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Measurement of amplitude probability distributions for photon-number-operator eigenstates

B. Yurke and D. Stoler
AT&T Bell Laboratories, Murray Hill, New Jersey 07974
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A homodyne detector measures a field-amplitude component of the incoming signal. If the incoming signal is in an n-photon eigenstate the homodyne detector's output probability distribution exhibits n fringes. Here the degree to which the visibility of these fringes is degraded by a homodyne detector with less than unit quantum efficiency is evaluated. An experiment employing conjugate pairs of photons generated via a parametric down conversion or four-wave mixing is proposed by which these fringes could be observed.

Because of their utility in squeezed-state detection homodyne detectors have received considerable attention.[1-5] It is now quite well known that a homodyne detector measures an amplitude component of the electromagnetic field. If the input to the homodyne detector consists of a photon-number eigenstate with n photons then the probability distribution for the homodyne detector will have the same form as the probability distribution for the x coordinate of a harmonic oscillator in its nth energy eigenstate. This probability distribution exhibits n peaks or fringes. If the homodyne detector has less than unit quantum efficiency these fringes will be blurred due to the noise associated with the detector losses. Here we calculate the degree to which the fringe visibility is degraded due to detector losses. It is shown that the homodyne-detector output probability distribution can be written as a convolution of the field-amplitude component probability distribution of the incoming signal with the field-amplitude component probability distribution of the vacuum field emitted by the loss.

It is known that the signal and idler photons produced in the parametric-down-conversion process are highly correlated.[6-9] It has in fact been experimentally demonstrated[10-12] that by measuring the arrival of the idler photons one can determine the positions of the photons in the signal beam to within a coherence time. The number of photons counted by a photodetector in the idler beam during a coherence time could thus be used to gate a homodyne detector in the signal beam so that the homodyne detector's integrated output over a coherence time is only recorded when a one-photon wave packet enters its input port. More generally, the homodyne detector could be gated in such a way that its output, integrated over a coherence time, is recorded only when m photons enter the idler photodetector in a coherence time. In this way an experiment could be performed to map out the probability distribution for a field-amplitude component of a number-operator eigenstate. The experiment described here is akin to other feed-forward schemes that have been proposed for generating photon-number eigenstates or antibunched light from frequency down converters.[13,14]

The analysis performed here is a single-mode analysis in which the effects of the random deletion noise of photodetectors with less than unit quantum efficiency are taken into account. We show that for one-photon wave packets the fringes are visible even when photodetectors with quantum efficiencies of 0.5 are used. The successful generation of squeezed states[15-19] with four-wave mixers and parametric down converters shows that signal and idler beams with a reasonable photon flux and (in the case of cavity devices[15,17]) coherence times long compared to photodetector response times can be realized. The calculations presented thus indicate that the proposed experiment should be feasible with state-of-the-art technology.

The response of a homodyne detector to photon-number eigenstates will now be calculated. The operator[1-3] that a homodyne detector measures is

$$\hat{x} = \frac{1}{\sqrt{2}}(e^{i\phi}a_s^\dagger + e^{-i\phi}a_s) \ , \tag{1}$$

where ϕ is the local oscillator phase and a_s is an annihilation operator for the signal mode. This is the operator for one amplitude component of the electromagnetic field entering the photodetector. The effects of loss[1,5] can be simulated by placing a beam splitter in the signal port of an ideal homodyne detector. The beam splitter will couple a vacuum mode a_N into the signal so that the mode a delivered to the ideal homodyne detector is

$$a = \eta^{1/2}a_s + (1-\eta)^{1/2}a_N \ , \tag{2}$$

where η is the transmission of the beam splitter, or the quantum efficiency of the homodyne detector. Let

$$\hat{y} = \frac{1}{\sqrt{2}}(e^{i\phi}a_N^\dagger + e^{i\phi}a_N) \ , \tag{3}$$

then the output delivered by the homodyne detector is proportional to the operator

$$\hat{q} = \hat{x} + \left(\frac{1-\eta}{\eta}\right)^{1/2}\hat{y} \ . \tag{4}$$

