and $\Delta\gamma_\alpha$ by

$$\Delta\gamma_\alpha(1;2) = (2i)^{-1} [\Delta q_\alpha^*(1) - \Delta q_\alpha(2)] \; . \tag{5.41}$$

Therefore, in Eq. (5.39)

$$\gamma_\alpha^{(q+b)} + \gamma_\alpha^{(a)} = \gamma^{(q+b)} + \Delta\gamma_\alpha^{(T)} \; , \tag{5.42}$$

where

$$\begin{aligned} \Delta\gamma_\alpha^{(T)}(1;2) &= (\gamma_\alpha^{(a)} + \Delta\gamma_\alpha)(1;2) \\ &= (2i)^{-1} [q_\alpha'^*(1) - q_\alpha'(2)] \; , \end{aligned} \tag{5.43}$$

which means a total dissipation coefficient of the scatterer for the coherent wave.

Comparison of the new relation (5.37) with the original (2.47) shows us an optical relation for the effective change Δq_α, as given, in terms of the quantity $\Delta\gamma_\alpha$ of Eq. (5.41), by

$$\begin{aligned} \Delta\gamma_\alpha(\hat{\mathbf{x}} \mid 1;2) = (2i)^{-1} (G^* - G)(\hat{\mathbf{x}} \mid 1;2) \Delta K_\alpha(1;2) \\ + (2i)^{-1} (G^* T^{(\alpha)*} G^* - G T^{(\alpha)} G)(\hat{\mathbf{x}} \mid 1;2) \\ \times K_\alpha(1;2) \; , \end{aligned} \tag{5.44}$$

where use has been made of Eq. (5.5). We note that, to the ladder approximation of Eq. (2.32a), the second term gives the leading term with $K_\alpha \simeq K^{(q)}$, and the result is consistent with Eq. (5.6).

In the same way, comparing the new power equation (5.39) with (2.51) we find a relation of purely coherent counterpart, as given by

$$\left[\frac{\partial}{\partial \hat{\mathbf{x}}}\cdot\hat{\alpha}+\gamma^{(q+b)}\right]U^{(C)}V^{(\alpha)}U^{(C)}(\hat{\mathbf{x}}\mid 1;2)$$
$$+\Delta\gamma_\alpha^{(T)}U^{(C,\alpha)}(\hat{\mathbf{x}}\mid 1;2)$$
$$=(2i)^{-1}(G^*T^{(\alpha)*}G^*-GT^{(\alpha)}G)(\hat{\mathbf{x}}\mid 1;2)\ , \tag{5.45}$$

which is reduced to a local relation for $T^{(\alpha)}$ [as is found by using Eqs. (2.51) and (5.11) on the left-hand side; see Eq. (E2) for a direct derivation] as

$$(2i)^{-1}\delta(\hat{\mathbf{x}}\mid 1;2)\{\,[G^*(1)-G(2)]T^{(\alpha)*}(1)T^{(\alpha)}(2)$$
$$-T^{(\alpha)*}(1)+T^{(\alpha)}(2)\}\,U^{(C)}(1;2)$$
$$+\Delta\gamma_\alpha^{(T)}U^{(C,\alpha)}(\hat{\mathbf{x}}\mid 1;2)=0\ . \tag{5.46}$$

That is, Eq. (5.45) is another version of the (local) optical relation subjected by the scattering matrix $T^{(\alpha)}$, defined by Eq. (5.8), in the effective medium $M^*\neq M$ for the coherent wave.

The optical relation (5.37) for the BS equation (5.12) can be written in terms of the incoherent scattering matrix $S_\alpha^{(K+\Delta K)}$, defined by

$$S_\alpha^{(K+\Delta K)}=K_\alpha(1+U^{(C,\alpha)}S_\alpha^{(K+\Delta K)}) \tag{5.47}$$

(which differs from S_α of Eq. (5.15) by $K\to K_\alpha$), as has been given by Eq. (2.56) when the scatterer is absent in terms of S. Hence,

$$\left[\frac{\partial}{\partial \hat{\mathbf{x}}}\cdot\hat{\alpha}+\gamma_\alpha^{(a)}\right]U^{(C,\alpha)}S_\alpha^{(K+\Delta K)}(\hat{\mathbf{x}}\mid 1;2)$$
$$+\frac{\partial}{\partial \boldsymbol{\rho}}\cdot\boldsymbol{\beta}^{(e,\alpha)}(\hat{\mathbf{x}}\mid 1;2)=\gamma_\alpha^{(q+b)}(\hat{\mathbf{x}}\mid 1;2)\ . \tag{5.48}$$

Here

$$\boldsymbol{\beta}^{(e,\alpha)}(\hat{\mathbf{x}}\mid 1;2)=\boldsymbol{\beta}(\hat{\mathbf{x}})(1+U^{(C,\alpha)}S_\alpha^{(K+\Delta K)})(1;2)\ , \tag{5.49}$$

where and hereafter $(\hat{\mathbf{x}})$ is to stand for $(\hat{\mathbf{x}}\mid 1;2)$. Relation (5.48) directly ensures, with Eq. (5.39) for the coherent wave, conservation of the entire power flux, $\langle\hat{\mathbf{W}}_\alpha(\hat{\mathbf{x}})\rangle$, given according to Eq. (2.54) by

$$\langle\hat{\mathbf{W}}_\alpha(\hat{\mathbf{x}})\rangle=(\hat{\boldsymbol{\alpha}}+\boldsymbol{\beta})(\hat{\mathbf{x}})I_\alpha(1;2)$$
$$=\langle\hat{\mathbf{W}}(\hat{\mathbf{x}})\rangle+\Delta\hat{\mathbf{W}}_\alpha(\hat{\mathbf{x}})\ , \tag{5.50}$$

where $\Delta\hat{\mathbf{W}}_\alpha$ is a change caused by the scatterer, and on using the second term in Eq. (5.35), can be written as

$$\Delta\hat{\mathbf{W}}_\alpha(\hat{\mathbf{x}})=\langle\hat{\mathbf{W}}(\hat{\mathbf{x}})\rangle V^{(\alpha,K+\Delta K)}I(1;2)\ , \tag{5.51}$$

in terms of the matrix $\langle\hat{\mathbf{W}}(\hat{\mathbf{x}})\rangle$ [Eq. (2.54)]. Hence

$$\frac{\partial}{\partial \hat{\mathbf{x}}}\cdot\langle\hat{\mathbf{W}}_\alpha(\hat{\mathbf{x}})\rangle+\gamma_\alpha^{(a)}(\hat{\mathbf{x}})I_\alpha(1;2)$$
$$=(2i)^{-1}(G^{(\alpha)*}-G^{(\alpha)})(\hat{\mathbf{x}}\mid 1;2)\ . \tag{5.52}$$

It may be remarked that an equation of continuity for the change $\Delta\hat{\mathbf{W}}_\alpha$ is found by substitution of Eq. (5.50) into Eq. (5.52) to be

$$\frac{\partial}{\partial \hat{\mathbf{x}}}\cdot\Delta\hat{\mathbf{W}}_\alpha(\hat{\mathbf{x}}\mid 1;2)+\gamma_\alpha^{(a)}(\hat{\mathbf{x}})I_\alpha(1;2)$$
$$=(2i)^{-1}\delta(\hat{\mathbf{x}}\mid 1;2)[G^*T^{(\alpha)*}G^*(1)-GT^{(\alpha)}G(2)] \tag{5.53}$$

in consequence of Eq. (5.7), and therefore, with $\Delta\hat{\mathbf{W}}_\alpha$ of (5.51),

$$\frac{\partial}{\partial \hat{\mathbf{x}}}\cdot\Delta\hat{\mathbf{W}}_\alpha+\gamma_\alpha^{(a)}I_\alpha=0$$

everywhere, as long as the right-hand side is zero, as realized whenever the scatterer is sufficiently separated from the source $j^*(1)j(2)$ to multiply (5.53) from the right. This gives a strict version of the proof for the same conclusion, preliminarily given by Eq. (5.26) when $\gamma_\alpha^{(a)}=0$ so that the scatterer is nondissipative.

Equation (5.53) can be regarded as another version of the optical relation (5.44) for the changes caused by the scatterer and, in fact, could be directly found from the latter relation (Appendix E).

VI. SPECIFIC EXPRESSIONS OF THE LADDER APPROXIMATION

Using the ladder approximation (2.32b) for $K^{(j)}(1;2)$, we here obtain specific expressions of various statistical parameters, typically for when the medium is free from the fluctuation. In spite of the simple approximation involved, this particular case is very illustrative because the $K^{(j)}(1;2)$ is strictly consistent with optical relation (2.48) with the $M^{(j)}$ of (2.24b), by virtue of the relation (2.32c), and also because $b^{(j)}$ is intrinsically dispersive [(2.3)].

It is straightforward to obtain the present $\tilde{K}^{(j)}_{ab;cd}(\boldsymbol{\lambda}_1;\boldsymbol{\lambda}_2\mid\boldsymbol{\lambda}_1';\boldsymbol{\lambda}_2')$ according to the definition (4.1) in the form (4.4), so that in (D30),

$$\tilde{K}^{(j)}(\mathbf{u}\mid\boldsymbol{\lambda}\mid\mathbf{u}')$$
$$=4^{-1}(2\pi)^2P_\zeta(\mathbf{u}'-\mathbf{u})$$
$$\times[-(\mathbf{u}-\tfrac{1}{2}\boldsymbol{\lambda})\cdot(\mathbf{u}'-\tfrac{1}{2}\boldsymbol{\lambda})+B_0^2+k_0^2]$$
$$\times[-(\mathbf{u}+\tfrac{1}{2}\boldsymbol{\lambda})\cdot(\mathbf{u}'+\tfrac{1}{2}\boldsymbol{\lambda})+B_0^2+k_0^2]\ , \tag{6.1}$$

in terms of the conventional "power" spectrum function $P_\zeta(\mathbf{u})$ of $\zeta(\boldsymbol{\rho})$, defined by

$$C_\zeta(\boldsymbol{\rho}-\boldsymbol{\rho}')=\langle\zeta(\boldsymbol{\rho})\zeta(\boldsymbol{\rho}')\rangle$$
$$=4^{-1}\int d\mathbf{u}\,e^{-i\mathbf{u}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}P_\zeta(\mathbf{u})\ . \tag{6.2}$$

The expression (6.1) is an even function of $\boldsymbol{\lambda}$, as it should be, and in terms of $\mathbf{u}$, $\mathbf{u}'$, and $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda}_1=\mathbf{u}-\boldsymbol{\lambda}/2,\quad \boldsymbol{\lambda}_1'=\mathbf{u}'-\boldsymbol{\lambda}/2\ ,$$
$$\boldsymbol{\lambda}_2=\mathbf{u}+\boldsymbol{\lambda}/2,\quad \boldsymbol{\lambda}_2'=\mathbf{u}'+\boldsymbol{\lambda}/2\ . \tag{6.3}$$

The Fourier transform of $M^{(j)}(\boldsymbol{\rho}_2\mid\boldsymbol{\rho}_2')$ with respect to $\boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_2'$ can be obtained, by using relation (2.32c), according to

$$(2\pi)^2\delta(\boldsymbol{\lambda}_2-\boldsymbol{\lambda}_2')\tilde{M}^{(j)}(\boldsymbol{\lambda}_2)=(2\pi)^{-2}\int d\boldsymbol{\lambda}'\tilde{K}^{(j)}(-\boldsymbol{\lambda}';\boldsymbol{\lambda}_2\mid -\boldsymbol{\lambda}_2';\boldsymbol{\lambda}')\tilde{G}(z\mid\boldsymbol{\lambda}'\mid z)\Big|_{z=d_j}\,, \tag{6.4}$$

with $\tilde{G}(z\mid\boldsymbol{\lambda}\mid z)$ given by (3.13). By using Eq. (4.4) with (6.1) and (6.3), we immediately find that

$$\tilde{M}^{(j)}(\boldsymbol{\lambda}_2)=4^{-1}\int d\boldsymbol{\lambda}'P_\zeta(\boldsymbol{\lambda}'-\boldsymbol{\lambda}_2)\tilde{G}(z\mid\boldsymbol{\lambda}'\mid z)\Big|_{z=d_j}(-\boldsymbol{\lambda}_2\cdot\boldsymbol{\lambda}'+B_0^2+k_0^2)^2\,, \tag{6.5}$$

which can be rewritten on setting $\boldsymbol{\lambda}'=\mathbf{u}'+\boldsymbol{\lambda}/2$ as

$$\tilde{M}^{(j)}(\mathbf{u}+\boldsymbol{\lambda}/2)=4^{-1}\int d\mathbf{u}'P_\zeta(\mathbf{u}'-\mathbf{u})\tilde{G}(z\mid\mathbf{u}'+\boldsymbol{\lambda}/2\mid z)\Big|_{z=d_j}[-(\mathbf{u}+\boldsymbol{\lambda}/2)\cdot(\mathbf{u}'+\boldsymbol{\lambda}/2)+B_0^2+k_0^2]^2\,. \tag{6.6}$$

