# Theory of the holographic laser: Correlated emission in a ring cavity

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We consider a ring laser whose counterpropagating modes are coupled by a spatial modulation in the gain medium. We derive a nonlinear Fokker-Planck equation for these modes. In this way we show that the diffusion coefficient of the relative phase angle may vanish in both linear and nonlinear theory.

## I. INTRODUCTION

The concept of the spontaneous-emission noise quenching in a correlated-emission laser (CEL) has been developed recently.[1] In such a device two laser modes are coherently coupled. In quantum-beat or Hanle-effect lasers three-level atoms sustain the two laser modes which correspond to transitions from two (coherently prepared) upper levels to a common ground level.[2] Diffusion in the relative phase angle between both laser modes can vanish under certain conditions. It has been shown that this noise quenching can be used to improve the quantum-noise-limited sensitivity of both laser gravity wave detectors and laser gyroscopes.[3]

With the laser-gyro problem in mind, it has recently been shown that for an active medium consisting of two-level atoms, CEL operation can be achieved in the two oppositely directed running waves via a spatial modulation of the active medium.[4] The noise due to spontaneous emission is then suppressed just as in the "three-level lasers" described above.

In an earlier work, the theory for the correlated-emission ring laser has been only formulated to first order in the atom-field interaction. Gain coefficients $\alpha_{11}$ and $\alpha_{22}$ for the two modes are obtained as well as coupling coefficients $\alpha_{12}$ and $\alpha_{21}$. Noise quenching in the relative phase angle is obtained, when these coefficients are all equal. Therefore a spatial modulation of the linear gain medium is required. However, there remains the important question "Will higher-order effects destroy this noise quenching?" That is, will the nonlinear modification lead to a spatial hole burning which tends to modify or degrade the linear CEL noise quenching? It is therefore the purpose of this paper to formulate a nonlinear theory that takes saturation in the atom-field interaction into account. As we shall see, the nonlinear theory indeed leads to a different type of noise quenching (different formal form for noise quenching) but the ultimate result is still valid. That is, CEL noise quenching is obtained even in the presence of higher-order contributions to the atom-field interaction. Since steady-state operation can only be described by the theory incorporating saturation effects,[5] this necessary condition shows that the correlated-emission ring laser is indeed a realizable situation.

The motivation for the present CEL device derives from the realm of coherent Fourier optics and holography.[6] In particular, we recall that in the process of preparing in a hologram, one radiates a film with two beams of light as indicated in Fig. 1(a). These two beams of light (the reference beam and the incident beam) interfere to produce a holographic grading or modulation in the film. We then read out the information stored in this film by probing with the original light beam which is now scattered from the striated layers of developed film to produce our new signal [Fig. 1(b)]. In this way we note that the Read beam scatters from the striated medium to produce the new signal of interest.

In a similar way one anticipates that a striated gain medium will produce a strong coupling between the two counterpropagating modes of the ring-laser gyro. This correlation will be such that the two modes are strongly correlated and this correlation is anticipated to carry over into the quantum character of the fields as well. We therefore call this type of ring laser a holographic laser (HL).

In the HL the active medium in the ring cavity consists of thin layers with a constant spacing (cf. Fig. 2). The

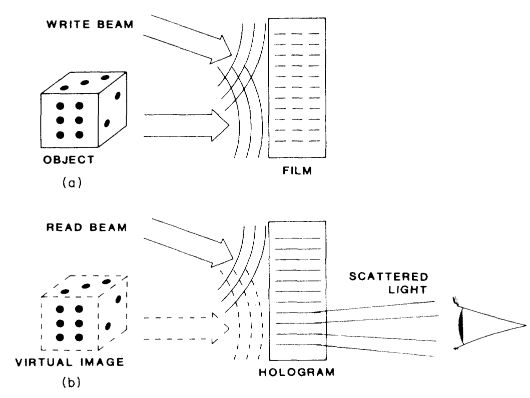


FIG. 1. (a) To create a hologram, an object beam and a reference (write) beam interfere. The interference pattern is recorded on the film. (b) After development the reference (now read) beam is scattered from the atomic layers in the hologram. From the scattered light a virtual image of the object is obtained. It should be pointed out, however, that the modulation in the hologram is varying like $\sin^2 z$ rather than having very narrow peaks.

 

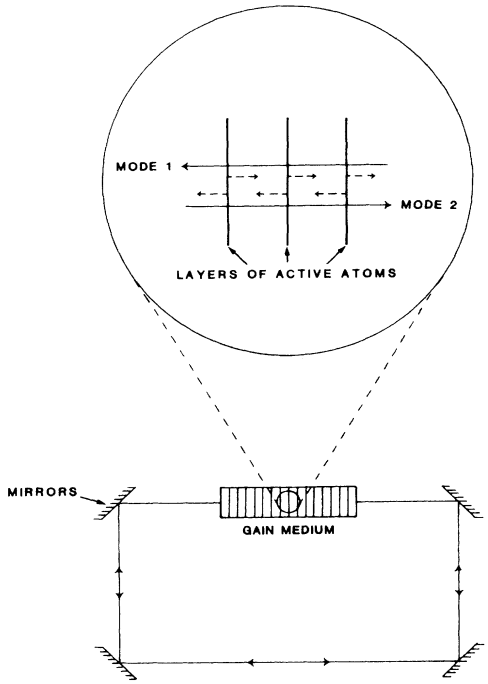


FIG. 2. In the holographic laser, each beam is reflected in part by the thin atomic layers of the gain medium (dotted arrows). When the reflected light interferes constructively with the light of the counterpropagating beam, noise quenching is achieved.

coherent coupling of the two counterpropagating modes in the ring cavity occurs by backscattering.[7,8] When light of a mode is backscattered from a layer of the gain medium, constructive interference is achieved when the phases of the reflected part of the beam and the counterpropagating beam match. The two modes in the ring cavity can be described by their wave vector $k_1, k_2$, their frequencies $\nu_1, \nu_2$, and their amplitude $E_0$ which we assume to be equal for both beams,

$$E_1 = E_0 \exp[i(k_1 z - \nu_1 t)]$$

and

$$E_2 = E_0 \exp[-i(k_2 z + \nu_2 t)] \ .$$

At the reflection at time $t_0$ the phases of both beams have to be equal up to an integer multiple of $2\pi$,

$$k_1 z_0 - \nu_1 t_0 - 2\pi j = -k_2 z_0 - \nu_2 t_0 (j = 0, \pm 1, \pm 2 \ldots) \ ,$$

where $z_0$ is the coordinate of the reflecting layer. From this we get

$$(k_1 + k_2) z_0 - (\nu_1 - \nu_2) t_0 = 2\pi j \ .$$

Since $\nu_1 - \nu_2$ contains the small signal, the second term on the left-hand side will be much smaller than the first term during the measurement time, so that the condition for $z_0$ is

$$z_0 \simeq 2\pi j / (k_1 + k_2) \simeq \pi j / k$$

since $k_1 \simeq k_2$.