The state vector $|\Psi\rangle$ has the form

$$|\Psi\rangle = |\Psi_x\rangle|\Psi_y\rangle \ , \tag{5}$$

where the signal state vector $|\Psi_x\rangle$ is operated on by a_s and the noise state vector $|\Psi_y\rangle$ (which will shortly be specialized to the vacuum state) is operated on by a_N. Since a_s and a_N are two independent modes, $\hat{x}$ and $\hat{y}$ commute and consequently the nth moment of $\hat{q}$ has the form

$$\langle \hat{q}^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1-\eta}{\eta} \right)^{k/2} \langle \Psi_x | \hat{x}^{n-k} | \Psi_x \rangle \langle \Psi_y | \hat{y}^k | \Psi_y \rangle \ . \tag{6}$$

The moments of $\hat{x}$ and $\hat{y}$ can be expressed as

$$\langle \Psi_x | \hat{x}^m | \Psi_s \rangle = \int_{-\infty}^{\infty} x^m P_s(x) dx \tag{7}$$

and

$$\langle \Psi_y | \hat{y}^m | \Psi_y \rangle = \int_{-\infty}^{\infty} x^m P_N(y) dy \ , \tag{8}$$

where $P_s(x)$ is the probability density for the observable x associated with the operator $\hat{x}$ and similarly $P_N(y)$ is the probability density for the observable y associated with the operator $\hat{y}$. Substituting Eqs. (7) and (8) into (6), one has

$$\langle \hat{q}^n \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[x + \left(\frac{1-\eta}{\eta} \right)^{1/2} y \right]^n P_s(x) P_N(y) \ . \tag{9}$$

Introducing the change of variables

$$q = x + \left(\frac{1-\eta}{\eta} \right)^{1/2} y \ , \tag{10}$$

Eq. (9) can be put into the form

$$\begin{aligned} \langle \hat{q}^n \rangle &\equiv \int_{-\infty}^{\infty} dq\, q^n P(q) \\ &= \int_{-\infty}^{\infty} dq\, q^n \int_{-\infty}^{\infty} dy\, P_s \left[q - \left(\frac{1-\eta}{\eta} \right)^{1/2} y \right] P_N(y) \ . \end{aligned} \tag{11}$$

Hence, the probability distribution for the random variable q measured by the homodyne detector is given by the convolution

$$P(q) = \int_{-\infty}^{\infty} dy\, P_s \left[q - \left(\frac{1-\eta}{\eta} \right)^{1/2} y \right] P_N(y) \ . \tag{12}$$

Specializing to the case when $|\Psi_y\rangle$ is the vacuum state $|0_y\rangle$,

$$a_N | 0_y \rangle = 0 \ , \tag{13}$$

it is a straightforward exercise in commutator algebra to show that

$$\langle 0_y | \hat{y}^{2k} | 0_y \rangle = \frac{(2k-1)!!}{2^k} \tag{14}$$

and

$$\langle 0_y | \hat{y}^{2k+1} | 0_y \rangle = 0 \ . \tag{15}$$

From Eqs. (14) and (15) it is easily established that the probability distribution $P_N(y)$ is

$$P_N(y) = \frac{1}{\sqrt{\pi}} e^{-y^2} \ . \tag{16}$$

In a similar manner, it is straightforward to show that the probability distribution $P_s(x)$ when $|\Psi_x\rangle$ is an n-photon eigenstate $|n_x\rangle$ is

$$P_S(x) = \frac{1}{\pi^{1/2} 2^n n!} H_n^2(x) e^{-x^2} \ , \tag{17}$$

where $H_n(x)$ is the nth Hermite polynomial. This result follows most readily by recognizing $\hat{x}$ as the position operator for a harmonic oscillator and transforming $|n_x\rangle$ into its x representation. Note that Eq. (17) is independent of the local oscillator phase. This is a consequence of the fact that the phase is completely uncertain for a number-operator eigenstate and, hence, it is irrelevant where the homodyne-detector phase is set. Substituting Eqs. (16) and (17) into Eq. (12) and making a suitable change of variables, the probability distribution $P_n(q)$ for the homodyne-detector output when the signal is in an n-photon eigenstate can be expressed as