In the same way

$$\tilde{M}^{(j)*}(\mathbf{u}-\boldsymbol{\lambda}/2)=4^{-1}\int d\mathbf{u}'P_\zeta(\mathbf{u}'-\mathbf{u})\tilde{G}^*(z\mid\mathbf{u}'-\boldsymbol{\lambda}/2\mid z)\Big|_{z=d_j}[-(\mathbf{u}-\boldsymbol{\lambda}/2)\cdot(\mathbf{u}'-\boldsymbol{\lambda}/2)+B_0^2+k_0^2]^2\,. \tag{6.7}$$

To obtain the factor $\boldsymbol{\beta}_{ab}^{(j,\lambda)}(\boldsymbol{\Omega})$ in (4.37) for the surface wave, we first observe that the Fourier transform of local optical relation (2.47) with respect to the $\boldsymbol{\rho}$ coordinates is presently written, except the factor $\delta_j(z)$, as

$$\begin{aligned}-i\boldsymbol{\lambda}\cdot\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u},\boldsymbol{\lambda})=&(2i)^{-1}[\tilde{M}^{(j)*}(\mathbf{u}-\boldsymbol{\lambda}/2)-\tilde{M}^{(j)}(\mathbf{u}+\boldsymbol{\lambda}/2)]\\&-(2\pi)^{-2}\int d\mathbf{u}'(2i)^{-1}[\tilde{G}^*(z\mid\mathbf{u}'-\boldsymbol{\lambda}/2\mid z)-\tilde{G}(z\mid\mathbf{u}'+\boldsymbol{\lambda}/2\mid z)]\Big|_{z=d_j}\tilde{K}^{(j)}(\mathbf{u}'\mid\boldsymbol{\lambda}\mid\mathbf{u})\,,\end{aligned} \tag{6.8}$$

by using (D9) and (D10), and the equation provides us with an equation to find the unknown $\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u},\boldsymbol{\lambda})$ explicitly. Hence, on using (6.1), (6.6), and (6.7) we obtain

$$\begin{aligned}\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u},\boldsymbol{\lambda})=8^{-1}\int d\mathbf{u}'(\mathbf{u}+\mathbf{u}')P_\zeta(\mathbf{u}'-\mathbf{u})\{&[-(\mathbf{u}-\boldsymbol{\lambda}/2)\cdot(\mathbf{u}'-\boldsymbol{\lambda}/2)+B_0^2+k_0^2]\tilde{G}^*(z\mid\mathbf{u}'-\boldsymbol{\lambda}/2\mid z)\\&+[-(\mathbf{u}+\boldsymbol{\lambda}/2)\cdot(\mathbf{u}'+\boldsymbol{\lambda}/2)+B_0^2+k_0^2]\tilde{G}(z\mid\mathbf{u}'+\boldsymbol{\lambda}/2\mid z)\}\Big|_{z=d_j}\,,\end{aligned} \tag{6.9}$$

which is, therefore, equal to $\tilde{\boldsymbol{\beta}}^{(j)*}(\mathbf{u},-\boldsymbol{\lambda})$, showing that its Fourier inversion with respect to $\boldsymbol{\lambda}$, $\boldsymbol{\beta}^{(j)}(\mathbf{u},\boldsymbol{\rho})$, is a real function of $\boldsymbol{\rho}$ and that $\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u},i\partial/\partial\boldsymbol{\rho})$ is a real operator. When $\boldsymbol{\lambda}=0$, Eq. (6.9) is reduced to

$$\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u},\boldsymbol{\lambda}=0)=8^{-1}\int d\mathbf{u}'(\mathbf{u}+\mathbf{u}')P_\zeta(\mathbf{u}'-\mathbf{u})(-\mathbf{u}\cdot\mathbf{u}'+B_0^2+k_0^2)(\tilde{G}^*+\tilde{G})(z\mid\mathbf{u}'\mid z)\Big|_{z=d_j}\,, \tag{6.10}$$

in which the contribution from the Green function is only the real part of $\tilde{G}(z\mid\mathbf{u}'\mid z)$ at $z=d_j$, as contrasted to (6.13) for $\gamma_{ab}^{(j,\lambda)}$ wherein only the imaginary part is involved.

The coefficients $\boldsymbol{\beta}_{ab}^{(j,\lambda)}(\boldsymbol{\Omega})$ are given according to (D10) and (D13) by

$$\boldsymbol{\beta}_{ab}^{(j,\lambda)}(\boldsymbol{\Omega})=\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u}=k_{ab}^{(T)}\boldsymbol{\Omega},\boldsymbol{\lambda})N_{ab}^{(T)}(\boldsymbol{\Omega},z=d_j)\,, \tag{6.11}$$

in terms of $\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u},\boldsymbol{\lambda})$, and $\boldsymbol{\beta}_{ab}^{(\lambda)}(\boldsymbol{\Omega},z)$ is by (D12). While $\gamma_{ab}^{(j,\lambda)}(\boldsymbol{\Omega},z)$ is given according to (D14) by

$$\gamma_{ab}^{(j,\lambda)}(\boldsymbol{\Omega},z)=(2ik_{ab}^{(T)})^{-1}[\tilde{M}^{(j)*}(\mathbf{u}-\boldsymbol{\lambda}/2)-\tilde{M}^{(j)}(\mathbf{u}+\boldsymbol{\lambda}/2)]\delta_j(z) \tag{6.12}$$

with $\mathbf{u}=k_{ab}^{(T)}\boldsymbol{\Omega}$ [where we note that $\gamma_{ab}^{(j,\lambda)}$ differs from $\gamma_{ab}^{(j)}$ of (3.19d) even when $\boldsymbol{\lambda}=0$], and use of Eqs. (6.6) and (6.7) yields

$$\gamma_{ab}^{(j,\lambda)}(\boldsymbol{\Omega},z)=(4k_{ab})^{-1}\int d\mathbf{u}'P_\zeta(\mathbf{u}'-k_{ab}\boldsymbol{\Omega})(B_0^2+k_0^2-k_{ab}\boldsymbol{\Omega}\cdot\mathbf{u}')^2(2i)^{-1}(\tilde{G}^*-\tilde{G})(z\mid\mathbf{u}'\mid z)\Big|_{z=d_j}\delta_j(z),\quad\boldsymbol{\lambda}=0 \tag{6.13}$$

which gives $\gamma_{ab}^{(j,\lambda)}(\boldsymbol{\Omega})$ except the factor $\delta_j(z)$. And $K_{ab;cd}^{(j,\lambda)}(\boldsymbol{\Omega}\mid\boldsymbol{\Omega}')$ is given by Eq. (D30) with (6.1), so that when $\boldsymbol{\lambda}=0$,

$$\begin{aligned}K_{ab;cd}^{(j,\lambda)}(\boldsymbol{\Omega}\mid\boldsymbol{\Omega}')=&\bar{N}_{ab}(\boldsymbol{\Omega},z=d_j)4^{-1}(2\pi)^2P_\zeta(k_{cd}\boldsymbol{\Omega}'-k_{ab}\boldsymbol{\Omega})\\&\times(B_0^2+k_0^2-k_{ab}k_{cd}\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')^2N_{cd}(\boldsymbol{\Omega}',z=d_j)\end{aligned} \tag{6.14}$$

For the last integral in Eq. (6.8) we can apply formula (4.6) by using mode series (3.13) and expression (3.10) for the factor $\delta_j(z)=\delta(z-d_j)$, so that we get the last term in local relation (D28) [cf. Eq. (D2)], while as to the variable $\mathbf{u}$ in (6.8), we can regard the whole terms, on multiplying with the additional factor $\phi_c^*\phi_d(z=d_j)$ as $f_{cd}(\mathbf{u},\boldsymbol{\lambda})$ in formula (4.6), so that its normal mode transform is given by $f_{cd}^{(\lambda)}(\boldsymbol{\Omega}')$ [(4.14b)]. Thus the local optical relation (D28) is reproduced with the terms of (6.11), (6.13), and (6.14), and consequently the integrated relation (D24), too.

In the relation (D28), the important mode waves are those of propagative nature, having the factors $\Delta_{ab}k_{ab}^{(e)}\sim\delta_{ab}k_{aa}$ of very small values for the nonpropagative mode waves. For the Green function $G(z\mid\boldsymbol{\rho}\mid z)$, $z=d_j$, involved in $M^{(j)}$ and $\boldsymbol{\beta}^{(j)}$, on the other hand, we observe that the important range of $\boldsymbol{\rho}$ is of the order of the boundary correlation distance l, which is assumed to be sufficiently small compared with the waveguide height d_2, so that the G can be approximated by that along an isolated planar boundary, in view of a negligible effect from the other boundary. In this case, it has been shown that[12]

$$\tilde{G}(\lambda) \equiv \tilde{G}(z \mid \lambda \mid z)\mid_{z=d_j}$$
$$=[i\tilde{h}(\lambda)+B_0+\tilde{M}^{(j)}(\lambda)]^{-1} . \quad (6.15)$$

Here

$$\tilde{h}(\lambda)=(k_0^2+\tilde{M}^{(q)}-\lambda^2)^{1/2}, \quad \mathrm{Im}(\tilde{h})<0 \quad (6.16)$$

where $\tilde{M}^{(q)}=0$ in the present case of $q=0$. Equation (6.15) shows that $\tilde{M}^{(j)}(\lambda)$ is involved in the $\tilde{G}(\lambda)$ in exactly the same way as $\tilde{M}^{(q)}(\lambda)$ is involved in Eq. (3.11) through $D_a(\lambda)$ of (3.8) and (3.9), and that for $\mid \mathrm{Im}(\lambda)\mid \to\infty$, $\mid \tilde{G}\mid \sim \mid \tilde{M}^{(j)}\mid^{-1}$ (which is a right asymptotic form also for the true $\tilde{G}$) and, therefore, gives asymptotic expressions similar to those in Appendix B given in terms of $\tilde{M}^{(q)}$.

Here, to evaluate $\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u},\lambda)$ by using (6.15), we limit ourselves to the case in which $B_0=0$ and $\tilde{M}^{(j)}$ is negligibly small so that

$$\tilde{G}(\lambda)\simeq[i\tilde{h}(\lambda)]^{-1} \quad (6.17a)$$

and, therefore,

$$(\tilde{G}^*+\tilde{G})(\lambda)\simeq\begin{cases}0, & \mid\lambda\mid=\mid\lambda\mid<k_0\\ 2\lambda^{-1}, & \mid\lambda\mid \gg k_0\end{cases} \quad (6.17b)$$

and also the boundary correlation distance is very small compared with the wavelength so that in (6.10),

$$P_\zeta(\mathbf{u}'-\mathbf{u})\simeq P_\zeta(\mathbf{u}') \quad (6.17c)$$

in view of $\mid\mathbf{u}\mid\sim k_0$. Hence, with $d\mathbf{u}'=u'du'd\boldsymbol{\Omega}'$ where $u'=\mid\mathbf{u}'\mid$, we obtain by using (6.17b),

$$\tilde{\boldsymbol{\beta}}^{(j)}(\mathbf{u},\lambda=0)$$
$$\simeq -4^{-1}\int_{2\pi}d\boldsymbol{\Omega}'\int_0^\infty du'\boldsymbol{\Omega}'(\mathbf{u}\cdot\boldsymbol{\Omega}')u'^2P_\zeta(\mathbf{u}') , \quad (6.18)$$

which, from (6.11), gives

$$\boldsymbol{\beta}_{ab}^{(j,\lambda)}(\boldsymbol{\Omega})\simeq -4^{-1}\pi k_{ab}\boldsymbol{\Omega}N_{ab}(\boldsymbol{\Omega},z=d_j)\times\int_0^\infty du'u'^2P_\zeta(\mathbf{u}') . \quad (6.19)$$

If we assume a correlation function $C_\zeta(\mathbf{r})$ of the form

$$C_\zeta(\mathbf{r})=\langle\zeta^2\rangle\exp[-\tfrac{1}{2}(\mathbf{r}/l)^2] \quad (6.20a)$$

so that

$$P_\zeta(\mathbf{u})=2\pi^{-1}\langle\zeta^2\rangle l^2\exp[-\tfrac{1}{2}(\mathbf{u}l)^2], \quad k_0l\ll 1 \quad (6.20b)$$

we obtain

$$\boldsymbol{\beta}_{ab}^{(j,\lambda)}(\boldsymbol{\Omega})\simeq -2^{-3/2}\pi^{1/2}\langle\zeta^2\rangle l^{-1}k_{ab}\boldsymbol{\Omega}N_{ab}(\boldsymbol{\Omega},z=d_j) , \quad (6.21)$$

which is directed opposite to $\boldsymbol{\Omega}$ so that it diminishes the unperturbed power flux along the boundary. This will work to increase the power density consistent with the fact that, as a result of $\mathrm{Re}(\tilde{M}^{(j)})>0$ [(6.22)], the wave intensity exponentially decreases with the distance from the boundary in its neighborhood [(2.2)], implying that the wave is partially trapped along the boundary.