We conclude from this heuristic derivation that the layers of the gain medium have to be located at $z = (\pi/k) j$ in order to get maximum coupling between the beams. In fact, the same result is obtained from our detailed analysis.

In this work we first derive the equation of motion for the field ensity operator, including saturation for the two counterpropagating beams. This equation is then transposed into a Fokker-Planck equation. Via a change of variables we then obtain a Fokker-Planck equation which involves the relative phase angle. From this expression we extract information concerning the drift and diffusion of the relative phase angle.

## II. THE MODEL FOR THE CORRELATED-EMISSION RING LASER

As in the preceding linear theory of the HL we describe both modes by annihilation and creation operators $a_1, a_1^\dagger$, and $a_2, a_2^\dagger$, respectively.[5] The modes have "bare cavity" eigenfrequencies $\Omega_1$ and $\Omega_2$, and their operating frequencies are $\nu_1$ and $\nu_2$. The losses are described by loss coefficients $\gamma_1$ and $\gamma_2$. Noting that both modes interact with the same atoms in the gain medium, the equation of motion for the field density operator is given by[4]

$$\dot{\rho}(a_1, a_1^\dagger, a_2, a_2^\dagger; t) = -i(\Omega_1 - \nu_1)[a_1^\dagger a_1, \rho] - i(\Omega_2 - \nu_2)[a_2^\dagger a_{21} \rho] - \sum_i \sum_{s_i} \langle s_i | [V^i(t), \rho^i(t)] | s_i \rangle + \mathcal{L}_1 \rho + \mathcal{L}_2 \rho \ . \quad (1)$$

The first two terms describe the "free" field oscillations, the last two terms the cavity losses with

$$\mathcal{L}_j \rho = -\tfrac{1}{2}\gamma_j (a_j^\dagger a_j \rho + \rho a_j^\dagger a_j - 2 a_j \rho a_j^\dagger), \quad j = 1,2 \ . \quad (2)$$

The interaction of the beams with the laser medium is given by the middle term in (1). $\rho^i(t)$ is the density operator for the $i$th atom and the field, obtained from the combined density operator for all atoms and the field by tracing over all atoms except the $i$th one,

$$\rho^i(t) = \mathrm{Tr}_{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_r} \rho_{\text{atom-field}}(t) \ . \quad (3)$$

The interaction potential of the $i$th atom and the field is given by

$$V^i(t) = g \, | b^i \rangle \langle a^i | \, A^\dagger(z,t) e^{-i\omega t} + (\text{adj.}) \ , \quad (4)$$

where $| a^i \rangle$ and $| b^i \rangle$ are the upper and lower state of the $i$th atom with energy separation $\omega$. $g$ is a coupling constant, and the combined operator

$$A(z,t) \equiv a_1 e^{-i\nu_1 t} u_1(z) + a_2 e^{-i\nu_2 t} u_2(z) \quad (5)$$

incorporates the normal mode functions $u_1(z)$ and $u_2(z)$. The coordinate $z$ is defined parallel to the beams so that the gain medium extends from $z = -l/2$ to $z = l/2$, and we assume that the extension of the medium perpendicular to the beam direction is independent of $z$.

Now the equation of motion for $\rho^i(t)$

$$\dot{\rho}^i(t) = -\frac{i}{\hbar}[V^i(t), \rho^i(t)] \quad (6)$$

is expanded perturbatively in the usual way,

$$\rho^i(t)=\rho^i(t_0)-\frac{i}{\hbar}\int_{t_0}^{t}dt'[V^i(t'),\rho_a^i(t_0)\otimes\rho(t)]-\frac{1}{\hbar^2}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''[V^i(t'),[V^i(t''),\rho_a^i(t_0)\otimes\rho(t)]]$$
$$+\frac{i}{\hbar^3}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\int_{t_0}^{t''}dt'''[V^i(t'),[V^i(t''),[V^i(t'''),\rho_a^i(t_0)\otimes\rho(t)]]]\ . \tag{7}$$

Note that the influence of a single atom on the field is very small. Therefore we have set $\rho(t_0)\simeq\rho(t)$ in Eq. (7).[9] The first order of this expansion gives the gain, and the third order the saturation, while the second order does not contribute because of the trace in Eq. (1).

With the help of this expansion the contribution from the gain medium in Eq. (1) can be rewritten and the equation of motion for the field density operator $\rho$ is then

$$\dot\rho(a_1,a_1^\dagger,a_2,a_2^\dagger;t)=-i(\Omega_1-\nu_1)[a_1^\dagger a_1,\rho]-i(\Omega_2-\nu_2)[a_2a_2^\dagger,\rho]$$
$$-\frac{\alpha_0}{2}\int_{-l/2}^{l/2}dz\,n(z)[A(z,t)A^\dagger(z,t)\rho+\rho A(z,t)A^\dagger(z,t)-2A^\dagger(z,t)\rho A(z,t)]$$
$$+\frac{\beta_0}{8}\int_{-l/2}^{l/2}dz\,n(z)[A(z,t)A^\dagger(z,t)A(z,t)A^\dagger(z,t)\rho+\rho A(z,t)A^\dagger(z,t)A(z,t)A^\dagger(z)$$
$$+6A(z,t)A^\dagger(z,t)\rho A(z,t)A^\dagger(z,t)$$
$$-4A^\dagger(z,t)\rho A(z,t)A^\dagger(z,t)A(z,t)-4A^\dagger(z,t)A(z,t)A^\dagger(z,t)\rho A(z,t)]+\mathcal{L}_1\rho+\mathcal{L}_2\rho\ . \tag{8}$$

Here $\alpha_0$ is a gain, $\beta_0$ is a saturation coefficient, and the summation over all atoms has been transformed into a spatial integration over the linear density $n(z)$ of gain atoms.[10]

