Structure of parameter space for a prototype nonlinear oscillator

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A prototype driven nonlinear two-dimensional oscillator with a limit cycle is studied. Three parameters α, β, and γ characterize the dynamics: the amplitude α of the external force, the frequency ratio β between the drive and the natural period, and the relaxation rate γ for perturbations off the limit cycle. Several limiting cases can be investigated analytically. The claim that this model is representative of a wide class of nonlinear oscillators is substantiated by showing that appropriate two-dimensional cross sections of the parameter space are qualitatively similar to known special cases.

Several models of nonlinear-driven oscillators have been discussed in great detail. The forced Brusselator[1,2]

$$\dot{x} = A - (B+1)x + x^2 y + \alpha\cos(\omega t)\ ,\qquad \dot{y} = Bx - x^2 y\ , \tag{1}$$

is probably the most thoroughly studied model, investigated by use of various numerical methods. A typical result is shown in Fig. 1, redrawn from Ref. 2. Another well-studied system is an exactly solvable nonlinear oscillator, subjected to periodic kicks,[3,4]

$$\ddot{x} + \dot{x}(4bx^2 - 2a) + b^2x^5 - 2abx^3 + (\omega_0^2 + a^2) = V_E \sum_n \delta(t - n\tau_E)\ , \tag{2}$$

whose dynamics can be transformed into a discrete mapping of two variables. Results for this model, qualitatively different from those for model (1), are shown in Fig. 2, reproduced from Ref. 3. In both these two models the free nonlinear oscillator has a limit cycle enclosing an unstable stationary point, while the external force is trying to drive the system off the limit cycle. Further investigations of this class of nonlinear-driven oscillators is desirable, with the purpose of uncovering similarities in the structure of parameter space.

In order to establish the connection between these different structures we have to consider a more suitably standardized parameter space. Three independent parameters in any driven nonlinear oscillator are important, i.e., the strength α of the external force, the ratio β between the two frequencies of the free oscillator and of the external force, and the relaxation rate γ for perturbations off the stable limit cycle. For example, for the Brusselator model (1) one may show that in case of $1+A^2 \lesssim B < (1+A)^2$ the latter two parameters are given by

$$\beta = \frac{[(A+1)^2 - B]^{1/2}[B-(A-1)^2]^{1/2}}{2\omega}\ ,\qquad \gamma = \frac{B - A^2 - 1}{2\omega}\ . \tag{3}$$

In the second model (2) with the values of the parameters a, b, and ω_0 used in Fig. 2, the relaxation is very fast (i.e., $1/\gamma \simeq 0$), and the parameters α and β are simply V_E and τ_E/τ_0, respectively. It emerges that the published studies for models (1) and (2) were done for different cross sections of the three-dimensional parameter space, viz., parallel or perpendicular to the γ axis, respectively.

In a recent study[5] a prototype nonlinear-driven oscillator has been proposed. This model is described by the following differential equations:

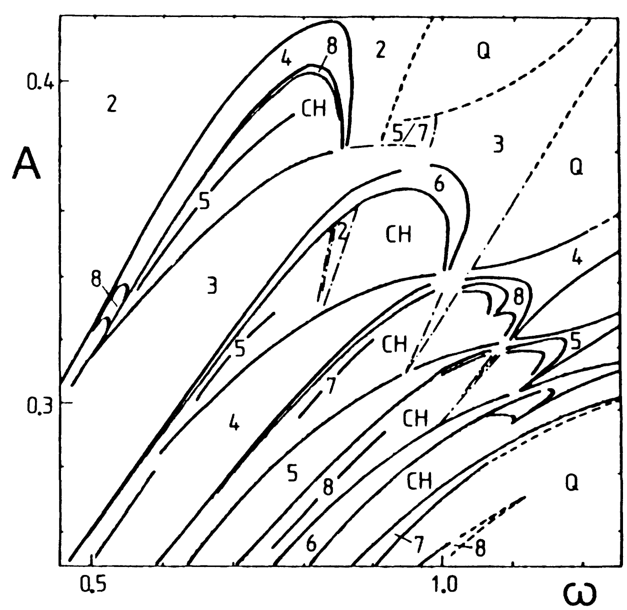

FIG. 1. Structure of parameter space of the forced Brusselator with $B = 1.2$ and $\alpha = 0.05$. The figure is redrawn from Ref. 2. The numbers in the figure indicate the periodicity which is stable in the corresponding zone, while Q indicates quasiperiodicity and CH indicates chaotic regions with embedded periodicities. A dashed line is a boundary between periodic and quasiperiodic regions, while a dot-dashed line is a boundary of unclear nature.