With the same procedure we obtain a similar expression for $\tilde{M}^{(j)}$ from (6.6) as

$$\tilde{M}^{(j)}(\lambda)\simeq 2^{-3/2}\pi^{1/2}\langle\zeta^2\rangle l^{-1}\lambda^2 , \quad (6.22)$$

which is real, giving $\gamma_{ab}^{(j,\lambda)}=0$ for $\lambda=0$ [(6.12)]. This indicates that to obtain $\gamma_{ab}^{(j,\lambda)}$ or the imaginary part of $\tilde{M}^{(j)}$, the approximation of using Eq. (6.15) completely fails and the original expression (3.13) for $\tilde{G}$ must be used. Here $\gamma_{ab}^{(j,\lambda)}$ can be obtained more directly by use of (D28); i.e., when $\lambda=0$ and dropping the factor $N_{cd}(\boldsymbol{\Omega},d_j)$ from the both sides by use of (D30) it gives, with $\tilde{K}^{(j)}(\mathbf{u}\mid\lambda=0\mid\mathbf{u}')$ of (6.1),

$$k_{cd}\gamma_{cd}^{(j,\lambda)}(\boldsymbol{\Omega}')=\sum_{a,b}\int d\boldsymbol{\Omega}\,\Delta_{ab}k_{ab}^{(e)}w_{ab}^{(e)}\tilde{N}_{ab}(\boldsymbol{\Omega},d_j)\times\tilde{K}^{(j)}(k_{ab}\boldsymbol{\Omega}\mid\lambda\mid k_{cd}\boldsymbol{\Omega}') , \quad (6.23)$$

which drastically depends on the mode function $\tilde{N}_{ab}(\boldsymbol{\Omega},d_j)$, in contrast to (6.22). Here the factors $\Delta_{ab}k_{ab}^{(e)}$ are very small for the nonpropagative mode waves, demonstrating that the extinction coefficients $\gamma_{ab}^{(j,\lambda)}$ are determined by the overall boundary conditions of the waveguide. While the attenuation coefficients γ_{ab} [(3.19b)] are connected with $\gamma_{ab}^{(j,\lambda)}$ by relation (D31), which, with $\lambda=0$, shows that

$$\gamma_{ab}\Delta_{ab}(\boldsymbol{\Omega})=(k_{ab}/k_{ab}^{(e)})\sum_{j=1}^{2}\gamma_{ab}^{(j,\lambda)}N_{ab}(\boldsymbol{\Omega},z=d_j) , \quad (6.24)$$

where $k_{ab}^{(e)}$ is defined by (D4) [the relation is not exactly the same as (3.25), as a consequence of $\gamma_{ab}^{(j,\lambda)}\neq\gamma_{ab}^{(j)}$].

On the other hand, Eqs. (6.21) and (6.22) can be connected through the relation

$$\boldsymbol{\beta}_{ab}^{(j,\lambda)}=-\frac{i}{2}\frac{\partial}{\partial\boldsymbol{\lambda}}\mathrm{Re}[\tilde{M}^{(j)}(\lambda)]N_{ab}(\boldsymbol{\Omega},z=d_j), \quad \boldsymbol{\lambda}=k_{ab}\boldsymbol{\Omega} \quad (6.25)$$

being another version of the approximation (D32).

The procedure of obtaining the medium counterparts $M_{ab}^{(q)}$ and $K_{ab;cd}^{(q)}$ is also the same, showing explicitly that $\hat{\boldsymbol{\beta}}_{ab}^{(q)}=0$ as a consequence of the nondispersive property.

VII. SUMMARY AND DISCUSSION

The wave equation and the two boundary equations can be unified by one wave Eq. (2.15) having the same form as that of a wave equation in an inhomogeneous random medium v, so that the deterministic Green function g can be defined by Eq. (2.19b) subjected to the new boundary condition that $\partial_n g$ be zero on S_1 and S_2. This enables the statistical equations also to be obtained with the same procedure and in the same form as when only the medium is random [(2.22) and (2.29)]. The BS equation for the coherence function is constructed in terms of two basic (coordinate) matrices M and K which are subject to a local (optical) relation ensuring power conservation at every point in the waveguide and on the boundaries [(2.47)]. Both the M and the K can be approximated by a sum of independent contributions from each of the medium and the boundaries, and an optical relation of exactly the same form holds true for each of the medium and the boundaries [(2.50)], independent of whether the other members are dissipative or not.

The power flux has a contribution also from the surface waves along the boundary surfaces, given in terms of the factor $\boldsymbol{\beta}$ [(2.54) and (2.46)]. If the medium were intrinsically dispersive [(2.14b)], the surface-wave terms would have an additional contribution from the medium to change the power flux to meet with the dispersive characteristic, resulting in a possible change of the (horizontal) direction of wave propagation, etc.

A mode theory of the coherent wave was developed based on the effective medium M of a general form [(3.1) and (3.2)]. The resulting set of eigenfunctions is an entire function of the Fourier variable $\boldsymbol{\lambda}$ of the horizontal coordinates, as the whole, and, in virtue of this, an exact (normal) mode expansion [(3.16)] is possible based on a Maclaurin expansion at the set of poles of the first-order Green function. A mode theory of the second-order Green function also can be developed with basically the same method, starting with the expansion of integral Eq. (4.5) for the incoherent scattering matrix S in a normal mode series (4.27), with a fundamental propagator $\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})$ for the coherent wave [(4.20)]. Again, the series is based on the Maclaurin expansion at the sets of poles of $G^*(1)G(2)$, which is possible only with the renormalized Green function (given in terms of $M^{(q)}$ and $M^{(j)}$) and not with the "bare" Green function (Appendix B). The mode summation is made over all the mode waves including, not only the nonprogatives, but also interfering mode waves (Appendix C); the last kind of terms are involved only in the present case of two-dimensional propagation, however, and are involved neither in case of one-dimensional (cylindrical waveguide) nor three-dimensional propagation. The mode waves other than the propagatives are negligible when change of the intensity is negligibly small within a separation of the order of the wavelength. The integral equation can eventually be converted to an equation of radiative transfer with a term of point source [(4.35)]. It may be remarked that the incoherent scattering matrix is important, not only because of giving the incoherent part of the wave, but also because of being a basic quantity to construct an effective scattering matrix of a fixed scatterer embedded in the waveguide [e.g., Eq. (5.19)]. Here a practical means of obtaining the scattering matrix may be to use expression (2.38), in terms of a known solution of the equation of radiative transfer, obtained by some means including the conventional method of eigenfunction expansion.

In Appendix D details of the power equations and related optical relations are given with a particular emphasis on the derivation of their mode expressions, including those of an integrated optical relation [(D22) and (D23)] which ensures the equation of radiative transfer to be consistent with power conservation; therein given are also similar relations that hold true for each of the boundaries. Here the factors of the surface waves, $\boldsymbol{\beta}_{ab}^{(j,\lambda)}$, play a role of supplementing a change caused by the dispersive nature of the original boundaries [(4.38)] in such a way that the total power integrated over the waveguide height is given by Eq. (4.39) in terms of a complex vector $\boldsymbol{\Omega}_{ab}$ [defined by Eq. (4.19) with (4.12)].

A composite system of the random waveguide with a fixed scatterer is particularly interesting and important, providing a typical problem of how to treat a fixed scatterer embedded in a random system in general or, specifically, how to give the solution of the BS equation (5.12) by a suitable expression, depending on different situations (location of source, distance, etc.) and information required, in the form of the conventional theory of scattering for a coherent wave whenever convenient. The BS equation is changed by the fixed scatterer through q'_α and ΔK_α [(5.3) and (5.13)], where the former causes a coherent scattering together with an incoherent scattering of small amount and plays a principal role in the scattering, and the latter ΔK_α is of purely incoherent characteristics, and mostly works as a higher-order correction. Three expressions were prepared for the solution, two for the case $\Delta K_\alpha=0$ and one for the general case: The first [(5.14)] is particularly convenient when the coherent wave is dominant, and the second [(5.20)] gives the scattered waves in the conventional form of scattering theory, in terms of an effective scattering matrix $V^{(\alpha,K)}$ of the scatterer; here the latter is obtained as the solution of an integral equation [(5.24)] and is subject to an important optical relation [(5.25c)] implying that the cross section has negative values in the shadow direction. The third expression [(5.35)] is for the general case $\Delta K_\alpha \neq 0$ and is the same as the second, except for the replacement of the effective scattering matrix with an exact one. A detailed structure of the related power equations, particularly of those changes caused by the scatterer, is very important to see a precise process of the scattering that takes place through the coherent and incoherent scatterings in a complex way. The power conservation is ensured by two optical relations of coherent and incoherent characteristics, respectively, for two quantities of the scatterer [Eq. (5.44) for ΔK_α and (5.46) for $T^{(\alpha)}$]. In Appendix E, optical expressions of related quantities are derived in some detail with the basic equations, and written in a general form so that they are applicable not only to the present waveguide but also to a wide class of random systems with a fixed scatterer.

Specific expressions of statistical parameters were obtained in Sec. VI to the ladder approximation (2.32b) for the boundaries provided that the medium is free from the fluctuation; this case is particularly illustrative because, in spite of the simple approximation involved, the M and K are strictly consistent with optical relation (2.48) and also because the scattering is intrinsically dispersive [(2.3)].

Finally, it may be remarked that the possible backscattering enhancement by the fixed scatterer is caused by ΔK_α, and the main contribution is made by the term with a diagram similar to the first one of Fig. 2(b) except having one triangle in every solid line. A closed-form equation can be constructed for ΔK_α including all the terms of this class.

APPENDIX A: AN INTEGRAL REPRESENTATION FOR $G(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}')$ IN ASYMPTOTIC RANGE

Let $f(\boldsymbol{\lambda})$ be a smooth and sufficiently slowly changing function of $\boldsymbol{\Omega}$ when changing the variable by $\boldsymbol{\lambda}=\lambda\boldsymbol{\Omega}$, $\boldsymbol{\Omega}^2=1$, with the element $d\boldsymbol{\lambda}=\lambda\, d\lambda\, d\boldsymbol{\Omega}$, in such a way that a dual integral

$$(2\pi)^{-2}\int_0^{\infty}d\lambda\,\lambda\int_{2\pi}d\mathbf{\Omega}\exp(-i\lambda\mathbf{\Omega}\cdot\boldsymbol{\rho})f(\lambda\mathbf{\Omega}) \tag{A1}$$

can be reduced, for $\rho=|\boldsymbol{\rho}|\to\infty$, to the single integral

$$(2\pi)^{-3/2}\rho^{-1/2}\int_0^{\infty}d\lambda\,\lambda^{1/2}\times[\exp(-i\lambda\rho+i\pi/4)f(\lambda\mathbf{\Omega}')+\exp(i\lambda\rho-i\pi/4)f(-\lambda\mathbf{\Omega}')]\ , \tag{A2}$$

where $\mathbf{\Omega}'=\boldsymbol{\rho}/\rho$, after the $\mathbf{\Omega}$ integration with the aid of the saddle point approximation. The above integral can be written by the infinite integral

$$(2\pi)^{-3/2}\rho^{-1/2}\int_{-\infty-i0}^{\infty-i0}d\lambda\,\lambda^{1/2}\times\exp(-i\lambda\rho+i\pi/4)f(\lambda\mathbf{\Omega}')\ , \tag{A3}$$

where $\arg(\lambda)=0$ for $\lambda>0$. In the special case where $f(\lambda\mathbf{\Omega})$ is independent of $\mathbf{\Omega}$, being given by $f(\lambda)$, an exact version of the integral is

$$(4\pi)^{-1}\int_{-\infty-i0}^{\infty-i0}d\lambda\,\lambda H_0^{(2)}(\lambda\rho)f(\lambda)\ , \tag{A4}$$

where $H_0^{(2)}(x)$ is the zeroth-order Hankel function of the second kind.

Hence the asymptotic expression (3.18) is obtained by regarding $f(\lambda)$ as each term of the series in Eq. (3.16). Whereas, when the effective medium M is anisotropic enough so that the $k_a(\mathbf{\Omega})$'s are appreciably dependent on $\mathbf{\Omega}$, the result is not quite right, because of an unnegligible error due to the failure of formula (A3) in the neighborhood of the pole of $f(\lambda\mathbf{\Omega}')$ at $\lambda=k_a(\mathbf{\Omega}')$.