## III. THE FOKKER-PLANCK EQUATION

We now transform the equation of motion for $\rho$ into a Fokker-Planck equation.[11,12] The density operator $\rho$ is expressed in the coherent state representation with a distribution function $P(\mathcal{E}_1,\mathcal{E}_2)$,

$$\rho\equiv\int d^2\mathcal{E}_1\int d^2\mathcal{E}_2P(\mathcal{E}_1,\mathcal{E}_2)\,|\,\mathcal{E}_1\rangle\langle\mathcal{E}_1\,|\otimes|\,\mathcal{E}_2\rangle\langle\mathcal{E}_2\,|\ . \tag{9}$$

$\mathcal{E}_1$ and $\mathcal{E}_2$ are complex numbers corresponding to mode 1 and 2, defined by

$$a_j\,|\,\mathcal{E}_j\rangle=\mathcal{E}_j\,|\,\mathcal{E}_j\rangle \quad\text{and}\quad a_j^+\,|\,\mathcal{E}_j\rangle=\left[\frac{\partial}{\partial\mathcal{E}_j}+\frac{\mathcal{E}_j^*}{2}\right]|\,\mathcal{E}_j\rangle\quad(\text{for } j=1,2)\ . \tag{10}$$

The Fokker-Planck equation for $P(\mathcal{E}_1,\mathcal{E}_2)$ is now obtained from a lengthy but straightforward calculation. We find

$$\dot P(\mathcal{E}_1,\mathcal{E}_1^*,\mathcal{E}_2,\mathcal{E}_2^*;t)=\frac{\alpha_{11}}{2}\frac{\partial^2P}{\partial\mathcal{E}_1\partial\mathcal{E}_1^*}+\frac{\alpha_{22}}{2}\frac{\partial^2P}{\partial\mathcal{E}_2\partial\mathcal{E}_2^*}+\alpha_{12}e^{-i\phi}\frac{\partial^2P}{\partial\mathcal{E}_1^*\partial\mathcal{E}_2}-i(\Omega_1-\nu_1)P+[i(\Omega_1-\nu_1)+\tfrac{1}{2}(\gamma_1-\alpha_{11})]\frac{\partial}{\partial\mathcal{E}_1}(\mathcal{E}_1P)$$
$$-i(\Omega_2-\nu_2)P+[i(\Omega_2-\nu_2)+\tfrac{1}{2}(\gamma_2-\alpha_{22})]\frac{\partial}{\partial\mathcal{E}_2}(\mathcal{E}_2P)-\tfrac{1}{2}\alpha_{12}e^{-i\phi}\left[\mathcal{E}_1\frac{\partial P}{\partial\mathcal{E}_2}+\mathcal{E}_2^*\frac{\partial P}{\partial\mathcal{E}_1^*}\right]$$
$$+\tfrac{1}{2}\beta_{11;11}\frac{\partial}{\partial\mathcal{E}_1}(\mathcal{E}_1^2\mathcal{E}_1^*P)-\tfrac{1}{8}\beta_{11;11}\left[3\mathcal{E}_1^2\frac{\partial^2P}{\partial\mathcal{E}_1^2}+5\mathcal{E}_1\mathcal{E}_1^*\frac{\partial^2P}{\partial\mathcal{E}_1\partial\mathcal{E}_1^*}\right]+\tfrac{1}{2}\beta_{22;22}\frac{\partial}{\partial\mathcal{E}_2}(\mathcal{E}_2^2\mathcal{E}_2^*P)$$
$$-\tfrac{1}{8}\beta_{22;22}\left[3\mathcal{E}_2^2\frac{\partial^2P}{\partial\mathcal{E}_2^2}+5\mathcal{E}_2\mathcal{E}_2^*\frac{\partial^2P}{\partial\mathcal{E}_2\partial\mathcal{E}_2^*}\right]+\beta_{12;12}\left[\mathcal{E}_1\mathcal{E}_1^*\frac{\partial}{\partial\mathcal{E}_2}(\mathcal{E}_2P)+\mathcal{E}_2\mathcal{E}_2^*\frac{\partial}{\partial\mathcal{E}_1}(\mathcal{E}_1P)\right]$$
$$-\tfrac{1}{8}\beta_{12;12}\left[5\mathcal{E}_1\mathcal{E}_1^*\frac{\partial^2P}{\partial\mathcal{E}_2\partial\mathcal{E}_2^*}+5\mathcal{E}_2\mathcal{E}_2^*\frac{\partial^2P}{\partial\mathcal{E}_1\partial\mathcal{E}_1^*}+12\mathcal{E}_1\mathcal{E}_2\frac{\partial^2P}{\partial\mathcal{E}_1\partial\mathcal{E}_2}+10\mathcal{E}_1\mathcal{E}_2^*\frac{\partial^2P}{\partial\mathcal{E}_1\partial\mathcal{E}_2^*}\right]$$
$$+\tfrac{1}{8}\beta_{11;12}e^{-i\phi}\left[8\frac{\partial}{\partial\mathcal{E}_1^*}(\mathcal{E}_1\mathcal{E}_1^*\mathcal{E}_2^*P)+4\frac{\partial}{\partial\mathcal{E}_1}(\mathcal{E}_1^2\mathcal{E}_2^*P)+4\mathcal{E}_1^2\mathcal{E}_1^*\frac{\partial P}{\partial\mathcal{E}_2}-10\mathcal{E}_1\mathcal{E}_2^*\frac{\partial^2P}{\partial\mathcal{E}_1\partial\mathcal{E}_1^*}-6\mathcal{E}_1^*\mathcal{E}_2^*\frac{\partial^2P}{\partial\mathcal{E}_1^{*2}}\right.$$
$$\left.-6\mathcal{E}_1^2\frac{\partial^2P}{\partial\mathcal{E}_1\partial\mathcal{E}_2}-10\mathcal{E}_1\mathcal{E}_1^*\frac{\partial^2P}{\partial\mathcal{E}_1^*\partial\mathcal{E}_2}\right]$$
$$+\tfrac{1}{8}\beta_{12;22}e^{-i\phi}\left[8\frac{\partial}{\partial\mathcal{E}_2}(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_2^*P)+4\frac{\partial}{\partial\mathcal{E}_2^*}(\mathcal{E}_1\mathcal{E}_2^{*2}P)+4\mathcal{E}_2\mathcal{E}_2^{*2}\frac{\partial P}{\partial\mathcal{E}_1^*}-10\mathcal{E}_1\mathcal{E}_2^*\frac{\partial^2P}{\partial\mathcal{E}_1^*\partial\mathcal{E}_2}-6\mathcal{E}_1\mathcal{E}_2\frac{\partial^2P}{\partial\mathcal{E}_2^2}\right.$$
$$\left.-6\mathcal{E}_2^{*2}\frac{\partial^2P}{\partial\mathcal{E}_1^*\partial\mathcal{E}_2^*}-10\mathcal{E}_2\mathcal{E}_2^*\frac{\partial^2P}{\partial\mathcal{E}_1^*\partial\mathcal{E}_2}\right]$$
$$+\tfrac{1}{8}\beta_{11;22}e^{-2i\phi}\left[4\mathcal{E}_1\mathcal{E}_2^*\left[\mathcal{E}_1\frac{\partial P}{\partial\mathcal{E}_2}+\mathcal{E}_2^*\frac{\partial P}{\partial\mathcal{E}_1^*}\right]-10\mathcal{E}_1\mathcal{E}_2^*\frac{\partial^2P}{\partial\mathcal{E}_1^*\partial\mathcal{E}_2}-3\mathcal{E}_2^{*2}\frac{\partial^2P}{\partial\mathcal{E}_1^{*2}}-3\mathcal{E}_1^2\frac{\partial^2P}{\partial\mathcal{E}_2^2}\right]+\text{c.c.} \tag{11}$$