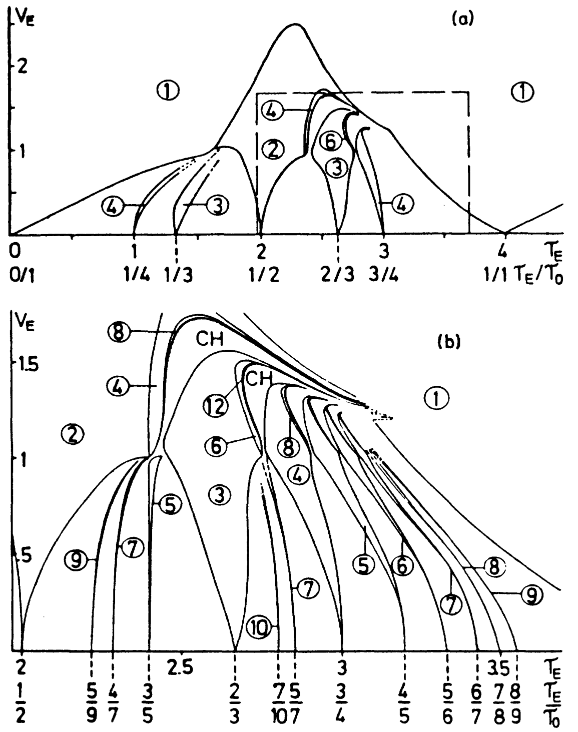

FIG. 2. Structure of parameter space of model (2). The figure is published in Ref. 3. Numbers inside circles indicate stable periods. (a) Stability zones for period solutions in the plane (τ_E, V_E) with $a=1.57079$, $b=15.7079$, and $\omega=1.57079$. (b) Enlargement of the rectangle marked in (a).

$$\dot{x} = sx(1-x^2-y^2) - y + 2\alpha \sum_n \delta(t-2\pi n\beta) ,$$
$$\dot{y} = x + sy(1-x^2-y^2) , \tag{4}$$

with $s \ge 0$, $0 < \beta \le 1$, and $\alpha \ge 0$. The limit cycle is the circle $x^2+y^2=1$. If one focuses interest on the state (x_n, y_n) of the system immediately after the nth kick, a two-dimensional return map

$$x_{n+1} = C[x_n \cos(2\pi\beta) - y_n \sin(2\pi\beta)] + 2\alpha ,$$
$$y_{n+1} = C[x_n \sin(2\pi\beta) + y_n \cos(2\pi\beta)] , \tag{5}$$

results, where

$$C = [(x_n^2+y_n^2) + (1-x_n^2-y_n^2)\exp(-4\pi\gamma)]^{-1/2} ,$$

with $\gamma = s\beta$. If the prototype oscillator (4) is representative of a wide class of driven nonlinear oscillators, one expects that it should be possible to show that appropriate cross sections of the parameter space are qualitatively equivalent to the results for models (1) and (2). In this Brief Report, I will show just that. Let us start, however, by considering several limiting cases which can be discussed analytically for the prototype model.

(i) $\gamma \to \infty$. This is the fast-relaxation limit, and the return map exhibits a dimensional reduction from $d=2$ to $d=1$. This nontrivial situation can be exactly reduced to a one-dimensional mapping. Detailed results for this case have already been published.[5,6] The parameter space, now two dimensional, can be divided into three regions. In the weak-force region $\alpha \le \frac{1}{2}$, the system displays mode-locking and quasiperiodic behavior. In the unimodal region the iteration is confined to an interval where the mapping is unimodal, and in the standard manner[7] period-doubling bifurcations and chaotic behavior are found. In the intermediate region the transition between the unimodal mapping and the mode-locking behavior takes place. The last two regions are separated by a line[5]

$$\alpha = \begin{cases} \frac{1}{2} & \text{if } |\frac{1}{2}-\beta| > \frac{1}{4} \\ \dfrac{1}{2\sin(\frac{1}{4}\pi + |\frac{1}{2}-\beta|\pi)} & \text{if } |\frac{1}{2}-\beta| < \frac{1}{4} . \end{cases} \tag{6}$$

Along this line the order of the period orbits is arranged according to the U sequences.[8]

