Free-electron-laser linewidth obtained from a master Fokker-Planck equation

M. Orszag*
Center for Advanced Studies and Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131
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The problem of the intrinsic linewidth of a free-electron laser is analyzed. As opposed to the traditional Schawlow-Townes linewidth in atomic lasers, which is a quantum-mechanical result, the free-electron-laser linewidth is classical, that is, independent of $\hbar$. The theory of the ultimate linewidth of a free-electron laser is developed here, using the master Fokker-Planck approach with the advantage, over previous work, that it does not make use of any ansatz.

I. INTRODUCTION

One of the most important characteristics of a laser oscillator is its narrow spectral linewidth. In an ordinary atomic laser, the Schawlow-Townes linewidth is given by[1,2]

$$D_{\text{atomiclaser}} = 1/2 \left[\frac{\nu}{Q} \right]^2 \frac{\hbar\nu}{P} , \tag{1}$$

where P is the total power in the oscillator. Of course, in most lasers, this limit is very difficult to achieve because of various inhomogeneous broadening mechanisms that would make the linewidth several orders of magnitude bigger. However, the ultimate linewidth of an atomic laser is a quantum-mechanical result (depends on $\hbar$), the reason for it being, of course, that the spontaneous emission in an atomic system is a purely quantum effect, and it is precisely the spontaneous emission noise source that originates this linewidth.

In sharp contrast with the atomic case, in the free-electron laser (FEL), the resulting linewidth does not depend on $\hbar$, therefore it is a classical effect, and it is related to the classical randomness at which the electrons are being injected in the laser. This is normally referred to as shot noise.

Although in many FEL's the frequency spectrum is determined by the finite duration of the laser pulse, typically picoseconds, the idea of the ultimate linewidth of a FEL could be interesting if one thinks in the eventual cw FEL's.

Recently Becker *et al.*[3] studied the intrinsic linewidth of a cw free-electron laser and compared it with the ordinary laser. In their treatment, the ansatz

$$\langle E(t)\rangle = \langle E(0)\rangle e^{-Dt/2} \tag{2}$$

was used. In their approach the evolution operator is calculated to first order in the recoil. Then making use of the ansatz (2) and Liouville's equation in the "course-grained" time-average approximation, they were able to calculate the intrinsic linewidth of a FEL.

Very recently[4] Gover *et al.* also attacked this problem, using purely classical methods. Since, as we mentioned before, the linewidth in a FEL is a classical phenomena, these calculations agree with Ref. 3.

In the present work, a fully quantum-mechanical model is used to derive a master and Fokker-Planck equation, in the interaction picture, *without* assuming any ansatz. The present calculations agree with Refs. 3 and 4, provided one neglects the laser intensity fluctuations. The assumption under which the results are obtained is the Born-Markov approximation. Since we have a cavity, one should assume a finite cavity Q in the master equation. However, although this term is important in the light-statistics and -photon distribution, it does not contribute to the linewidth and can be dropped in the present calculation.[3]

II. THE MODEL

Let's consider a helical wiggler and a circularly polarized laser field, namely,

$$\begin{aligned} \mathbf{A}_W &= A_W^* \hat{\mathbf{e}} e^{ik_q z} + A_W \hat{\mathbf{e}}^* e^{-ik_q z} , \\ \mathbf{A}_L &= A_L^* \hat{\mathbf{e}} e^{i\theta_L} + A_L \hat{\mathbf{e}}^* e^{-i\theta_L} , \end{aligned} \tag{3}$$

with $\theta_L = k_L z - \omega_L t$.

The Hamiltonian of the problem is

$$H = [(p - eA)^2 c^2 + m_0^2 c^4]^{1/2} , \tag{4}$$

where A is the sum of both the laser and wiggler fields.