APPENDIX B: ASYMPTOTIC FORM OF INTEGRAND FOR EQS. (3.11) AND (4.6)

To the first order approximation of Eq. (2.24a), $M^{(q)}(\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}')=0$ for $|\boldsymbol{\rho}-\boldsymbol{\rho}'|>l$ where l is the correlation distance of the medium q. Hence the Fourier transform $\widetilde{M}^{(q)}(\boldsymbol{\lambda},a^2)$ defined by Eq. (3.5a) is an entire function of $\boldsymbol{\lambda}$ and a^2 being given by

$$\widetilde{M}^{(q)}(\boldsymbol{\lambda},a^2)=\int_{|\rho|\le l}d\boldsymbol{\rho}\,e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\rho}}M_a^{(q)}(\boldsymbol{\rho})\ , \tag{B1}$$

where

$$M_a^{(q)}(\boldsymbol{\rho})=\int dz\,e^{iaz}M^{(q)}(\boldsymbol{\rho},|z|) \tag{B2}$$

and the eigenvalue a^2 is bounded for $|\boldsymbol{\lambda}|\to\infty$ [Eqs. (3.7)]. Here we set $\boldsymbol{\rho}=\rho\mathbf{\Omega}$, $\mathbf{\Omega}^2=1$, and investigate an asymptotic form of Eq. (B1) for $|\lambda|\to\infty$, which can be obtained with the same procedure as when deriving Eq. (A3), as

$$\widetilde{M}^{(q)}(\boldsymbol{\lambda},a^2)\sim(2\pi)^{1/2}\lambda^{-1/2}\times\int_{-l+i0}^{l+i0}d\rho\,\rho^{1/2}e^{i\lambda\rho-i\pi/4}M_a^{(q)}(\rho\mathbf{\Omega}) \tag{B3}$$

which, by partial integration, gives

$$\widetilde{M}^{(q)}(\boldsymbol{\lambda},a^2)\sim(2\pi l)^{1/2}\lambda^{-1/2}(i\lambda)^{-1}\times[e^{i\lambda l-i\pi/4}M_a^{(q)}(l\mathbf{\Omega})-e^{-i\lambda l+i\pi/4}M_a^{(q)}(-l\mathbf{\Omega})]\ . \tag{B4}$$

Except for a numerical factor,

$$|\widetilde{M}^{(q)}(\boldsymbol{\lambda},a^2)|_{|\lambda|\to\infty}\sim|\lambda|^{-3/2}\exp[+|\mathrm{Im}(\lambda)|\,l]\ . \tag{B5}$$

Thus, from Eqs. (3.8) and (3.9),

$$|D_a(\boldsymbol{\lambda})|_{|\lambda|\to\infty}\sim\begin{cases}|\lambda|^{-3/2}\exp[+|\mathrm{Im}(\lambda)|\,l] & \text{for } |\mathrm{Im}(\lambda)|\sim\infty\\ |\lambda|^{2}\ \text{ for } \mathrm{Im}(\lambda)=0\end{cases} \tag{B6}$$

showing that $\widetilde{G}(z\,|\,\boldsymbol{\lambda}\,|\,z')\to0$ by Eq. (3.13).

To investigate the asymptotic form of the integrand in Eq. (4.6) to the approximation of using Eq. (2.32a), i.e.,

$$K^{(q)}(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2\,|\,\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')=K^{(q)}(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2)\delta(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_1')\delta(\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_2')\ , \tag{B7}$$

$$K^{(q)}(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2)=\langle q(\hat{\mathbf{x}}_1)q(\hat{\mathbf{x}}_2)\rangle\ , \tag{B8}$$

we first observe that, in Eq. (4.4),

$$\widetilde{K}_{ab;cd}(\mathbf{u}\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}')=\widetilde{K}_{ab;cd}(\mathbf{u}-\mathbf{u}')\equiv\int_{|\mathbf{r}|\le l}d\mathbf{r}\,K_{ab;cd}(\mathbf{r})e^{i(\mathbf{u}-\mathbf{u}')\cdot\mathbf{r}} \tag{B9}$$

where

$$K_{ab;cd}(\mathbf{r})=\int dz_1dz_2\bar{\phi}_a^*(z_1)\bar{\phi}_b(z_2)\times K^{(q)}(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2)\phi_c^*(z_1)\phi_d(z_2)\ . \tag{B10}$$

Here, obtaining the asymptotic form of $\widetilde{K}_{ab;cd}(\mathbf{u})$ for $u=|\mathbf{u}|\to\infty$ is straightforward, and the result is the same as given by Eq. (B5) with $\lambda\to u$. Hence, from Eq. (4.7),

$$|f_{ab}(\mathbf{u},\boldsymbol{\lambda})|_{|u|\to\infty}\sim|u|^{-3}\exp[+2\,|\mathrm{Im}(u)|\,l]\ , \tag{B11}$$

which shows that, since $u=(u_1+u_2)/2$ and $\boldsymbol{\lambda}_1=u_1\mathbf{\Omega}-\boldsymbol{\lambda}_T/2$ in Eq. (4.6),

$$|[D_a^*(\boldsymbol{\lambda}_1)]^{-1}f_{ab}(\mathbf{u},\boldsymbol{\lambda})|_{|u_1|\to\infty}\sim\begin{cases}|u_1|^{-3/2}, & |\mathrm{Im}(u_1)|\sim\infty\\ |u_1|^{-5}, & \mathrm{Im}(u_1)=0\ .\end{cases} \tag{B12}$$

Also for $|u_2|\to\infty$, the same asymptotic form is obtained with $D_a^*(\boldsymbol{\lambda}_1)\to D_b(\boldsymbol{\lambda}_2)$ in Eq. (B12). Thus the integrand tends to zero both for $|u_1|\to\infty$ and $|u_2|\to\infty$.

Also it may be remarked that $M^{(q)}$ plays an essential role to determine the asymptotic form (B6) for D_a and, consequently, also (B12); that is, the Maclaurin expansion (4.13) is possible only with the renormalized G and not with the Green function in the ideal waveguide free from the fluctuation. The above conclusion remains unchanged even for the replacement of $q(\hat{\mathbf{x}})$ by $(\partial/\partial\hat{\mathbf{x}})^n q(\hat{\mathbf{x}})(\partial/\partial\hat{\mathbf{x}})^n$, as $b^{(j)}$ given by (2.3), for example; the proof is straightforward with the same procedure. Although we have considered only the medium so far, the situation is also the same for the boundary counterpart. More precisely, $\tilde{G}(z \mid \lambda \mid z) \mid_{z=d_j}$ from (3.13) tends to (6.15) for $\mid \lambda \mid \to \infty$, showing that $\tilde{M}^{(j)}(\lambda)$ is involved in $\tilde{G}$ asymptotically in the same form as $\tilde{M}^{(q)}(\lambda)$ is in $D_a(\lambda)$ through (3.8) and (3.9). Therefore, even when $q=0$ so that $M^{(q)}=0$, an asymptotic form similar to (B12) holds true, tending to zero for either of $\mid u_1 \mid \to \infty$ and $\mid u_2 \mid \to \infty$.

APPENDIX C: EVALUATION OF $\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})$

We first introduce the notations $k_{-a}^{(T)}(\boldsymbol{\Omega}) = -k_a^{(T)}(-\boldsymbol{\Omega})$ and $k_{-a}^{*(T)}(\boldsymbol{\Omega}) = -k_a^{*(T)}(-\boldsymbol{\Omega})$ for those sets of roots of Eq. (4.10) with negative real values, so that

$$k_{-a,-b}^{(T)}(-\boldsymbol{\Omega}) = -k_{ab}^{(T)}(\boldsymbol{\Omega}) ,$$

$$\gamma_{-a,-b}^{(T)}(-\boldsymbol{\Omega}) = -\gamma_{ab}^{(T)}(\boldsymbol{\Omega}) , \tag{C1}$$

$$w_{-a,-b}^{(T)}(-\boldsymbol{\Omega}) = -w_{ab}^{(T)}(\boldsymbol{\Omega}) ,$$

$$f_{-a,-b}^{(\lambda)}(-\boldsymbol{\Omega}) = +f_{ab}^{(\lambda)}(\boldsymbol{\Omega}) \tag{C2}$$

[Eqs. (4.14) and (4.15)]. To evaluate the integral (4.16), we change the path of integration to take it along the positive imaginary axis; hence,

$$\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda}) = \frac{1}{2\pi}\int_0^{i\infty} du + \begin{cases} w_{ab}^{(T)}(\gamma_{ab}^{(T)} - i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})^{-1} & \text{for } +a \\ 0 \quad \text{for } -a , \end{cases} \tag{C3}$$

in terms of the reside value at the pole $u_1 = k_{+a}^{*(T)}$ existing in the upper half-plane of u. Here $\gamma_{ab}^{(T)}$ is defined by Eq. (4.19) and the symbols $\pm a$ represent the cases of the poles $u_1 = k_{\pm a}^{*(T)}(\boldsymbol{\Omega})$, respectively.

The integral in Eq. (C3) is elementary and is given, when the members of the mode waves are either $(+a,+b)$ or $(-a,-b)$, by

$$(2\pi)^{-1}\int_0^{i\infty} du = (2\pi i)^{-1} w_{ab}^{(T)}(\gamma_{ab}^{(T)} - i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})^{-1} \times \ln\left[\frac{(k_b^{(T)} - \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2)}{(k_a^{*(T)} + \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2)}\right] \tag{C4}$$

(where ln takes the principal values), which remains unchanged against the replacement of $a \to -a$, $b \to -b$, and $\boldsymbol{\Omega} \to -\boldsymbol{\Omega}$; while, for the other combinations of the mode waves, the ln factor is replaced by

$$\ln \mid_{+a,-b} = -\pi i + \ln\left[\frac{k_b^{(T)}(-\boldsymbol{\Omega}) + \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2}{k_a^{*(T)}(\boldsymbol{\Omega}) + \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2}\right] , \tag{C5}$$

$$\ln \mid_{-a,+b} = +\pi i + \ln\left[\frac{k_b^{(T)}(\boldsymbol{\Omega}) - \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2}{k_a^{*(T)}(-\boldsymbol{\Omega}) - \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2}\right] , \tag{C6}$$

wherein the ln terms are interchanged for $\boldsymbol{\Omega} \to -\boldsymbol{\Omega}$.

Hence since both (C3) and (C4) are invariant against the simultaneous replacement of $a \to -a$, $b \to -b$, and $\boldsymbol{\Omega} \to -\boldsymbol{\Omega}$, the sum of the series (4.13) over the mode waves of members $(+a,+b)$ and $(-a,-b)$ can be obtained by using an effective $\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})$ of Eq. (C3) given, on doubling the integral part (C4), by

$$\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda}) = w_{ab}^{(e)}(\boldsymbol{\Omega},\boldsymbol{\lambda})[\gamma_{ab}^{(T)}(\boldsymbol{\Omega}) - i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}]^{-1} \tag{C7}$$

with the elements $w_{+a,+b}^{(e)}$ of $w_{ab}^{(e)}$

$$w_{+a,+b}^{(e)} = w_{+a,+b}^{(T)}\left[1 + \frac{1}{\pi i}\ln\left[\frac{k_b^{(T)}(\boldsymbol{\Omega}) - \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2}{k_a^{*(T)}(\boldsymbol{\Omega}) + \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2}\right]\right] , \tag{C8}$$

in view of the following $\boldsymbol{\Omega}$ integration.

For the sum of the series over the mode members $(+a,-b)$ and $(-a,+b)$, on the other hand, we observe that the ln factor in Eq. (C4) can be replaced, effectively, by two of the ln terms in Eq. (C5), because of being equal to the sum of (C5) and (C6) with $\boldsymbol{\Omega} \to -\boldsymbol{\Omega}$ in the latter; that is, Eq. (C7) holds true by defining the elements $w_{+a,-b}^{(e)}$, according to

$$w_{+a,-b}^{(e)} = w_{+a,-b}^{(T)}\left[1 + \frac{1}{\pi i}\ln\left[\frac{k_b^{(T)}(-\boldsymbol{\Omega}) + \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2}{k_a^{*(T)}(\boldsymbol{\Omega}) + \boldsymbol{\Omega}\cdot\boldsymbol{\lambda}/2}\right]\right] , \tag{C9}$$

which still has a form similar to Eq. (C8). Thus the result is finally given by the series (4.17) with a summation restricted to the mode waves $(+a,\pm b)$.

Equation (C9) first suggests to us that the terms of $(+a,-b)$ are caused by an interference of the mode waves propagating in opposite directions. However, it may be remarked that this happens only in the case of two-dimensional propagation, and happens neither in one-dimensional nor in three-dimensional propagation. A mathematical aspect of this fact in the three-dimensional case is as follows: The original series (4.13) formally remains unchanged (with the understanding of $\boldsymbol{\Omega}$ and $\boldsymbol{\lambda}$ as the corresponding variables in three-dimensional space), and a change is caused only through the factor $w_{ab}^{(T)}$, which is still given by Eq. (4.14a) except the replacement of $k_{ab}^{(T)} \to (k_{ab}^{(T)})^2$. However, this leads to the relation

$$w_{-a,-b}^{(T)}(-\boldsymbol{\Omega}) = +w_{ab}^{(T)}(+\boldsymbol{\Omega}) ,$$

as contrasted to that in Eq. (C2), resulting in several important changes, as follows. From Eq. (C4), it follows that

$$\left[\sum_{+a,+b,+\boldsymbol{\Omega}} + \sum_{-a,-b,-\boldsymbol{\Omega}}\right]\int_0^{i\infty} du = 0$$

and using Eqs. (C5) and (C6), also that, effectively,

$$\left[\sum_{+a,-b,+\boldsymbol{\Omega}} + \sum_{-a,+b,-\boldsymbol{\Omega}}\right]\frac{1}{2\pi}\int_0^{i\infty} du = -\sum_{+a,-b} w_{ab}^{(T)}\left[\gamma_{ab}^{(T)} - i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}\right]^{-1} ,$$

which perfectly cancels those from the second term in Eq. (C3) for the mode waves $(+a,-b)$. Thus we find the simple result that the series (4.17) still holds true with a new $\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})$, defined by Eq. (C7) with the replacement of $w_{ab}^{(e)} \to w_{ab}^{(T)}$, and also with a new summation $\sum_{a,b}$ extended only over the mode waves $(+a,+b)$ (without any interference term). Also for the case of one-dimensional propagation, the situation is exactly the same with $k_{ab}^{(T)} \to 1$ in Eq. (4.14a).