$$P_n(q) = \frac{\eta^{1/2} e^{-\eta q}}{\pi 2^n n!} \int_{-\infty}^{\infty} dy\, H_n^2((1-\eta)^{1/2} y + \eta q) e^{-y^2} \ . \tag{18}$$

The integral can be evaluated to yield[20]

$$P_n(q) = \left(\frac{\eta}{\pi} \right)^{1/2} e^{-\eta q^2} \sum_{k=0}^{n} \frac{\eta^{n-k}}{2^{n-k}(n-k)!} \times \binom{n}{k} H_{2(n-k)}(\eta^{1/2} q) \ . \tag{19}$$

The first three cases are

$$P_0(q) = \left(\frac{\eta}{\pi} \right)^{1/2} e^{-\eta q^2} \ , \tag{20}$$

$$P_1(q) = \left(\frac{\eta}{\pi} \right)^{1/2} e^{-\eta q^2} (1 - \eta + 2\eta^2 q^2) \ , \tag{21}$$

and

$$P_2(q) = \frac{1}{2} \left(\frac{\eta}{\pi} \right)^{1/2} e^{-\eta q^2} [3\eta^2 - 4\eta + 2 + (8\eta^2 - 12\eta^3) q^2 + 4\eta^4 q^4] \ . \tag{22}$$

The probability distributions Eqs. (21) and (22) are plotted in Figs. 1 and 2, respectively, the cases (a), (b), and (c) being respectively $\eta = 1$, $\eta = 0.75$, and $\eta = 0.5$. For Fig. 1 ($n = 1$) the fringes are still visible when $\eta = 0.5$.

More generally, one can take the asymptotic form for $H_n(x)$ to calculate the fringe visibility for large n. In particular, one has, asymptotically,[21]

$$H_{2n}^2(x) \simeq 2^{2n} [(2n-1)!!]^2 e^{x^2} \cos^2(\sqrt{4n+1}\, x) \ , \tag{23}$$

which, upon substitution into Eq. (18), yields the asymptotic form for the probability distribution $P_{2n}(q)$:

$$P_{2n}(q) = \frac{2^{n-1} [(2n-1)!!]^2}{\sqrt{\pi} n!} \left\{ 1 + \exp\left[- \left(\frac{1-\eta}{\eta} \right) (4n+1) \right] \cos(2\sqrt{4n+1}\, q) \right\} \ . \tag{24}$$