The integration in Eq. (9) involves the normal mode functions and the density of atoms in the gain medium. The coefficients in (11) are therefore found to be (for $i,j,k,m=1,2$)

$$\alpha_{ij}\equiv\alpha_0\int_{-l/2}^{l/2}u_i(z)u_j^*(z)n(z)dz\ ,$$
$$\beta_{ij;km}\equiv\beta_0\int_{-l/2}^{l/2}u_i(z)u_j(z)u_k^*(z)u_m^*(z)n(z)dz\ . \tag{12}$$

The angle $\phi$ is given by

$$\phi=(\nu_1-\nu_2)t\ . \tag{13}$$

To obtain the Fokker-Planck equation (12) we have assumed

$$|\beta_{ij;km}|\ll|\alpha_{i'j'}|\ ,$$

but

$$|\beta_{ij;km}|\,|\mathcal{E}_1|^2\simeq|\beta_{ij;km}|\,|\mathcal{E}_2|^2\lesssim|\alpha_{i'j'}|\ .$$

In this approximation, we obtain also the mean time derivative of $\mathcal{E}_1$ and $\mathcal{E}_2$ from the drift coefficients of the Fokker-Planck equation. The noise-induced drift can be neglected (cf. Appendix). We then have

$$\dot{\mathcal{E}}_1=-i(\Omega_1-\nu_1)\mathcal{E}_1+\tfrac{1}{2}(\alpha_{11}-\gamma_1)\mathcal{E}_1+\tfrac{1}{2}\alpha_{12}^*e^{i\phi}\mathcal{E}_2-\tfrac{1}{2}\beta_{11;11}\mathcal{E}_1^2\mathcal{E}_1^*-\beta_{12;12}\mathcal{E}_1\mathcal{E}_2\mathcal{E}_2^*$$
$$-\tfrac{1}{2}\beta_{11;12}e^{-i\phi}\mathcal{E}_1^2\mathcal{E}_2^*-\beta_{11;12}^*e^{i\phi}\mathcal{E}_1\mathcal{E}_1^*\mathcal{E}_2-\tfrac{1}{2}\beta_{12;22}^*e^{i\phi}\mathcal{E}_2^2\mathcal{E}_2^*-\tfrac{1}{2}\beta_{11;22}^*e^{2i\phi}\mathcal{E}_1^*\mathcal{E}_2^2 \tag{14a}$$

and

$$\dot{\mathcal{E}}_2=-i(\Omega_2-\nu_2)\mathcal{E}_2+\tfrac{1}{2}(\alpha_{22}-\gamma_2)\mathcal{E}_2+\tfrac{1}{2}\alpha_{12}e^{-i\phi}\mathcal{E}_1-\tfrac{1}{2}\beta_{22;22}\mathcal{E}_2^2\mathcal{E}_2^*-\beta_{12;12}\mathcal{E}_1\mathcal{E}_1^*\mathcal{E}_2$$
$$-\tfrac{1}{2}\beta_{11;12}^*e^{i\phi}\mathcal{E}_1^*\mathcal{E}_2^2-\beta_{11;12}e^{-i\phi}\mathcal{E}_1\mathcal{E}_2\mathcal{E}_2^*-\tfrac{1}{2}\beta_{12;22}e^{-i\phi}\mathcal{E}_1^2\mathcal{E}_1^*-\tfrac{1}{2}\beta_{11;22}e^{-2i\phi}\mathcal{E}_1^2\mathcal{E}_2^*\ . \tag{14b}$$

Both differential equations are coupled in the linear theory (by $\alpha_{12}$) as well as in the nonlinear theory (by $\beta_{12;12}$, $\beta_{11;12}$, $\beta_{12;22}$, and $\beta_{11;22}$, these coefficients take account of the mutual saturation of both beams). In addition, we have the usual gain and saturation terms for the two beams.

Since we are interested in the relative phase angle between the modes, we substitute new variables for $\mathcal{E}_1$ and $\mathcal{E}_2$ in a next step. From the definitions

$$\mathcal{E}_1\equiv\rho_1e^{-i\theta_1}\ \text{and}\ \mathcal{E}_2\equiv\rho_2e^{-i\theta_2} \tag{15}$$

we take $\rho_1,\rho_2,\varphi\equiv\frac{1}{2}(\theta_1+\theta_2)$, and $\theta\equiv\theta_1-\theta_2$ as new variables. Expressing the derivatives in Eq. (12) in terms of the new variables, we obtain after another straightforward calculation the following Fokker-Planck equation for the distribution function $P(\rho_1,\rho_2,\varphi,\theta;t)$:

$$\dot P(\rho_1,\rho_2,\varphi,\theta;t)=d_0P-d(\rho_1)\frac{\partial P}{\partial\rho_1}-d(\rho_2)\frac{\partial P}{\partial\rho_2}-d(\varphi)\frac{\partial P}{\partial\varphi}-d(\theta)\frac{\partial P}{\partial\theta}+D(\rho_1)\frac{\partial^2P}{\partial\rho_1^2}+D(\rho_2)\frac{\partial^2P}{\partial\rho_2^2}$$
$$+D(\varphi)\frac{\partial^2P}{\partial\varphi^2}+D(\theta)\frac{\partial^2P}{\partial\theta^2}+D(\rho_1,\rho_2)\frac{\partial^2P}{\partial\rho_1\partial\rho_2}+D(\rho_1,\varphi)\frac{\partial^2P}{\partial\rho_1\partial\varphi}+D(\rho_1,\theta)\frac{\partial^2P}{\partial\rho_1\partial\theta}$$
$$+D(\rho_2,\varphi)\frac{\partial^2P}{\partial\rho_2\partial\varphi}+D(\rho_2,\theta)\frac{\partial^2P}{\partial\rho_2\partial\theta}+D(\varphi,\theta)\frac{\partial^2P}{\partial\varphi\,\partial\theta}\ . \tag{16}$$