To estimate $\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})$ for the propagative mode waves in the present case of two-dimensional propagation, we use $k_a^{*(T)} = k_{ab}^{(T)} + i\gamma_{ab}^{(T)}$ and a similar expression for $k_b^{(T)}$ in Eq. (C8); hence,

$$w_{+a,+b}^{(e)} \simeq w_{ab}^{(T)}[1-(\pi k_{ab}^{(T)})^{-1}(\gamma_{ab}^{(T)} - i\mathbf{\Omega}\cdot\boldsymbol{\lambda})] \tag{C10}$$

(where $|\gamma_{ab}^{(T)}| \ll |k_{ab}^{(T)}| \simeq |k_{ab}|$ and $w_{ab}^{(T)} \simeq w_{ab}$), showing that

$$\tilde{U}_{+a,+b}(\mathbf{\Omega},\boldsymbol{\lambda}) \simeq w_{ab}[(\gamma_{ab}^{(T)} - i\mathbf{\Omega}\cdot\boldsymbol{\lambda})^{-1} - (\pi k_{ab})^{-1}] \ . \tag{C11}$$

In a similar fashion we find for the interfering mode waves that

$$\tilde{U}_{+a,-b}(\mathbf{\Omega},\boldsymbol{\lambda}) \simeq w_{ab}(\gamma_{ab})^{-1}[1+(4/\pi)(k_{ab}/\gamma_{ab})] \ , \tag{C12}$$

where

$$|k_{+a,-b}| \ll |\gamma_{+a,-b}| \sim |k_a^*|, |k_b| \quad (\gg |\boldsymbol{\lambda}|) \ ,$$

whereas for the nonpropagative mode waves with $|k_{ab}| \ll |\gamma_{ab}|$,

$$\tilde{U}_{+a,\pm b}(\mathbf{\Omega},\boldsymbol{\lambda}) \simeq (i\pi)^{-1} w_{ab}(\gamma_{ab})^{-1} \ln|k_b/k_a^*| \ , \tag{C13}$$

which is close to zero whenever $|k_b/k_a^*| \sim 1$.

In the range of $|\boldsymbol{\lambda}| \lesssim \gamma_{aa} \ll |k_a|$ where the spatial change of the average intensity is so slow that the change becomes appreciable only after the propagation over a distance of the order of the coherence distances of the mode waves, $\tilde{\Omega}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda})$ can actually be given by Eq. (C11), on neglect of the second term in the bracket and also the other terms from (C12) and (C13).

APPENDIX D: DETAILS OF OPTICAL RELATIONS

The basic relation is given by Eq. (2.47), which shows a local relation inherent between the fundamental matrices M and K in the BS equation (2.29) and ensures power conservation of the entire system at every point in the waveguide and on the boundaries.

1. Local optical relations

We begin with the second term on the right-hand side, i.e.,

$$(2i)^{-1}\delta(\hat{\mathbf{x}}\,|\,1;2)[G^*(1)-G(2)]K(1;2) \ , \tag{D1}$$

whose Fourier transform with respect to the horizontal coordinates can be expanded in an eigenfunction series of the form (4.6); where the factor $f_{ab}(\mathbf{u},\boldsymbol{\lambda})$ is presently [Eq. (3.11)] (see also Refs. 17 and 16)

$$f_{ab}(\mathbf{u},\boldsymbol{\lambda}) = (2i)^{-1}[D_b(\boldsymbol{\lambda}+\mathbf{u}/2) - D_a^*(\boldsymbol{\lambda}-\mathbf{u}/2)] \times \phi_a^*(\boldsymbol{\lambda}_1,z)\phi_b(\boldsymbol{\lambda}_2,z)\tilde{K}_{ab;cd}(\mathbf{u}\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}') \ . \tag{D2}$$

The result can be given by Eq. (4.17) with the $f_{ab}^{(\lambda)}(\mathbf{\Omega})$ of Eq. (4.24), in which

$$f_{ab}(\mathbf{\Omega}) \propto D_b(k_b^{(T)}\mathbf{\Omega}+\boldsymbol{\lambda}_T/2) - D_a^*(k_a^{*(T)}\mathbf{\Omega}-\boldsymbol{\lambda}_T/2) = 0$$

from Eq. (4.10) and, using Eqs. (3.17) in (4.26),

$$f'_{ab}(\mathbf{\Omega}) = -2iN_{ab}^{(T)}(\mathbf{\Omega},z)k_{ab}^{(e)}\tilde{K}_{ab;cd}(k_{ab}^{(T)}\mathbf{\Omega}\,|\,\boldsymbol{\lambda}\,|\,\mathbf{u}') \ . \tag{D3}$$

Here

$$k_{ab}^{(e)} = 2^{-1}[k_a^{*(T)}(1-\kappa_a^{*\prime(T)}) + k_b^{(T)}(1-\kappa_b^{\prime(T)})] \ , \tag{D4}$$

$$N_{ab}^{(T)}(\mathbf{\Omega},z) = \phi_a^*(\boldsymbol{\lambda}_1,z)\phi_b(\boldsymbol{\lambda}_2,z)\,|_{a,b}^{(T)} \ , \tag{D5}$$

where

$$\begin{aligned} \boldsymbol{\lambda}_1\,|_{a,b}^{(T)} &= k_a^{*(T)}\mathbf{\Omega} - \boldsymbol{\lambda}_T/2 \ , \\ \boldsymbol{\lambda}_2\,|_{a,b}^{(T)} &= k_b^{(T)}\mathbf{\Omega} + \boldsymbol{\lambda}_T/2 \ . \end{aligned} \tag{D6}$$

The substitution into (4.17) through (4.24) finally yields a mode transform of (D1), after setting $\mathbf{u}' = k_{cd}^{(T)}\mathbf{\Omega}'$, by

$$\sum_{a,b} N_{ab}^{(T)}(\mathbf{\Omega},z)k_{ab}^{(e)}w_{ab}^{(e)}K_{ab;cd}^{(\lambda)}(\mathbf{\Omega}\,|\,\mathbf{\Omega}') \ , \tag{D7}$$

in terms of the notation $K_{ab;cd}^{(\lambda)}(\mathbf{\Omega}\,|\,\mathbf{\Omega}')$ defined by Eq. (4.28).

The corresponding transform of the remaining terms of Eq. (2.47), i.e.,

$$\frac{\partial}{\partial\boldsymbol{\rho}}\cdot\boldsymbol{\beta}(\hat{\mathbf{x}}\,|\,1;2) - \delta(\hat{\mathbf{x}}\,|\,1;2)(2i)^{-1}[M^*(1)-M(2)] \ , \tag{D8}$$

also is obtained in the same way, by the multiplication to the right with $U^{(C)}(1;2)$ and the subsequent mode transformation to write it in the form (4.17). Here the result can be given in a simple form by writing

$$\boldsymbol{\beta}(\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}'_1;\hat{\mathbf{x}}'_2) = \sum_{j=1}^{2}\delta_j(z)\boldsymbol{\beta}^{(j)}(\boldsymbol{\rho}-\boldsymbol{\rho}'\,|\,\mathbf{r}',z'_1,z'_2) \tag{D9}$$

with the mode transform $\tilde{\boldsymbol{\beta}}_{cd}^{(j)}(\mathbf{u},\boldsymbol{\lambda})$ defined by

$$\tilde{\boldsymbol{\beta}}_{cd}^{(j)}(\mathbf{u},\boldsymbol{\lambda}) = \int d\boldsymbol{\rho}' e^{i\boldsymbol{\lambda}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} \int d\mathbf{r}' e^{-i\mathbf{u}\cdot\mathbf{r}'} \int dz'_1 dz'_2 \boldsymbol{\beta}^{(j)}(\boldsymbol{\rho}-\boldsymbol{\rho}'\,|\,\mathbf{r}',z'_1,z'_2)\phi_c^*(\boldsymbol{\lambda}_1,z'_1)\phi_d(\boldsymbol{\lambda}_2,z'_2) \ . \tag{D10}$$

Thus we obtain the normal-mode series for the terms (D8) as

$$\sum_{c,d}\int d\mathbf{\Omega}[-i\boldsymbol{\lambda}\cdot\boldsymbol{\beta}_{cd}^{(\lambda)}(\mathbf{\Omega},z) - k_{cd}^{(T)}\gamma_{cd}^{(a+b,\lambda)}(\mathbf{\Omega},z)N_{cd}^{(T)}(\mathbf{\Omega},z)]\tilde{U}_{cd}(\mathbf{\Omega},\boldsymbol{\lambda}) \ , \tag{D11}$$

in terms of the notations

$$\boldsymbol{\beta}_{cd}^{(\lambda)}(\mathbf{\Omega},z) = \sum_{j=1}^{2}\delta_j(z)\boldsymbol{\beta}_{cd}^{(j,\lambda)}(\mathbf{\Omega}) \ , \tag{D12}$$

$$\boldsymbol{\beta}_{cd}^{(j,\lambda)}(\mathbf{\Omega}) = \tilde{\boldsymbol{\beta}}_{cd}^{(j)}(k_{cd}^{(T)}\mathbf{\Omega},\boldsymbol{\lambda}) \ , \tag{D13}$$

and

$$\gamma_{cd}^{(q+b,\lambda)}(\mathbf{\Omega},z)=(2ik_{cd}^{(T)})^{-1}[\tilde{M}^*(\mathbf{u}-\boldsymbol{\lambda}/2)-\tilde{M}(\mathbf{u}+\boldsymbol{\lambda}/2)] \tag{D14}$$

with $\mathbf{u}=k_{cd}^{(T)}\mathbf{\Omega}$, where both $\tilde{M}^*$ and $\tilde{M}$ are generally z matrices to be multiplied with the z vectors $\phi_c^*(\boldsymbol{\lambda}_1,z_1)$ and $\phi_d(\boldsymbol{\lambda}_2,z_2)$, followed by setting $z_1=z_2=z$ [cf. Eqs. (3.12) and (3.31)]. We note that even when $\boldsymbol{\lambda}=0$, $\gamma_{ab}^{(q+b,\lambda)}$ are not exactly the same as the $\gamma_{ab}^{(q+b)}$ defined according to (3.19c) and (3.19d).

We finally obtain, as a normal mode transformed version of Eq. (2.47), the relation

$$-i\boldsymbol{\lambda}\cdot\boldsymbol{\beta}_{cd}^{(\lambda)}(\mathbf{\Omega},z)=k_{cd}^{(T)}\gamma_{cd}^{(q+b,\lambda)}(\mathbf{\Omega},z)N_{cd}^{(T)}(\mathbf{\Omega},z)-\sum_{a,b}\int d\mathbf{\Omega}'N_{ab}^{(T)}(\mathbf{\Omega}',z)k_{ab}^{(e)}w_{ab}^{(e)}K_{ab;cd}^{(\lambda)}(\mathbf{\Omega}'\mid\mathbf{\Omega})\ , \tag{D15}$$

which gives a local optical relation, in the sense of depending on both the coordinate z and the Fourier variable $\boldsymbol{\lambda}$, and ensures power conservation at every point in the waveguide and on the boundaries.

Equation (D15) can be rewritten in terms of more accessible physical parameters by using a relation corresponding to Eq. (3.26), which is presently given, as is proved later, by

$$k_{ab}^{(T)}\left[[-i\mathbf{\Omega}\cdot\boldsymbol{\lambda}+\gamma_{ab}^{(q+b,\lambda)}(\mathbf{\Omega},z)]N_{ab}^{(T)}(\mathbf{\Omega},z)+\frac{\partial}{\partial z}N_{ab}^{(3,T)}(\mathbf{\Omega},z)\right]=(\gamma_{ab}^{(T)}-i\mathbf{\Omega}\cdot\boldsymbol{\lambda})k_{ab}^{(e)}N_{ab}^{(T)}(\mathbf{\Omega},z)\ , \tag{D16}$$

in terms of $N_{ab}^{(T)}(\mathbf{\Omega},z)$ and $k_{ab}^{(e)}$ defined by (D4). The superscript (λ) is to be attached for every quantity after the setting (4.23).

By using (D16) to delete the term of $\gamma_{cd}^{(q+b,\lambda)}$ in (D15), we obtain another expression of the local relation, as

$$\begin{aligned}-i\boldsymbol{\lambda}\cdot[\boldsymbol{\beta}_{cd}^{(\lambda)}(\mathbf{\Omega},z)+\mathbf{\Omega}k_{cd}^{(T)}N_{cd}^{(T)}(\mathbf{\Omega},z)]+\frac{\partial}{\partial z}[k_{cd}^{(T)}N_{cd}^{(3,T)}(\Omega,z)]\\=[\gamma_{cd}^{(T)}(\mathbf{\Omega})-i\mathbf{\Omega}\cdot\boldsymbol{\lambda}]k_{cd}^{(e)}N_{cd}^{(T)}(\mathbf{\Omega},z)-\sum_{a,b}\int d\mathbf{\Omega}'N_{ab}^{(T)}(\mathbf{\Omega}',z)k_{ab}^{(e)}w_{ab}^{(e)}K_{ab;cd}^{(\lambda)}(\mathbf{\Omega}'\mid\mathbf{\Omega})\ ,\end{aligned} \tag{D17}$$

written in terms of $\partial N_{cd}^{(3,T)}(\mathbf{\Omega},z)/\partial z$.