If one now performs a Lorentz transformation, considering that (a) in FEL's a small fraction of the electron's energy is converted to light, (b) the transverse canonical momentum of the electrons is a constant of motion, then the Hamiltonian becomes nonrelativistic and can be expanded (the prime denotes with respect to the moving frame)

$$H' = mc^2 + \frac{(p')^2}{2m} + \frac{e^2}{m} (\mathbf{A}_W \cdot \mathbf{A}_L)' , \tag{5}$$

where

$$(\mathbf{A}_W \cdot \mathbf{A}_L)' = A_W^* A_L e^{i(k'z' - \Delta\omega t')} + \text{H.c.} , \tag{6}$$

and

$$\begin{aligned} k' &= k'_W + k'_L , \\ \Delta\omega &= \omega'_L - \omega'_W . \end{aligned} \tag{7}$$

In Eq. (5), m is the renormalized mass given by

$$m = m_0(1+K_1)^{1/2} , \tag{8}$$

with

$$K_1 = \frac{e^2 B}{m_0^2 c^2 k_q^2} . \tag{9}$$

The fact that one consider's the longitudinal motion produces an apparent larger mass due to the transverse wiggling of the electron. For the present purposes, it is convenient to go to the interaction picture.

The interaction Hamiltonian can be written in the intersection picture as[5] [in the Bambini-Revieri (BR) frame]

$$H_1(t) = \hbar g \sqrt{N_W}(a e^{i(2kz)} e^{it\beta} + \text{H.c.}) , \tag{10}$$

where k is the laser-field wave number (in the BR frame, we drop the primes from now on), g the coupling constant, $\sqrt{N_W}$ the wiggler field amplitude, assumed classical and much stronger than the laser field, a is the annihilation operator for the laser field, and β is defined as

$$\beta \equiv \frac{\hbar(2k)^2 + 4kp}{2m} \approx \frac{2kp}{m} . \tag{11}$$

III. THE INTRINSIC LINEWIDTH OF A FEL

Under the coarse-graining time approximation, one can write[6]

$$\dot{\rho}_{\text{ph}} = r\,\text{Tr}_e[\rho(t+T) - \rho(t)] + \text{loss} , \tag{12}$$

where T is the interaction time between the electrons and the wiggler field, ρ,ρ_{ph} are the electron-photon and photon density operator, respectively, r the electron injection rate, and Tr_e is the trace over the electron operators. As mentioned above, the losses can be dropped.

From Liouville's equation,

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_1(t),\rho] , \tag{13}$$

and iterating twice, one obtains

$$\rho(t+T) = \rho(t) - \frac{i}{\hbar}\int_t^{t+T} dt'[H_1(t'),\rho(t)] - \frac{1}{\hbar^2}\int_t^{t+T} dt' \int_t^{t'} dt''[H_1(t'),[H_1(t''),\rho(t)]] . \tag{14}$$

In Eq. (14) we have made use of the Born-Markov approximation by replacing in the last term $\rho(t'')$ by $\rho(t)$.

Next we have to trace over the electrons. The second term on the right-hand side of Eq. (14) does not contribute since $H_1(t)$ has no diagonal terms and one can always assume that initially the density matrix can be written as a product of the electron and the photon part and that the electron part is diagonal.

Thus, replacing Eq. (14) into (12), one gets

$$\frac{d\rho_{\text{ph}}}{dt} = -\frac{r}{\hbar^2}\int_t^{t+T} dt' \int_t^{t'} dt'' \text{Tr}_e[H_1(t'),[H_1(t''),\rho(t)]] . \tag{15}$$

A straightforward calculation leads to the following master equation: $(\rho \equiv \rho_{\text{ph}})$

$$\frac{d\rho}{dt} = -rg^2 N_W [I(aa^\dagger\rho - a\rho a^\dagger + \rho a^\dagger a) + \text{H.c.}] , \tag{16}$$

where

$$\text{Re}I = \text{Re}\int_t^{t+T} dt' \int_t^{t'} dt'' e^{i\beta(t'-t'')} = \frac{1}{2}\left[\frac{\sin(\beta T/2)}{\beta/2}\right]^2 . \tag{17}$$

Now we proceed in the standard way to derive the Fokker-Planck equation from the master equation.