The coefficients of Eq. (16) are given in Table I, where we have defined $\delta_0$, $\delta_1$, $\delta_2$, and $2\delta_3$ as the arguments of the complex numbers $\alpha_{12}$, $\beta_{11;12}$, $\beta_{12;22}$, and $\beta_{11;22}$, respectively, and the angle $\psi$ as

$$\psi\equiv\theta+\phi=\theta+(\nu_1-\nu_2)t\ . \tag{17}$$

## IV. DIFFUSION OF THE RELATIVE PHASE ANGLE

We now take a closer look to the diffusion coefficient of the relative phase angle $\theta$ (Table I). From now on we assume for simplicity that all gain coefficients and all saturation coefficients are real, i.e., $\delta_0=\delta_1=\delta_2=\delta_3=0$. Then we have

$$D(\theta)=\frac{\alpha_{11}}{4\rho_1^2}+\frac{\alpha_{22}}{4\rho_2^2}-\frac{\alpha_{12}}{2\rho_1\rho_2}\cos\psi-\tfrac{1}{8}\left\{\beta_{11;11}+\beta_{22;22}+\beta_{12;12}\left[\frac{5}{2}\left[\frac{\rho_1^2}{\rho_2^2}+\frac{\rho_2^2}{\rho_1^2}\right]+1\right]-\beta_{11;22}\left[5+\frac{3}{2}\left[\frac{\rho_1^2}{\rho_2^2}+\frac{\rho_2^2}{\rho_1^2}\right]\right]\cos(2\psi)\right\}$$
$$+\tfrac{1}{4}(\beta_{11;12}-\beta_{12;22})\left[\frac{\rho_1}{\rho_2}-\frac{\rho_2}{\rho_1}\right]\cos\psi\ , \tag{18}$$

TABLE I. The coefficients of Eq. (16).

$$d_0=\gamma_1-\alpha_{11}+\gamma_2-\alpha_{22}+2\beta_{11;11}\rho_1^2+2\beta_{22;22}\rho_2^2+2\beta_{12;12}(\rho_1^2+\rho_2^2)+4\,|\beta_{11;12}|\,\rho_1\rho_2\cos(\psi-\delta_1)+4\,|\beta_{12;22}|\,\rho_1\rho_2\cos(\psi-\delta_2)\ ,$$

$$d(\rho_1)=\tfrac{1}{2}(\alpha_{11}-\gamma_1)\rho_1+\tfrac{1}{2}\,|\alpha_{12}|\,\rho_2\cos(\psi-\delta_0)-\tfrac{1}{2}\beta_{11;11}\rho_1^3-\tfrac{3}{2}\,|\beta_{11;12}|\,\rho_1^2\rho_2\cos(\psi-\delta_1)-\tfrac{1}{2}\,|\beta_{12;22}|\,\rho_2^3\cos(\psi-\delta_2)-\beta_{12;12}\rho_1\rho_2^2$$
$$-\tfrac{1}{2}\,|\beta_{11;22}|\,\rho_1\rho_2^2\cos(2\psi-2\delta_3)\ ,$$

$$d(\rho_2)=\tfrac{1}{2}(\alpha_{22}-\gamma_2)\rho_2+\tfrac{1}{2}\,|\alpha_{12}|\,\rho_1\cos(\psi-\delta_0)-\tfrac{1}{2}\beta_{22;22}\rho_2^3-\tfrac{1}{2}\,|\beta_{11;12}|\,\rho_1^3\cos(\psi-\delta_1)$$
$$-\tfrac{3}{2}\,|\beta_{12;22}|\,\rho_1\rho_2^2\cos(\psi-\delta_2)-\beta_{12;12}\rho_1^2\rho_2-\tfrac{1}{2}\,|\beta_{11;22}|\,\rho_1^2\rho_2\cos(2\psi-2\delta_3)\ ,$$

$$d(\varphi)=\tfrac{1}{2}(\nu_1+\nu_2-\Omega_1-\Omega_2)+\tfrac{1}{4}\,|\alpha_{12}|\left[\frac{\rho_1}{\rho_2}-\frac{\rho_2}{\rho_1}\right]\sin(\psi-\delta_0)-\tfrac{1}{4}\,|\beta_{11;12}|\,\frac{\rho_1}{\rho_2}(\rho_1^2-\rho_2^2)\sin(\psi-\delta_1)$$
$$-\tfrac{1}{4}\,|\beta_{12;22}|\,\frac{\rho_2}{\rho_1}(\rho_1^2-\rho_2^2)\sin(\psi-\delta_2)-\tfrac{1}{4}\,|\beta_{11;22}|\,(\rho_1^2-\rho_2^2)\sin(2\psi-2\delta_3)\ ,$$

$$d(\theta)=\Omega_1-\nu_1-(\Omega_2-\nu_2)-\tfrac{1}{2}\,|\alpha_{12}|\left[\frac{\rho_1}{\rho_2}+\frac{\rho_2}{\rho_1}\right]\sin(\psi-\delta_0)+\tfrac{1}{2}\,|\beta_{11;12}|\,\frac{\rho_1}{\rho_2}(\rho_1^2+\rho_2^2)\sin(\psi-\delta_1)$$
$$+\tfrac{1}{2}\,|\beta_{12;22}|\,\frac{\rho_2}{\rho_1}(\rho_1^2+\rho_2^2)\sin(\psi-\delta_2)+\tfrac{1}{2}\,|\beta_{11;22}|\,(\rho_1^2+\rho_2^2)\sin(2\psi-2\delta_3)\ ,$$