2. Local power conservation

It is straightforward to find the mode expression of power equation (2.51) for the coherent wave, with the same procedure of using formula (4.17). Hence, by using definition (D14) for $\gamma_{ab}^{(q+b,\lambda)}$, we obtain

$$\begin{aligned}\sum_{a,b}\int d\mathbf{\Omega}\,k_{ab}^{(T)}\left[[-i\boldsymbol{\lambda}\cdot\mathbf{\Omega}+\gamma_{ab}^{(q+b,\lambda)}(\mathbf{\Omega},z)]N_{ab}^{(T)}(\mathbf{\Omega},z)+\frac{\partial}{\partial z}N_{ab}^{(3,T)}(\mathbf{\Omega},z)\right]\tilde{U}_{ab}(\mathbf{\Omega},\boldsymbol{\lambda})\bar{N}_{ab}^{(T)}(\mathbf{\Omega},z')\\=\sum_{a,b}\int d\mathbf{\Omega}\,k_{ab}^{(e)}w_{ab}^{(e)}N_{ab}^{(T)}(\mathbf{\Omega},z)\bar{N}_{ab}^{(T)}(\mathbf{\Omega},z')\ ,\end{aligned} \tag{D18}$$

where the right-hand side is the expression for $\delta(\hat{\mathbf{x}}\mid 1;2)(2i)^{-1}[G^*(1)-G(2)]$ and is the same as (D7) except the replacement of $K_{ab;cd}^{(\lambda)}(\mathbf{\Omega}\mid\mathbf{\Omega}')$ by $\bar{N}_{ab}^{(T)}(\mathbf{\Omega},z')$, and relation (D16) is derived therefrom, by using (4.18).

The consistency of the equation of radiative transfer (4.35) with power conservation is shown by multiplying both sides of its Fourier transform, i.e.,

$$[\gamma_{ab}-i\boldsymbol{\lambda}\cdot\mathbf{\Omega}_{ab}]\tilde{I}_{ab;cd}(\mathbf{\Omega}\mid\boldsymbol{\lambda}\mid\mathbf{\Omega}')=w_{ab}^{(e)}\left[\delta_{ac}\delta_{bd}\delta(\mathbf{\Omega}-\mathbf{\Omega}')+\sum_{i,j}\int d\mathbf{\Omega}''K_{ab;ij}^{(\lambda)}(\mathbf{\Omega}\mid\mathbf{\Omega}'')\tilde{I}_{ij;cd}(\mathbf{\Omega}''\mid\boldsymbol{\lambda}\mid\mathbf{\Omega}')\right]\ , \tag{D19}$$

with $\sum_{ab}\int d\mathbf{\Omega}\,k_{ab}^{(e)}N_{ab}^{(T)}(\mathbf{\Omega},z)$ and the following use of the local relation (D17); hence,

$$\sum_{a,b}\int d\mathbf{\Omega}\left[-i\boldsymbol{\lambda}\cdot[\boldsymbol{\beta}_{ab}^{(\lambda)}(\mathbf{\Omega},z)+\mathbf{\Omega}k_{ab}^{(T)}N_{ab}^{(T)}(\mathbf{\Omega},z)]+\frac{\partial}{\partial z}[k_{ab}^{(T)}N_{ab}^{(3,T)}(\mathbf{\Omega},z)]\right]\tilde{I}_{ab;cd}(\mathbf{\Omega}\mid\boldsymbol{\lambda}\mid\mathbf{\Omega}')=k_{cd}^{(e)}w_{cd}^{(e)}N_{cd}^{(T)}(\mathbf{\Omega}',z)\ , \tag{D20}$$

which, by the further multiplication with $\sum_{c,d}\int d\mathbf{\Omega}'\bar{N}_{cd}^{(T)}(\mathbf{\Omega}',z')$, precisely reproduces Eq. (2.55) in terms of the mode functions in view of the right-hand side representing $(2i)^{-1}\delta(\hat{\mathbf{x}}\mid 1;2)[G^*(1)-G(2)]$ [cf. Eq. (D18)].

3. Integrated optical relation and an approximate $\boldsymbol{\beta}_{ab}(\mathbf{\Omega})$

By the z integration of the local relation (D17) over $\infty\geq z\geq-\infty$, an integrated relation is obtained to ensure only the integrated power over the waveguide height, and is given in terms of

$$\Delta_{cd}^{(T)}=\int dz\,N_{ab}^{(T)}(\mathbf{\Omega},z)\ , \tag{D21}$$

$$\boldsymbol{\beta}_{cd}^{(\lambda)}(\mathbf{\Omega})=\int dz\,\boldsymbol{\beta}_{cd}^{(\lambda)}(\mathbf{\Omega},z)=\sum_{j=1}^{2}\boldsymbol{\beta}_{cd}^{(j,\lambda)}(\mathbf{\Omega})+\boldsymbol{\beta}_{cd}^{(q,\lambda)}(\mathbf{\Omega}) \tag{D22}$$

(where $\boldsymbol{\beta}_{cd}^{(q,\lambda)}=0$ in the present case of an originally nondispersive medium), and $\mathbf{\Omega}_{ab}$ [Eq. (4.19)] by

$$-i\boldsymbol{\lambda}\cdot[\boldsymbol{\beta}_{cd}^{(\lambda)}(\boldsymbol{\Omega})+\boldsymbol{\Omega}k_{cd}^{(T)}\Delta_{cd}^{(T)}]$$
$$=[\gamma_{cd}-i\boldsymbol{\Omega}_{cd}\cdot\boldsymbol{\lambda}]k_{cd}^{(e)}\Delta_{cd}^{(T)}$$
$$-\sum_{a,b}\int d\boldsymbol{\Omega}'\Delta_{ab}^{(T)}k_{ab}^{(e)}w_{ab}^{(e)}K_{ab;cd}^{(\lambda)}(\boldsymbol{\Omega}'\mid\boldsymbol{\Omega}) , \quad (D23)$$

which is reduced, when $\boldsymbol{\lambda}=0$, to

$$\gamma_{cd}(\boldsymbol{\Omega})k_{cd}^{(e)}\Delta_{cd}=\sum_{a,b}\int d\boldsymbol{\Omega}'\Delta_{ab}k_{ab}^{(e)}w_{ab}^{(e)}K_{ab;cd}(\boldsymbol{\Omega}'\mid\boldsymbol{\Omega}) , \quad (D24)$$

where $\Delta_{cd}\sim\delta_{cd}$ [Eq. (3.22)].

As long as the $\boldsymbol{\lambda}$ dependence is negligible for both the incoherent term [given by the last integral in (D23)] and the $k_{cd}^{(e)}$ (because of being even functions of $\boldsymbol{\lambda}$ as realized when the system is isotropic in the horizontal direction; see also Sec. VI), $\boldsymbol{\beta}_{cd}^{(\lambda)}(\boldsymbol{\Omega})$ for $\boldsymbol{\lambda}=0$, i.e., $\boldsymbol{\beta}_{cd}(\boldsymbol{\Omega})$, is given by

$$\boldsymbol{\beta}_{cd}(\boldsymbol{\Omega})\simeq\Delta_{cd}(\boldsymbol{\Omega}_{cd}k_{cd}^{(e)}-\boldsymbol{\Omega}k_{cd}) \quad (D25)$$
$$=\Delta_{cd}[(\boldsymbol{\Omega}_{cd}-\boldsymbol{\Omega})k_{cd}-\tfrac{1}{2}\boldsymbol{\Omega}_{cd}(k_c^*\kappa_c^{*\prime}+k_d\kappa_d')] , \quad (D26)$$

which means a correction of both the $\boldsymbol{\Omega}$ and the k_{cd} to meet with the dispersive characteristic of the system so that the total (integrated) power flux is given by Eq. (4.39). These equations hold true even in the more general case of when the original medium is intrinsically dispersive [Eq. (2.14c)].

A correction term to $\boldsymbol{\beta}_{cd}(\boldsymbol{\Omega})$ is given from (D23) by

$$i\frac{\partial}{\partial\boldsymbol{\lambda}}\left\{\Delta_{cd}^{(T)}k_{cd}^{(e)}\gamma_{cd}(\boldsymbol{\Omega})-\sum_{a,b}\int d\boldsymbol{\Omega}'\Delta_{ab}^{(T)}k_{ab}^{(e)}w_{ab}^{(e)}K_{ab;cd}^{(\lambda)}(\boldsymbol{\Omega}'\mid\boldsymbol{\Omega})\right\}\Bigg|_{\lambda=0} \quad (D27)$$

which is zero whenever the factor in large parentheses is an even function of $\boldsymbol{\lambda}$.

4. Optical relation for each of the boundaries

Considered so far are optical relations for the entire system, whereas there exist optical relations also for each of the medium and the boundaries [Eq. (2.50)], which hold true independently of those of the other members which may even be dissipative. By applying exactly the same procedure as when deriving Eq. (D15) to each of the boundaries, we obtain a local relation of the same form and, by the z integration of the result, we also obtain an integrated relation of each, as

$$-i\boldsymbol{\lambda}\cdot\boldsymbol{\beta}_{cd}^{(j,\lambda)}(\boldsymbol{\Omega})=k_{cd}^{(T)}\gamma_{cd}^{(j,\lambda)}(\boldsymbol{\Omega})N_{cd}^{(T)}(\boldsymbol{\Omega},d_j)-\sum_{a,b}\int d\boldsymbol{\Omega}'\Delta_{ab}^{(T)}k_{ab}^{(e)}w_{ab}^{(e)}K_{ab;cd}^{(j,\lambda)}(\boldsymbol{\Omega}'\mid\boldsymbol{\Omega}) ,$$
$$j=1,2 . \quad (D28)$$

Here $\boldsymbol{\beta}_{cd}^{(j,\lambda)}$ is defined by Eqs. (D13) and $\gamma_{cd}^{(j,\lambda)}$ is by Eq. (D14) for $\gamma_{ab}^{(q+b,\lambda)}$ with $q+b\to j$ and $\tilde{M}\to\tilde{M}^{(j)}$, and

$$K_{ab;cd}^{(\lambda)}(\boldsymbol{\Omega}\mid\boldsymbol{\Omega}')=K_{ab;cd}^{(q,\lambda)}(\boldsymbol{\Omega}\mid\boldsymbol{\Omega}')+\sum_{j=1}^{2}K_{ab;cd}^{(j,\lambda)}(\boldsymbol{\Omega}\mid\boldsymbol{\Omega}') \quad (D29)$$

with

$$K_{ab;cd}^{(j,\lambda)}(\boldsymbol{\Omega}\mid\boldsymbol{\Omega}')=\bar{N}_{ab}^{(T)}(\boldsymbol{\Omega},d_j)\tilde{K}^{(j)}(k_{ab}^{(T)}\boldsymbol{\Omega}\mid\boldsymbol{\lambda}\mid k_{cd}^{(T)}\boldsymbol{\Omega}')\times N_{cd}^{(T)}(\boldsymbol{\Omega}',d_j) . \quad (D30)$$

In the same way, a medium counterpart of optical relation similar to Eq. (D28) can be obtained and the sum of the two relations reproduces Eq. (D23), as may be directly shown by using the relation

$$k_{ab}^{(T)}\left[[-i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}+\gamma_{ab}^{(q,\lambda)}]\Delta_{ab}^{(T)}(\boldsymbol{\Omega})+\sum_{j=1}^{2}\gamma_{ab}^{(j,\lambda)}N_{ab}^{(T)}(\boldsymbol{\Omega},d_j)\right]$$
$$=[\gamma_{ab}^{(T)}-i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}]k_{ab}^{(e)}\Delta_{ab}^{(T)}(\boldsymbol{\Omega}) , \quad (D31)$$

which is the z-integrated version of Eqs. (D16), and is reduced to Eq. (3.25) when $i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}=\gamma_{ab}^{(T)}$ and $\boldsymbol{\lambda}_T=0$.

Equation (D28) provides $\boldsymbol{\beta}_{cd}^{(j,\lambda)}(\boldsymbol{\Omega})$ directly in terms of $\tilde{M}^{(j)}(\boldsymbol{\lambda})$, when neglecting the contribution from the incoherent term given by the last integral; that is, $\boldsymbol{\beta}_{cd}^{(j,\lambda)}\simeq\boldsymbol{\beta}_{cd}^{(j)}$ where

$$\boldsymbol{\beta}_{cd}^{(j)}(\boldsymbol{\Omega})\simeq i\frac{\partial}{\partial\boldsymbol{\lambda}}[k_{cd}^{(T)}\gamma_{cd}^{(j,\lambda)}(\boldsymbol{\Omega})N_{cd}^{(T)}(\boldsymbol{\Omega},d_j)]\,|_{\lambda=0} , \quad (D32)$$

which can be evaluated by using an expression similar to Eq. (D14). The same is also true for the medium counterpart $\boldsymbol{\beta}_{cd}^{(q,\lambda)}$, which should be zero, however, whenever the medium is originally nondispersive.

APPENDIX E: DETAILS OF OPTICAL EQUATIONS FOR A FIXED SCATTERER

A direct proof of the (local) relation (5.46) for $T^{(\alpha)}$ can be given from its original definition (5.8). By substitution of the expressions

$$q'_\alpha=T^{(\alpha)}[1+GT^{(\alpha)}]^{-1} ,$$
$$q'^{*}_\alpha=T^{(\alpha)*}[1+G^{*}T^{(\alpha)*}]^{-1} \quad (E1)$$

into the third equality of Eq. (5.43), we directly find a relation

$$\delta(\hat{\mathbf{x}}\mid 1;2)\Delta\gamma_\alpha^{(T)}(1;2)G^{(\alpha)*}(1)G^{(\alpha)}(2)$$
$$=(2i)^{-1}\delta(\hat{\mathbf{x}}\mid 1;2)\times\{T^{(\alpha)*}(1)[1+G(2)T^{(\alpha)}(2)]-[1+G^{*}(1)T^{(\alpha)*}(1)]T^{(\alpha)}(2)\}G^{*}(1)G(2) \quad (E2)$$

by using Eq. (5.7), and the relation (5.46) is reproduced therefrom.