First, we write the density matrix in the P representation

$$\rho = \int d^2\alpha\, P(\alpha,\alpha^*,t)\,|\alpha\rangle\langle\alpha| . \tag{18}$$

Next, we use the following properties:[6]

$$\begin{aligned} a^\dagger|\alpha\rangle\langle\alpha| &= \left[\frac{\partial}{\partial\alpha} + \alpha^*\right]|\alpha\rangle\langle\alpha| , \\ |\alpha\rangle\langle\alpha|a^\dagger &= \alpha^*|\alpha\rangle\langle\alpha| , \\ |\alpha\rangle\langle\alpha|a &= \left[\frac{\partial}{\partial\alpha^*} + \alpha\right]|\alpha\rangle\langle\alpha| , \\ a|\alpha\rangle\langle\alpha| &= \alpha|\alpha\rangle\langle\alpha| . \end{aligned} \tag{19}$$

If we combine Eq. (16), (18), and (19), we easily obtain

$$\frac{\partial P(\alpha,\alpha^*,t)}{\partial t} = 2rg^2 N_W \,\text{Re}(I)\frac{\partial^2 P(\alpha,\alpha^*,t)}{\partial\alpha\partial\alpha^*} . \tag{20}$$

For the purpose of determining the FEL's linewidth, it is convenient to express Eq. (20) in polar coordinates.

If we write

$$\alpha = re^{i\theta} , \tag{21}$$

it is simple to prove that[6]

$$\frac{\partial^2}{\partial\alpha\partial\alpha^*} = \frac{1}{4r^2}\left[r\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial\theta^2}\right] , \tag{22}$$

so that Eq. (20) in polar coordinates reads

$$\frac{\partial P(|\alpha|,\theta,t)}{\partial t}=\frac{rg^2N_W}{4|\alpha|^2}\left[\frac{\sin\beta T/2}{\beta/2}\right]^2 \times\left[|\alpha|\frac{\partial}{\partial|\alpha|}|\alpha|\frac{\partial}{\partial|\alpha|}+\frac{\partial^2}{\partial\theta^2}\right] \times P(|\alpha|,\theta,t)\ . \qquad (23)$$

IV. COMMENTS AND DISCUSSIONS

Equation (23) is the central result of this paper. If we assume an extremely stabilized free-electron laser, that is, if we neglect the intensity fluctuations and make $|\alpha|^2=n_s$, one gets

$$\frac{\partial P}{\partial t}=\frac{D}{2}\frac{\partial^2 P}{\partial\theta^2}\ , \qquad (24)$$

where

$$D=\frac{rg^2N_WT^2}{2n_s}\left[\frac{\sin(\beta T/2)}{\beta T/2}\right]^2 \qquad (25)$$

is the ultimate linewidth of the free-electron laser. This result agrees with previous calculations.[3,4]

We also may write

$$D=\frac{r_{sp}}{2n_s}\ , \qquad (26)$$

where r_{sp} is the rate of spontaneous emission. By spontaneous emission in the FEL is meant, magnetic scattering into the vacuum, with no prior laser photons present. It is of interest to note that the FEL linewidth given by Eq. (25) is classical. This linewidth is wider than the atomic one when one is in the classical regime.

If one defines the recoil energy by[7]

$$\hbar\omega=(\hbar k)^2/2m\ ,$$

then one can speak of a classical or quantum regime depending on whether $\omega T \ll 1$ or $\omega T \gtrsim 1$, respectively. Defining $\omega T=\epsilon$, it has been shown[8] that the ratio of the atomic and FEL linewidth is (4ϵ). So for $\epsilon \ll 1$, that is in the classical regime, the FEL linewidth is much larger than the atomic one. However, it still can be as narrow as 10^{-3} to 10^{-5} Hz in long-pulse or cw future FEL's.

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*Permanent address: Facultad de Fisica, Pontificia Universidad Católica de Chile, Casilla 6177, Santiago 22, Chile.

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