$$D(\rho_1)=\tfrac{1}{4}\alpha_{11}-\tfrac{1}{2}\beta_{11;11}\rho_1^2-\tfrac{5}{16}\beta_{12;12}\rho_2^2-|\beta_{11;12}|\,\rho_1\rho_2\cos(\psi-\delta_1)-\tfrac{3}{16}\,|\beta_{11;22}|\,\rho_2^2\cos(2\psi-2\delta_3)\ ,$$

$$D(\rho_2)=\tfrac{1}{4}\alpha_{22}-\tfrac{1}{2}\beta_{22;22}\rho_2^2-\tfrac{5}{16}\beta_{12;12}\rho_1^2-|\beta_{12;22}|\,\rho_1\rho_2\cos(\psi-\delta_2)-\tfrac{3}{16}\,|\beta_{11;22}|\,\rho_1^2\cos(2\psi-2\delta_3)\ ,$$

$$D(\varphi)=\frac{1}{16}\left[\frac{\alpha_{11}}{\rho_1^2}+\frac{\alpha_{22}}{\rho_2^2}+\frac{2\,|\alpha_{12}|}{\rho_1\rho_2}\cos(\psi-\delta_0)\right]-\frac{1}{32}\left\{\beta_{11;11}+\beta_{22;22}+\beta_{12;12}\left[\frac{5}{2}\left[\frac{\rho_1^2}{\rho_2^2}+\frac{\rho_2^2}{\rho_1^2}\right]-1\right]\right.$$
$$+|\beta_{11;22}|\left[5-\frac{3}{2}\left[\frac{\rho_1^2}{\rho_2^2}+\frac{\rho_2^2}{\rho_1^2}\right]\right]\cos(2\psi-2\delta_3)$$
$$\left.+2[\,|\beta_{11;12}|\cos(\psi-\delta_1)+|\beta_{12;22}|\cos(\psi-\delta_2)]\left[\frac{\rho_1}{\rho_2}+\frac{\rho_2}{\rho_1}\right]\right\}$$

$$D(\theta)=\frac{1}{4}\left[\frac{\alpha_{11}}{\rho_1^2}+\frac{\alpha_{22}}{\rho_2^2}-\frac{2\,|\alpha_{12}|}{\rho_1\rho_2}\cos(\psi-\delta_0)\right]-\frac{1}{8}\left\{\beta_{11;11}+\beta_{22;22}+\beta_{12;12}\left[\frac{5}{2}\left[\frac{\rho_1^2}{\rho_2^2}+\frac{\rho_2^2}{\rho_1^2}\right]+1\right]\right.$$
$$\left.-|\beta_{11;22}|\left[5+\frac{3}{2}\left[\frac{\rho_1^2}{\rho_2^2}+\frac{\rho_2^2}{\rho_1^2}\right]\right]\cos(2\psi-2\delta_3)\right\}$$
$$+\tfrac{1}{4}[\,|\beta_{11;12}|\cos(\psi-\delta_1)-|\beta_{12;22}|\cos(\psi-\delta_2)]\left[\frac{\rho_1}{\rho_2}-\frac{\rho_2}{\rho_1}\right]\ ,$$

$$D(\rho_1,\rho_2)=\tfrac{1}{2}\,|\alpha_{12}|\cos(\psi-\delta_0)-\tfrac{1}{8}[11\beta_{12;12}\rho_1\rho_2+8\,|\beta_{11;12}|\,\rho_1^2\cos(\psi-\delta_1)+8\,|\beta_{12;22}|\,\rho_2^2\cos(\psi-\delta_2)+5\,|\beta_{11;22}|\,\rho_1\rho_2\cos(2\psi-2\delta_3)]\ ,$$

$$D(\rho_1,\varphi)=\frac{|\alpha_{12}|}{4\rho_2}\sin(\psi-\delta_0)-\frac{1}{8}\left[\frac{4\rho_1^2-3\rho_2^2}{\rho_2}\,|\beta_{11;12}|\sin(\psi-\delta_1)+|\beta_{12;22}|\,\rho_2\sin(\psi-\delta_2)+\frac{5\rho_1^2-3\rho_2^2}{2\rho_1}\,|\beta_{11;22}|\sin(2\psi-2\delta_3)\right]\ ,$$

$$D(\rho_2,\varphi)=-\frac{|\alpha_{12}|}{4\rho_1}\sin(\psi-\delta_0)+\frac{1}{8}\left[\,|\beta_{11;12}|\,\rho_1\sin(\psi-\delta_1)+\frac{4\rho_2^2-3\rho_1^2}{\rho_1}\,|\beta_{12;22}|\sin(\psi-\delta_2)+\frac{5\rho_2^2-3\rho_1^2}{2\rho_2}\,|\beta_{11;22}|\sin(2\psi-2\delta_3)\right]\ ,$$

$$D(\rho_1,\theta)=-\frac{|\alpha_{12}|}{2\rho_2}\sin(\psi-\delta_0)+\frac{1}{8}\left[2\rho_2\,|\beta_{12;22}|\sin(\psi-\delta_2)+\frac{8\rho_1^2+6\rho_2^2}{\rho_2}\,|\beta_{11;12}|\sin(\psi-\delta_1)+\frac{5\rho_1^2+3\rho_2^2}{\rho_1}\,|\beta_{11;22}|\sin(2\psi-2\delta_3)\right]\ ,$$

$$D(\rho_2,\theta)=-\frac{|\alpha_{12}|}{2\rho_1}\sin(\psi-\delta_0)+\frac{1}{8}\left[2\rho_1\,|\beta_{11;12}|\sin(\psi-\delta_1)+\frac{8\rho_2^2+6\rho_1^2}{\rho_1}\,|\beta_{12;22}|\sin(\psi-\delta_2)+\frac{5\rho_2^2+3\rho_1^2}{\rho_2}\,|\beta_{11;22}|\sin(2\psi-2\delta_3)\right]\ ,$$

$$D(\varphi,\theta)=\frac{1}{4}\left[\frac{\alpha_{11}}{\rho_1^2}-\frac{\alpha_{22}}{\rho_2^2}\right]+\frac{1}{8}\left[-\beta_{11;11}+\beta_{22;22}+\tfrac{5}{2}\beta_{12;12}\left[\frac{\rho_1^2}{\rho_2^2}-\frac{\rho_2^2}{\rho_1^2}\right]-2\,|\beta_{11;12}|\,\frac{\rho_2}{\rho_1}\cos(\psi-\delta_1)\right.$$
$$\left.+2\,|\beta_{12;22}|\,\frac{\rho_1}{\rho_2}\cos(\psi-\delta_2)-\tfrac{3}{2}\,|\beta_{11;22}|\left[\frac{\rho_1^2}{\rho_2^2}-\frac{\rho_2^2}{\rho_1^2}\right]\cos(2\psi-2\delta_3)\right]\ .$$