1. Direct proof of the power equation (5.53)

The power equation can be derived as a direct consequence of local optical relation (5.44) for the scatterer, and this is demonstrated below to the approximation of $\Delta K_\alpha=0$ where I_α is given by Eq. (5.20). Using Eq. (5.51) with (2.55),

$$\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot\Delta\hat{\mathbf{W}}_\alpha(\hat{\mathbf{x}}\mid 1;2)=\delta(\hat{\mathbf{x}}\mid 1;2)(2i)^{-1}[G^{*}(1)-G(2)]\times V^{(\alpha,K)}(1;2)I(1;2) . \quad (E3)$$

Here it holds the relation

$$V^{(\alpha,K)}I = V^{(\alpha)}U^{(C)}(1+KI_\alpha)\ , \tag{E4}$$

which is proved, with the aid of the relation

$$V^{(\alpha,K)}U^{(C)}S = V^{(\alpha)}U^{(C)}S_\alpha \tag{E5}$$

[which is evident by comparison of Eq. (5.16) with (5.18)], by substitution of expression (5.21) for S_α into the right-hand side and expression (2.37) for $U^{(C)}S$ into the left-hand side, and followed dropping of the common factor K from both sides.

Hence, using the expression (E4) in Eq. (E3) with the relation

$$\begin{aligned}\delta(\hat{\mathbf{x}}\mid 1;2)(2i)^{-1}&[G^*(1)-G(2)]V^{(\alpha)}(1;2)U^{(C)}(1;2)\\ &=\delta(\hat{\mathbf{x}}\mid 1;2)(2i)^{-1}[G^*T^{(\alpha)*}G^*(1)-GT^{(\alpha)}G(2)]\\ &\quad -\Delta\gamma_\alpha^{(T)}(\hat{\mathbf{x}}\mid 1;2)U^{(C,\alpha)}(1;2)\end{aligned} \tag{E6}$$

[which is derived straightforwardly by use of relation (E2) for $T^{(\alpha)}$], we find that

$$\begin{aligned}\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot\Delta\hat{\mathbf{W}}_\alpha(\hat{\mathbf{x}}\mid 1;2)&=(2i)^{-1}\delta(\hat{\mathbf{x}}\mid 1;2)\\ &\quad\times[G^*T^{(\alpha)*}G^*(1)-GT^{(\alpha)}G(2)]\\ &\quad+(\Delta\gamma_\alpha-\Delta\gamma_\alpha^{(T)})(\hat{\mathbf{x}}\mid 1;2)I_\alpha(1;2)\ ,\end{aligned} \tag{E7}$$

by virtue of the optical relation (5.44) for $\Delta\gamma_\alpha$ ($\Delta K_\alpha=0$), and, therefore, the power equation (5.53) is reproduced in view of (5.43).

Also, Eq. (E6) directly reproduces the coherent power equation (5.45) by rewritting the left-hand side by use of Eq. (2.51). Here the right-hand side has the same form as that of Eq. (E7) except the dissipation factor $\Delta\gamma_\alpha^{(T)}$, which is replaced by $\Delta\gamma_\alpha^{(T)}-\Delta\gamma_\alpha=\gamma_\alpha^{(a)}$ in the latter, meaning that an incoherent wave is scattered by the deterministic scatterer at the additional dissipation of the incident coherent wave by the factor $\Delta\gamma_\alpha$.

2. Cross section $V^{(\alpha)}_{ab;cd}(\mathbf{\Omega}'\mid\mathbf{\Omega}'')$

In expression (5.20) for I_α with $V^{(\alpha,K)}$ of Eq. (5.19), the coordinate matrix $V^{(\alpha)}$ is involved only through the combination

$$\begin{aligned}U^{(C)}V^{(\alpha)}U^{(C)}(1;2)&=G^*(1)G(2)[T^{(\alpha)*}(1)T^{(\alpha)}(2)G^*(1)G(2)\\ &\quad+T^{(\alpha)*}(1)G^*(1)+T^{(\alpha)}(2)G(2)]\ ,\end{aligned} \tag{E8}$$

with the factor $K(1;2)$ on one or both sides. The cross section (5.25a) of $V^{(\alpha)}$ is found from an optical expression of Eq. (E8) which is obtained according to the procedure of deriving Eq. (4.27) as follows.

We begin with the first term in the bracket of Eq. (E8) when multiplied with $K(1;2)$ from the left, i.e.,

$$K(1;2)G^*(1)G(2)T^{(\alpha)*}(1)T^{(\alpha)}(2)\ , \tag{E9}$$

whose eigenfunction transform can be written in the form (4.6) with

$$\begin{aligned}f_{ab}(\mathbf{u},\boldsymbol{\lambda})&=\tilde{K}_{ij;ab}(\,|\,\boldsymbol{\lambda}\mid\mathbf{u})\tilde{T}^{(\alpha)*}_{ac}(\boldsymbol{\lambda}_1\mid\boldsymbol{\lambda}_1'')\\ &\quad\times\tilde{T}^{(\alpha)}_{bd}(\boldsymbol{\lambda}_2\mid\boldsymbol{\lambda}_2'')\exp[i(\boldsymbol{\lambda}-\boldsymbol{\lambda}'')\cdot\boldsymbol{\rho}_\alpha]\ .\end{aligned} \tag{E10}$$

Here $\tilde{T}^{(\alpha)}_{bd}(\boldsymbol{\lambda}\mid\boldsymbol{\lambda}'')$ is a mode transform of $T^{(\alpha)}(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}'')$, defined by

$$\begin{aligned}\tilde{T}^{(\alpha)}_{bd}&(\boldsymbol{\lambda}\mid\boldsymbol{\lambda}'')\\ &=\int d\hat{\mathbf{x}}\,d\hat{\mathbf{x}}''\bar{\phi}_b(z)\exp[i\boldsymbol{\lambda}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_\alpha)]T^{(\alpha)}(\hat{\mathbf{x}}\mid\hat{\mathbf{x}}'')\\ &\quad\times\phi_d(z'')\exp[-i\boldsymbol{\lambda}''\cdot(\boldsymbol{\rho}''-\boldsymbol{\rho}_\alpha)]\ ,\end{aligned} \tag{E11}$$

where $\boldsymbol{\rho}_\alpha$ is the center coordinates of the scatterer, and $\boldsymbol{\lambda}_1=\mathbf{u}-\boldsymbol{\lambda}/2$, $\boldsymbol{\lambda}_2=\mathbf{u}+\boldsymbol{\lambda}/2$, and $\boldsymbol{\lambda}''=\boldsymbol{\lambda}_2''-\boldsymbol{\lambda}_1''$ [Eq. (4.2b)].

Hence, the result of the $\mathbf{u}$ integration in Eq. (4.6) can be given by the mode series (4.17) with $f_{ab}^{(\lambda)}(\mathbf{\Omega})$ defined by Eq. (4.14b), provided that the scatterer size is smaller than half of the medium correlation distance (Appendix B); otherwise, another mode expansion must be found. To this end, we first introduce a formula of dividing any analytic function of a variable u, say $\tilde{f}(u)$ where $\tilde{f}(u)\to 0$ for $u\to\pm\infty$, into three parts, as[5]

$$\tilde{f}(u)=\tilde{f}^{\,+}(u)+\tilde{f}^{\,-}(u)+\tilde{f}^{\,0}(u)\ ,$$

with

$$\begin{aligned}\tilde{f}^{\,\pm}(u)&=\frac{(\pm)}{2\pi i}\int_{-\infty}^{\infty}du'(u'-u\mp i0)^{-1}\\ &\quad\times\exp[\mp i(u'-u)\Delta]\tilde{f}(u')\ ,\\ \tilde{f}^{\,0}(u)&=\pi^{-1}\int_{-\infty}^{\infty}du'(u'-u)^{-1}\sin[(u'-u)\Delta]\tilde{f}(u')\ ,\end{aligned}$$

where Δ is a positive and arbitrarily large number. The functions $\tilde{f}^{\,\pm}(u)$ are analytic on the upper and lower half-planes of u, respectively, and $\tilde{f}^{\,0}(u)$ is an entire function which tends to zero as $\Delta\to 0$. The Fourier inversions of the respective functions, $f^{\pm}(\rho)$ and $f^0(\rho)$, are different from zero only in the ranges of $\rho\gtrless\Delta$ and $|\rho|<\Delta$, respectively. This shows that if we are not interested in the function $f(\rho)$ within the range $|\rho|<\Delta$, the transforms $\tilde{f}^{\,\pm}(u)$ are sufficient that have the additional factors $\exp(\pm i\Delta u)$ compared with those for $\Delta=0$, to change their asymptotic forms drastically. The situation is also the same in the present two-dimensional case, by setting $\mathbf{u}=u\mathbf{\Omega}$ with a unit vector $\mathbf{\Omega}$. The range $|\boldsymbol{\rho}|<\Delta$ can be chosen to be the interior of the scatterer, where the fluctuating part of the medium should really not exist [$K(1;2)=0$], and should be excluded from the integration range, although this was preliminary included by assuming the basic wave equation (5.1).

Equation (4.13) is now replaced, by dividing the integrand into the three parts with respect to each of u_1 and u_2, independently, by

$$F(\boldsymbol{\lambda})=\int_{2\pi} d\boldsymbol{\Omega}\frac{1}{2\pi}\int_0^\infty du \sum_{a,b}(u_1-k_a^{*(T)})^{-1}(u_2-k_b^{(T)})^{-1}\exp[\mp i\Delta(\boldsymbol{\Omega})(u_1-u_2-k_a^{*(T)}+k_b^{(T)})]w_{ab}^{(T)}f_{ab}^{(\lambda)}(\boldsymbol{\Omega}) . \tag{E12}$$

The upper signs of $\Delta(\boldsymbol{\Omega})$ are for the poles k_{+a}^* and k_{+b}, and the lower signs are for the k_{-a}^* and k_{-b}; the u integrals here involved are still elementary, as in the case of $\Delta=0$ (Appendix C). With Eq. (4.9), formula (4.17) is now replaced when the interference terms are neglected by

$$F(\boldsymbol{\lambda})=\int d\boldsymbol{\Omega}\sum_{+a,+b}\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})\exp[-\Delta(\boldsymbol{\Omega})(\gamma_{ab}^{(T)}-i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})]f_{ab}^{(\lambda)}(\boldsymbol{\Omega}) , \tag{E13}$$

in terms of $\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})$ and $\gamma_{ab}^{(T)}$, defined by Eqs. (4.18) and (4.19).

Equation (E13) shows that the necessary change can be achieved by multiplying $\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})$ with the factor exp[ ] which is unity at the pole $i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}=\gamma_{ab}^{(T)}$. Here

$$(2\pi)^{-2}\int d\boldsymbol{\lambda}\,\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})\exp[-\Delta(\boldsymbol{\Omega})(\gamma_{ab}^{(T)}-i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})]e^{-i\boldsymbol{\lambda}\cdot\boldsymbol{\rho}} = \begin{cases}0, & \boldsymbol{\Omega}\cdot\boldsymbol{\rho}<\Delta(\boldsymbol{\Omega})\\ U_{ab}(\boldsymbol{\Omega},\boldsymbol{\rho}), & \boldsymbol{\Omega}\cdot\boldsymbol{\rho}>\Delta(\boldsymbol{\Omega}) ,\end{cases} \tag{E14}$$

with $U_{ab}(\boldsymbol{\Omega},\boldsymbol{\rho})$ given by Eq. (4.21b); that is, the factor exp[ ] serves only to make the propagator U_{ab} zero inside the scatterer $[\boldsymbol{\Omega}\cdot\boldsymbol{\rho}<\Delta(\boldsymbol{\Omega})]$.[18]

The situation is also the same for the $\mathbf{u}$ integration involved in the product

$$T^{(\alpha)*}(1)T^{(\alpha)}(2)G^*(1)G(2)K(1;2) .$$

Thus we obtain a mode transform of the first term in the bracket of Eq. (E8), as [Eq. (4.17)]

$$\tilde{U}_{ab}(\boldsymbol{\Omega}',\boldsymbol{\lambda}')\sigma_{ab;cd}^{(\alpha)}(\boldsymbol{\Omega}'\mid\boldsymbol{\Omega}'')\tilde{U}_{cd}(\boldsymbol{\Omega}'',\boldsymbol{\lambda}'')\exp[i(\boldsymbol{\lambda}'-\boldsymbol{\lambda}'')\cdot\boldsymbol{\rho}_\alpha] \tag{E15}$$

[where the phase factor is from Eq. (E10)]. Here the two exponential factors of Δ have been included in $\tilde{U}_{ab}$ and $\tilde{U}_{cd}$, and

$$\sigma_{ab;cd}^{(\alpha)}(\boldsymbol{\Omega}'\mid\boldsymbol{\Omega}'')=\tilde{T}_{ac}^{(\alpha)*}(k_a^*\boldsymbol{\Omega}'\mid k_c^*\boldsymbol{\Omega}'')\tilde{T}_{bd}^{(\alpha)}(k_b\boldsymbol{\Omega}'\mid k_d\boldsymbol{\Omega}'') , \tag{E16}$$

in which $\boldsymbol{\lambda}_T'/2$ and $\boldsymbol{\lambda}_T''/2$ (of the order of magnitude of γ_{aa}^{-1} or smaller) have been neglected with the assumption that the scatterer size is very small compared with the coherence distances of the mode waves.