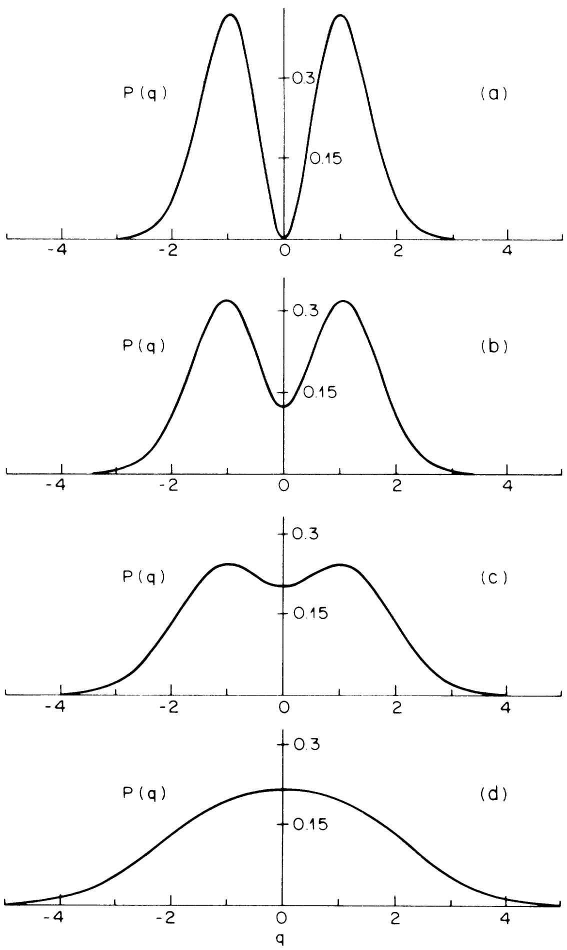

FIG. 1. The probability distribution for the charge q delivered by a homodyne detector observing a single-photon eigenstate. The cases when the quantum efficiency η of the homodyne detector is 1, 0.75, 0.5, and 0.25 are depicted in (a), (b), (c), and (d), respectively. Note that the fringes are still visible for $\eta=0.5$ but are washed out when $\eta=0.25$.

This is a sinusoidal function of q and a fringe visibility can be defined in the usual way:[22]

$$V_{2n}=\frac{P_{2n}^{\max}(q)-P_{2n}^{\min}(q)}{P_{2n}^{\max}(q)+P_{2n}^{\min}(q)}\ , \tag{25}$$

where $P_{2n}^{\max}(q)$ and $P_{2n}^{\min}(q)$ are the values of $P_{2n}(q)$ at the fringe maxima and minima, respectively. Equation (24) substituted into Eq. (25) yields

$$V_{2n}=\exp\left[-\left(\frac{1-\eta}{\eta}\right)(4n+1)\right]\ . \tag{26}$$

When n is large one can see that the fringes will wash out when the loss $L=1-\eta$ is of order

$$L=\frac{1}{(4n+1)}\ . \tag{27}$$

An experimental arrangement by which one might try to observe the probability distribution for an amplitude component of a number-operator eigenstate is shown in Fig. 3. A four-wave-mixing medium or parametric-down-conversion medium, NL, is pumped with an intense coherent-state source. Noncollinear signal and idler beams can be produced in a number of ways. Here, for conceptual simplicity, the signal photons with center frequency $\omega_p-\omega_s$ and the conjugate idler photons with center frequency $\omega_p+\omega_s$ are spatially separated using a dispersive element, the prism P. The signal and idler are frequency selected with knife edges S1 and S2. A photodetector D1 in the idler beam gates the output of the homodyne detector which consists of the local oscillator, the beam splitter BS, and detector D2. Since it was shown that the probability distribution is independent of the local oscillator phase the local oscillator does not need to be phase locked to the pump (in contrast to what is done in squeezed-state detection). It is only required that the local oscillator emit light at frequency $\omega_p-\omega_s$ and that it remain stable in phase over the coherence time determined by the inverse bandwidth of the frequencies that are allowed to pass through S2. We now point out that the effects of losses in D1 can be made negligible by operating at a sufficiently low photon flux. It has been shown[15] that when a detector of efficiency η_d is used in the idler beam, the probability $P_{d,m}(n)$ that there are n photons present in the idler when only m are counted is