In addition we require that both modes have the same average number of photons, i.e., $\rho_1^2=\rho_2^2\equiv\rho^2$. With this condition, Eq. (18) simplifies to

$$D(\theta)=\frac{1}{4\rho^2}[\alpha_{11}+\alpha_{22}-2\alpha_{12}\cos\psi] -\tfrac{1}{8}[\beta_{11;11}+\beta_{22;22}+6\beta_{12;12} -8\beta_{11;22}\cos(2\psi)] \ . \tag{18a}$$

How the coefficients $\alpha_{11}$, $\alpha_{22}$, $\alpha_{12}$, and $\beta_{11;11}$, $\beta_{22;22}$, $\beta_{12;12}$, and $\beta_{11;22}$ are related to each other depends on the modulation of the gain medium, i.e., on the spatial variation of $n(z)$. Our aim is to obtain equality of all gain coefficients and equality of all saturation coefficients, i.e., $\alpha_{11}=\alpha_{22}=\alpha_{12}\equiv\alpha$ and $\beta_{11;11}=\beta_{22;22}=\beta_{12;12}=\beta_{11;22}\equiv\beta$. How we are able to achieve this will be investigated in the following chapter. Here we assume now that this condition is fulfilled, and Eq. (18a) simplifies to

$$\begin{aligned} D(\theta)&=\frac{\alpha}{2\rho^2}(1-\cos\psi)-\beta(1-\cos2\psi)\\ &=\frac{\alpha}{2\rho^2}(1-\cos\psi)-2\beta(1-\cos^2\psi)\\ &=\left[\frac{\alpha}{2\rho^2}-2\beta(1+\cos\psi)\right](1-\cos\psi)\ . \end{aligned} \tag{18b}$$

We see that the diffusion coefficient $D(\theta)$ vanishes for $\psi=0$ also when nonlinear saturation effects are taken into account. Thus we see that the quantum noise quenching obtained for the Hanle effect and quantum beat CEL's is recovered and remains valid in the nonlinear regime.

## V. MODULATION OF THE GAIN MEDIUM

For traveling waves the normal mode functions $u_1(z)$ and $u_2(z)$ are given by

$$u_1(z)=e^{ikz} \text{ and } u_2(z)=e^{-ikz}\ . \tag{19}$$

Then from Eq. (13), using the fact that $|u_1|^2=|u_2|^2=1$, we have

$$\begin{aligned} \alpha_{jj}&=\alpha_0\int_{-l/2}^{l/2} n(z)\,|u_j(z)|^2dz\\ &=\alpha_0\int_{-l/2}^{l/2} n(z)dz \quad (j=1,2)\ ,\\ \alpha_{12}&=\alpha_0\int_{-l/2}^{l/2} n(z)u_1(z)u_2^*(z)dz=\alpha_0\int_{-l/2}^{l/2} n(z)e^{2ikz}dz\ ,\\ \beta_{11;11}&=\beta_{12;12}=\beta_{22;22}=\beta_0\int_{-l/2}^{l/2} n(z)dz\ ,\\ \beta_{11;12}&=\beta_{12;22}=\beta_0\int_{-l/2}^{l/2} n(z)e^{2ikz}dz\ ,\\ \beta_{11;22}&=\beta_0\int_{-l/2}^{l/2} n(z)u_1^2(z)u_2^{*2}(z)dz=\beta_0\int_{-l/2}^{l/2} n(z)e^{4ikz}dz\ . \end{aligned} \tag{20}$$

If the gain medium is not modulated, i.e., $n(z)=n_0=$ const., $\alpha_{12}$ and $\beta_{11;22}$ are of the order of $1/k$, which is very small at optical frequencies. In this case, we no longer obtain noise quenching in the relative phase angle, and from Eq. (18a) we have

$$D(\theta)=\frac{\alpha}{2\rho^2}-\beta\ . \tag{18a$'$}$$

From Eq. (20) we conclude that a modulation for which the gain and saturation coefficients are equal must have the density of atoms mainly at points where $e^{2ikz}\simeq 1$, at $z=\pi j/k$ where $j$ is an integer. In principle we would like to have $\delta$ functions there but in reality there will be peaks in the density with a width $\Delta$. When we assume that these peaks are of Gaussian type, the density function $n(z)$ between $z=-l/2$ and $l/2$ looks as follows:

$$n(z)=\sum_{j=-j_0}^{j_0}\frac{n_0}{2j_0+1}\left[\frac{\ln2}{\pi}\right]^{1/2}\frac{1}{\Delta}\times\exp\left[-\frac{\ln2}{\Delta^2}\left[z-\frac{\pi}{k}j\right]^2\right] \tag{21}$$

with $j=[lk/2\pi]$ and $n_0=$ const.

With this equation we obtain for the integrals of Eq. (20)

$$\int_{-l/2}^{l/2} n(z)dz=n_0\ ,$$

$$\int_{-l/2}^{l/2} n(z)e^{2ikz}dz=n_0\exp\left[-\frac{\Delta^2k^2}{\ln2}\right]\ ,$$

and

$$\int_{-l/2}^{l/2} n(z)e^{4ikz}dz=n_0\exp\left[-\frac{4\Delta^2k^2}{\ln2}\right]\ . \tag{22}$$

We see that all integrals are approximately equal if the width $\Delta$ is much smaller than $1/k$. This result [Eq. (22)] is still valid when only every $n$th peak in Eq. (21) is nonzero ($n$ positive integer). From Eq. (22) we see immediately that all coefficients for gain are equal ($\alpha_{11}=\alpha_{22}=\alpha_{12}$) as well as all coefficients for saturation ($\beta_{11;11}=\beta_{22;22}=\beta_{12;12}=\beta_{11;12}=\beta_{12;22}=\beta_{11;22}$), which is required to obtain a vanishing diffusion coefficient $D(\theta)$.

As an example of a possible device, one could envision optically pumping narrow regions of the active medium via a combination of interfering pump beams.