To obtain a corresponding expression from the second term in Eq. (E8), we consider the product

$$K(1;2)G^*(1)T^{(\alpha)*}(1)G^*(1)G(2)K(1;2) ,$$

whose eigenfunction transform is

$$\sum_{a,b,c}(2\pi)^{-2}\int d\boldsymbol{\lambda}_2\tilde{K}_{ij;ab}(\,|\,\boldsymbol{\lambda}'\mid\mathbf{u}')D_a^{*-1}(\boldsymbol{\lambda}_1')\tilde{T}_{ac}^{(\alpha)*}(\boldsymbol{\lambda}_1'\mid\boldsymbol{\lambda}_1'') \times D_c^{*-1}(\boldsymbol{\lambda}_1'')D_b^{-1}(\boldsymbol{\lambda}_2)\tilde{K}_{cb;ef}(\mathbf{u}''\mid\boldsymbol{\lambda}''\mid\,) \times e^{i(\boldsymbol{\lambda}'-\boldsymbol{\lambda}'')\cdot\boldsymbol{\rho}_\alpha} . \tag{E17}$$

$\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$ are regarded as constants, and $\boldsymbol{\lambda}_1'$, $\boldsymbol{\lambda}_1''$, $\mathbf{u}'$, and $\mathbf{u}''$ are functions of $\boldsymbol{\lambda}_2=\boldsymbol{\lambda}_2'=\boldsymbol{\lambda}_2''$ through the relations

$$\boldsymbol{\lambda}_1'=\boldsymbol{\lambda}_2-\boldsymbol{\lambda}',\quad \boldsymbol{\lambda}_1''=\boldsymbol{\lambda}_2-\boldsymbol{\lambda}'' . \tag{E18}$$

On setting $\boldsymbol{\lambda}_2=u_2\boldsymbol{\Omega}$, $\boldsymbol{\Omega}^2=1$, with $d\boldsymbol{\lambda}_2=u_2du_2d\boldsymbol{\Omega}$, we first rewrite Eq. (E18) as

$$\boldsymbol{\lambda}_1'=u_1'\boldsymbol{\Omega}-\boldsymbol{\lambda}_T',\quad \boldsymbol{\lambda}_1''=u_1''\boldsymbol{\Omega}-\boldsymbol{\lambda}_T'' , \tag{E19}$$

with the components $\boldsymbol{\lambda}_T',\boldsymbol{\lambda}_T''$ of $\boldsymbol{\lambda}',\boldsymbol{\lambda}''$ orthogonal to $\boldsymbol{\Omega}$, respectively, so that

$$\begin{aligned}\mathbf{u}'&=2^{-1}[(u_1'+u_2)\boldsymbol{\Omega}-\boldsymbol{\lambda}_T'] ,\\ \mathbf{u}''&=2^{-1}[(u_1''+u_2)\boldsymbol{\Omega}-\boldsymbol{\lambda}_T''] .\end{aligned} \tag{E20}$$

Then for given $\boldsymbol{\Omega}$, we regard the entire integrand of Eq. (E17) as a function of independent variables u_1', u_1'', and u_2, and expand it at their poles $k_a^{*(T)}\simeq k_a^*$, $k_c^{*(T)}\simeq k_c^*$, and k_b, given by

$$\begin{aligned}D_a^*(k_a^{*(T)}\boldsymbol{\Omega}-\boldsymbol{\lambda}_T')&=D_c^*(k_c^{*(T)}\boldsymbol{\Omega}-\boldsymbol{\lambda}_T'')\\ &=D_b(k_b\boldsymbol{\Omega})=0\end{aligned} \tag{E21}$$

[cf. Eq. (4.10)]. Hence, Eq. (E17) can be given by a mode series of the form

$$\sum_{a,b,c}\int d\boldsymbol{\Omega}(2\pi)^{-2}\int_0^\infty du_2k_b(u_1'-k_a^*)^{-1}(u_1''-k_c^*)^{-1}(u_2-k_b)^{-1}\left[\left.\frac{\partial D_a^*}{\partial u_1'}\right|_a\left.\frac{\partial D_c^*}{\partial u_1''}\right|_c\left.\frac{\partial D_b}{\partial u_2}\right|_b\right]^{-1} \times\tilde{T}_{ac}^{(\alpha)*}(k_a^*\boldsymbol{\Omega}\mid k_c^*\boldsymbol{\Omega})e^{i(\boldsymbol{\lambda}'-\boldsymbol{\lambda}'')\cdot\boldsymbol{\rho}_\alpha}f_{abc}(\boldsymbol{\Omega}) . \tag{E22}$$

Here

$$f_{abc}(\boldsymbol{\Omega})=\tilde{K}_{ij;ab}(\,|\,\boldsymbol{\lambda}'\mid k_{ab}\boldsymbol{\Omega})\tilde{K}_{cb;ef}(k_{cb}\boldsymbol{\Omega}\mid\boldsymbol{\lambda}''\mid\,) , \tag{E23}$$

which should have two more factors of exp[ ] in Eq. (E13), though not written here explicitly.

Since from Eqs. (E19) and (E18)

$$u_1'=u_2-\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}',\quad u_1''=u_2-\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}'' , \tag{E24}$$

the u_2 integration in Eq. (E22) is elementary and can be given, to a good approximation, by taking the residue values at the poles k_b existing on the lower half-plane of u_2 and by neglecting those of the nonpropagative mode waves. Hence, by using Eq. (4.18) for $\tilde{U}_{ab}(\boldsymbol{\Omega},\boldsymbol{\lambda})$ with

$w_{ab}^{(e)} \to w_{ab}$ and $\gamma_{ab}^{(T)} \to \gamma_{ab}$, Eq. (E22) can be written as

$$\sum_{a,b,c}{}' \int d\Omega\, i4\pi(1-\kappa_b')k_b^2(k_{ab}k_{cb})^{-1}\tilde{U}_{ab}(\Omega,\lambda')\tilde{U}_{cb}(\Omega,\lambda'') \times \tilde{T}_{ac}^{(\alpha)*}(k_a^*\Omega \mid k_c^*\Omega)e^{i(\lambda'-\lambda'')\cdot\rho_\alpha} f_{abc}(\Omega) . \quad \text{(E25)}$$

Also from the third term in Eq. (E1), we obtain a similar expression.

Thus the mode transform of the entire product $KU^{(C)}V^{(\alpha)}U^{(C)}K$ is obtained finally in the form

$$\sum_{a,b,c,d}{}' \int d\Omega' d\Omega'' K_{ij;ab}^{(\lambda')}(\ \mid \Omega')\tilde{U}_{ab}(\Omega',\lambda')V_{ab;cd}^{(\alpha)}(\Omega' \mid \Omega'') \times \exp[i(\lambda'-\lambda'')\cdot\rho_\alpha]\tilde{U}_{cd}(\Omega'',\lambda'')K_{cd;ef}^{(\lambda'')}(\Omega'' \mid \ ) \quad \text{(E26)}$$

with

$$V_{ab;cd}^{(\alpha)}(\Omega' \mid \Omega'') = \sigma_{ab;cd}^{(\alpha)}(\Omega' \mid \Omega'') - \gamma_{ab;cd}^{(\alpha)}(\Omega')\delta(\Omega'-\Omega'') . \quad \text{(E27)}$$

$\sigma_{ab;cd}^{(\alpha)}$ is defined by Eq. (E16) and

$$\gamma_{ab;cd}^{(\alpha)}(\Omega) = (2i)^{-1}(k_{ab}^{(e)}w_{ab})^{-1} \times [f_{a,cd}^*(\Omega)\delta_{bd}\tilde{T}_{ac}^{(\alpha)*}(k_a^*\Omega \mid k_c^*\Omega) - f_{b,dc}(\Omega)\delta_{ac}\tilde{T}_{bd}^{(\alpha)}(k_b\Omega \mid k_d\Omega)] , \quad \text{(E28)}$$

where $w_{ab} = w_{ab}^{(T)} \mid_{\lambda=0}$ and $k_{ab}^{(e)}$ are given by Eqs. (4.14a) and (D4), respectively, and

$$f_{a,cd}^*(\Omega) = (1+i2^{-1}\gamma_{ad}^{(e)}/k_{ad}^{(e)})^{-1}(1-i2^{-1}\gamma_{cd}/k_{cd}) \quad \text{(E29)}$$

with

$$\gamma_{ad}^{(e)} = [k_a^*(1-\kappa_a'^*) - k_d(1-\kappa_d')]/i = \gamma_{da}^{(e)*} . \quad \text{(E30)}$$

As long as $|\gamma_{ab}| \ll |k_{ab}|$ and $|\gamma_{cd}| \ll |k_{cd}|$ we have $f_{a,cd}^* \simeq f_{b,dc} \simeq 1$ in Eq. (E28), resulting in

$$\sum_{a,b}{}' \int d\Omega' \Delta_{ab}k_{ab}^{(e)}w_{ab}V_{ab;cd}^{(\alpha)}(\Omega' \mid \Omega'') \simeq 0 , \quad \text{(E31)}$$

by virtue of the optical relation

$$\sum_{a,b}{}' \int d\Omega' \Delta_{ab}k_{ab}^{(e)}w_{ab}\sigma_{ab;cd}^{(\alpha)}(\Omega' \mid \Omega'') \simeq (2i)^{-1}[\tilde{T}_{dc}^{(\alpha)*}(k_d^*\Omega'' \mid k_c^*\Omega'') - \tilde{T}_{cd}^{(\alpha)}(k_c\Omega'' \mid k_d\Omega'')] , \quad \text{(E32)}$$

which is just the mode transformed version of the $\hat{\mathbf{x}}$ integrated relation of Eq. (5.46), in the particular case $\Delta\gamma_\alpha^{(T)}=0$; the proof is straightforward with the same procedure as when deriving the mode transform (D7) of Eq. (D1).

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$$\sum_B [G^*(1)K^{(BA)}(1;2) - G(2)K^{(AB)}(1;2)] .$$

Here the summation is over all the q, b_1, and b_2, with the abbreviation $K^{(AA)}(1;2)=K^{(A)}(1;2)$.
[15]The matrix elements (3.29) for $\gamma_{a'b'}^{(q)}$ have the same form as the right-hand side of Eq. (2.9b) for $\mathbf{s}^{(j)}(\rho \mid \rho_1;\rho_2)$. Hence, writing $\tilde{M}^{(q)} = \tilde{M}_R^{(q)} - i\tilde{M}_I^{(q)}$ in terms of the Hermitian and anti-Hermitian parts $\tilde{M}_R^{(q)}$ and $\tilde{M}_I^{(q)}$, respectively, the contribution of $\tilde{M}_R^{(q)}$ to $\gamma_{a'b'}^{(q)}$ can be written in the divergence form

$$\partial_z s_{a'b'}^{(qM)}(\Omega,z \mid z_1;z_2)/k_{a'b'} ,$$

so that the substitution in Eq. (3.28) results in the replacement of the left-hand side by

$$\frac{\partial}{\partial z}[N_{a'b'}^{(3)}(\Omega,z) + k_{a'b'}^{-1}s_{a'b'}^{(qM)}(\Omega,z)N_{a'b'}(\Omega,z)]$$

and the right-hand side by what is given with $\tilde{M}^{(q)} \to \tilde{M}_I^{(q)}$. Here the term of $s_{a'b'}^{(qM)}$ means a correction of the power flux due to the dispersive nature of an effective medium $\tilde{M}_R^{(q)}$ [cf.

the surface-wave term in (2.13)].

[16]The eigenfunctions involved in the transform $\tilde{K}_{ef;ab}$ and $\tilde{\mathbf{S}}_{ab;cd}$ given by $f_{ab}^{(\lambda)}(\mathbf{\Omega})$ [(4.14b)] are to have their values at the poles $u_1=k_a^{*(T)}$ and $u_2=k_b^{(T)}$.

[17]In Eq. (D2) we could make an alternative choice of writing the factor $D_b-D_a^*$ by

$$D_b(u_2\mathbf{\Omega}+\boldsymbol{\lambda}_T/2)-D_a^*(u_1\mathbf{\Omega}-\boldsymbol{\lambda}_T/2)$$

[Eqs. (4.8)]. But the resulting integrand in formula (4.6) would not tend to zero for $|u_1|,|u_2|\to\infty$, so that the expansion according to (4.13) would not be possible; that is, the result (D7) can be obtained only with the choice of Eq. (D2).

[18]The surface $\mathbf{\Omega}\cdot\boldsymbol{\rho}=\Delta(\mathbf{\Omega})$ does not need to be exactly the real boundary of the scatterer, and may be any surface slightly bigger than that because the scatterer is assumed to be very small compared with the coherence distances of the mode waves, so that the difference makes no appreciable effect.