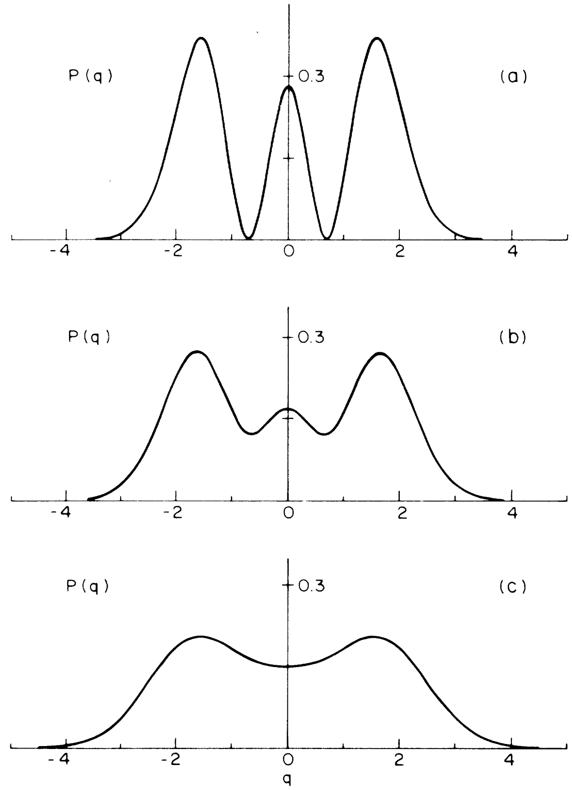

FIG. 2. The probability distribution for the charge q delivered by a homodyne detector observing a two-photon eigenstate. The cases when the quantum efficiency η of the detector is 1, 0.75, and 0.5 are depicted in (a), (b), and (c), respectively.

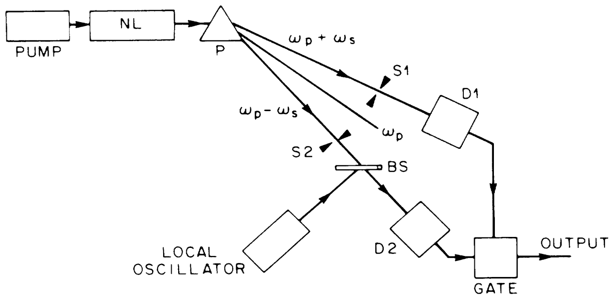

FIG. 3. A feed-forward scheme for gating the output of a homodyne detector so that its output is only observed when an n-photon wave packet enters the input port. The homodyne detector consists of the local oscillator, the beam splitter BS, and the photodetector D2. The strong correlations between the signal and idler photons coming from the parametric medium NL are exploited to gate the homodyne detector. The gate is opened only when photodetector D1 in the idler beam reports the arrival of n photons in the coherence time determined by the inverse bandwidth of the frequencies of light allowed to pass through knife edges S1 and S2. For further details see the text.

given by

$$P_{d,m}(n) = \begin{cases} \begin{pmatrix} n \\ m \end{pmatrix} (1-\gamma)^{m+1}\gamma^{n-m} \text{ for } n > m \ , \\ 0 \text{ for } n < m \ , \end{cases} \tag{28}$$

where

$$\gamma = (1-\eta_d)\tanh^2\left(\frac{\beta}{2}\right) \ , \tag{29}$$

and β is related to the conversion gain G_{si} (in power) of incoming signal to outgoing idler via

$$G_{si} = \sinh^2\left(\frac{\beta}{2}\right) \ . \tag{30}$$

The probability distribution $P_{\eta_d,m}(q)$ of the homodyne-detector output when the homodyne detector is gated by a photodetector with efficiency η_d is, thus,

$$P_{\eta_d,m}(q) = \sum_{n=m}^{\infty} P_{d,m}(n)P_n(q) \ . \tag{31}$$

For small γ, as can be obtained by reducing the parametric gain, Eq. (28) yields

$$P_{d,m}(m) \rightarrow 1$$

and $P_{d,m}(n)$ is of order γ^{n-m}. Hence, in the low-intensity limit only the first term of the sum (31) contributes significantly and $P_{\eta_d,m}(q)$ reduces to $P_n(q)$.

In summary, we have evaluated the effects of loss on a homodyne detector's ability to observe fringes in the probability distribution of an amplitude component of the electromagnetic field when the field is in a number-operator eigenstate. For the $n=1$ eigenstate the fringes are still visible when the homodyne-detector efficiency $\eta = 0.5$. An experiment was proposed in which the conjugate pairs of photons produced in parametric-down conversion are employed to gate a homodyne detector, so that its output is recorded only when an n-photon wave packet (where n has been predetermined) enters the homodyne detector.

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