## VI. MEAN MOTION OR DRIFT OF AMPLITUDE AND RELATIVE PHASE ANGLE

We now examine the drift coefficient $d(\theta)$ for the relative phase angle (cf. Table I). In the Langevin equation the time development of $\theta$ is given by a velocity and a noise term. The drift coefficient in the corresponding Fokker-Planck equation contains the velocity, but in general there is also a noise-induced drift.[13] However, this noise-induced drift can be neglected here as is shown in the Appendix.

Therefore we can write

$$\begin{aligned} \dot\theta\simeq d(\theta)=\Omega_1-\Omega_2-\nu_1+\nu_2-\tfrac{1}{2}\alpha_{12}\left[\frac{\rho_1}{\rho_2}+\frac{\rho_2}{\rho_1}\right]\sin\psi &+\tfrac{1}{2}\beta_{11;12}\frac{\rho_1}{\rho_2}(\rho_1^2+\rho_2^2)\sin\psi\\ &+\tfrac{1}{2}\beta_{12;22}\frac{\rho_2}{\rho_1}(\rho_1^2+\rho_2^2)\sin\psi\\ &+\tfrac{1}{2}\beta_{11;22}(\rho_1^2+\rho_2^2)\sin(2\psi)+\mathcal{F}(t)\ , \end{aligned} \tag{23}$$

where $\mathcal{F}(t)$ is the Langevin noise source defined by $\langle \mathcal{F}^\dagger(t)\mathcal{F}(t')\rangle = 2D(t-t')$ with a diffusion constant $D$.

For $\rho_1=\rho_2=\rho$ we obtain the following equation for the angle $\psi=\theta+(\nu_1-\nu_2)t$:

$$\dot\psi=\Omega_1-\Omega_2-\alpha_{12}\sin\psi+(\beta_{11;12}+\beta_{12;22})\rho^2\sin\psi + \beta_{11;22}\rho^2\sin(2\psi) \ . \tag{23a}$$

Assuming $\alpha_{11}=\alpha_{22}=\alpha_{12}\equiv\alpha$ and $\beta_{11;11}=\beta_{22;22}=\beta_{11;12}=\beta_{12;22}=\beta_{11;22}\equiv\beta$ (in the case of the discussed modulation), we have

$$\dot\psi=\Omega_1-\Omega_2-[\alpha-2\beta\rho^2(1+\cos\psi)]\sin\psi \ . \tag{23b}$$

For small $\psi$, we have $\sin\psi\simeq\psi$ and $\cos\psi\simeq 1$, and we obtain

$$\dot\psi\simeq\Omega_1-\Omega_2-(\alpha-4\beta\rho^2)\psi \ . \tag{23c}$$

From Table I we have in the steady state ($\gamma_1=\gamma_2\equiv\gamma$)

$$\begin{aligned}\dot\rho=\theta&=\tfrac{1}{2}(\alpha-\gamma)\rho+\tfrac{1}{2}\alpha\rho\cos\psi-\tfrac{3}{2}\beta\rho^3-2\beta\rho^2\cos\psi-\tfrac{1}{2}\beta\rho^3\cos(2\psi)\\ &=\tfrac{1}{2}(\alpha-\gamma)\rho+\tfrac{1}{2}\alpha\rho\cos\psi-\beta\rho^3(1+\cos\psi)^2 \ ,\end{aligned}$$

i.e.,

$$\rho^2=\frac{1}{2}\frac{\alpha(1+\cos\psi)-\gamma}{\beta(1+\cos\psi)^2}\simeq\frac{2\alpha-\gamma}{8\beta} \ .$$

Therefore, we have

$$\alpha-2\beta\rho^2(1+\cos\psi)=\frac{\gamma}{1+\cos\psi}\simeq\tfrac{1}{2}\gamma \ ,$$

so that

$$\dot\psi\simeq\Omega_1-\Omega_2-\tfrac{1}{2}\gamma\psi \ , \tag{24}$$

i.e., locking occurs for

$$\psi=\frac{2}{\gamma}(\Omega_1-\Omega_2) \ ,$$

which gives us the phase shift associated with the measurement of small rotation rates.

## CONCLUSIONS

The two counterpropagating laser modes in a ring laser can be coupled via a modulation of the active medium. For certain conditions, the diffusion of the relative phase angle vanishes, and this is still true in the nonlinear regime.

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## APPENDIX: CONNECTION BETWEEN LANGEVIN EQUATION AND FOKKER-PLANCK EQUATION

It is well known[13] that a Fokker-Planck equation for $N$ variables $\xi_i$ ($i=1,\ldots,N$)

$$\frac{\partial P(\{\xi\},t)}{\partial t}=-\sum_i\frac{\partial}{\partial\xi_i}[d(\xi_i)P]+\sum_{i,j}\frac{\partial^2}{\partial\xi_i\partial\xi_j}[D(\xi_i,\xi_j)P] \tag{A1}$$

is equivalent to $N$ Langevin equations

$$\dot\xi_1=h_i(\{\xi\},t)+\sum_j g_{ij}(\{\xi\},t)\Gamma_j(t) \ , \tag{A2}$$

where the $\Gamma_j(t)$ are random variables. The connection between the coefficients is

$$D(\xi_i,\xi_j)=\sum_k g_{ik}(\{\xi\},t)g_{jk}(\{\xi\},t) \tag{A3}$$

and

$$d(\xi_i)=h_i(\{\xi\},t)+\sum_k g_{kj}(\{\xi\},t)\frac{\partial}{\partial\xi_k}g_{ij}(\{\xi\},t) \ . \tag{A4}$$

The mean motion of the variable $\xi_i$, is not only given by the drift coefficient but there are additional terms,

$$\begin{aligned}h_i(\{\xi_i,\},t)&=\langle\dot\xi_i\rangle\\ &=d(\xi_i)-\sum_k g_{kj}(\{\xi\},t)\frac{\partial}{\partial\xi_k}g_{ij}(\{\xi\},t) \ .\end{aligned} \tag{A5}$$

The second term is the so-called noise-induced drift.

In addition we have to rewrite Eq. (16) to obtain an equation of the form (A1). The interchange of diffusion coefficients and derivatives results in a contribution to the drift coefficients that is the derivative of the diffusion coefficient.

In our special case, however, where all coefficients are given by Table I, the derivatives of the diffusion coefficients are a factor $1/\rho^2$ smaller than the drift coefficients and therefore negligible. The same is true for the noise-induced drift. Because of Eq. (A3) the noise-induced-drift term has the same order of magnitude as the derivative of the diffusion coefficient.

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