(ii) $\gamma = 0$. In this case we have $C=1$, and the two-dimensional return mapping (5) takes the following form:

$$x_{n+1} = x_n \cos(2\pi\beta) - y_n \sin(2\pi\beta) + 2\alpha ,$$
$$y_{n+1} = x_n \sin(2\pi\beta) + y_n \cos(2\pi\beta) . \tag{7}$$

The unique fixed point is then $x^* = \alpha$ and $y^* = \alpha \cot(\pi\beta)$. Letting $X_n = x_n - x^*$ and $Y_n = y_n - y^*$, we get

$$X_{n+1} = X_n \cos(2\pi\beta) - Y_n \sin(2\pi\beta) ,$$
$$Y_{n+1} = X_n \sin(2\pi\beta) + Y_n \cos(2\pi\beta) , \tag{8}$$

which is nothing but a one-dimensional circle map[9,10]

$$\theta_{n+1} = \theta_n + \beta \pmod 1 ,$$

with $\tan\theta_n = Y_n/X_n$. The stable state of the system is a periodic or quasiperiodic orbit according to whether the value of β is rational or irrational (the center of rotation remains, of course, a fixed point).

(iii) For very small γ the fixed point moves to

$$x^* = \alpha - \gamma \frac{\pi\alpha}{\sin^2(\pi\beta)} \left[1 - \frac{\alpha^2}{\sin^2(\pi\beta)} \right] + O(\gamma^2) ,$$
$$y^* = \alpha \cot(\pi\beta) + O(\gamma^2) . \tag{9}$$

In the case of $\alpha > |\sin(\pi\beta)|$, the fixed point is outside the limit cycle $x^2+y^2=1$, and one can show that its neighborhood is length contracting in any direction. The stable state of the return map is then always a period-1 orbit. By comparison with case (ii) it is evident that the system for $\alpha > |\sin(\pi\beta)|$ displays singular behavior when $\gamma \to 0$.

(iv) For $\alpha \gg 1$ and $\gamma > 0$, the fixed point is also outside the limit cycle $x^2+y^2=1$. The unique fixed point is stable since its neighborhood is length contracting in any direction.

(v) $\alpha = 0$. With no external force, the stroboscopic period β has no physical relevance. The return map simply exhibits periodic or quasiperiodic behavior according to whether β is rational or irrational.

Apart from the above limiting cases the two-dimensional map must be analyzed numerically. Let us

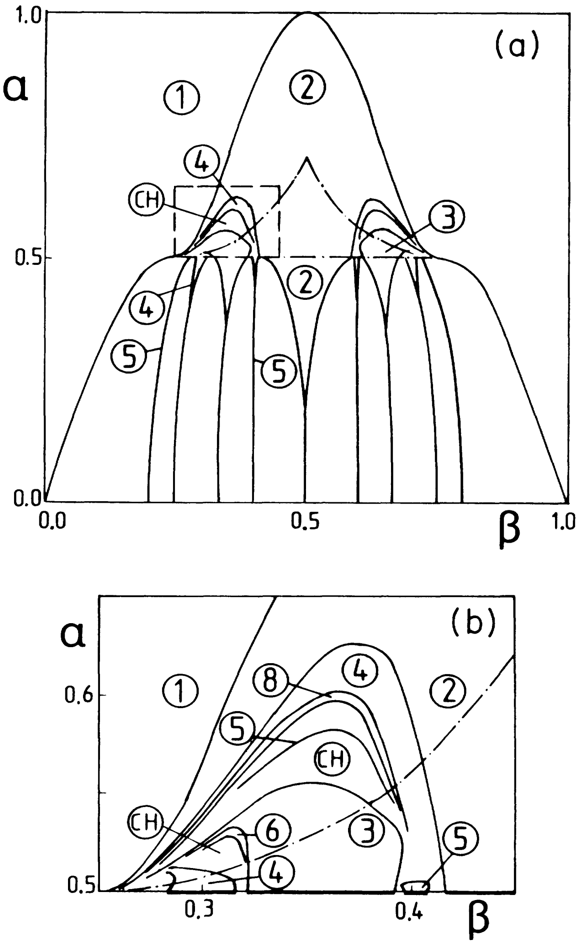

FIG. 3. (a) Structure of parameter space of the prototype model (4) for $\gamma=1$. Numbers inside circles indicate stable periods. In the region marked by dot-dashed line the state may depend on the initial values, which here are taken to be $x=0.1$ and $y=0.1$. (b) Enlargement of the dashed rectangle in (a).

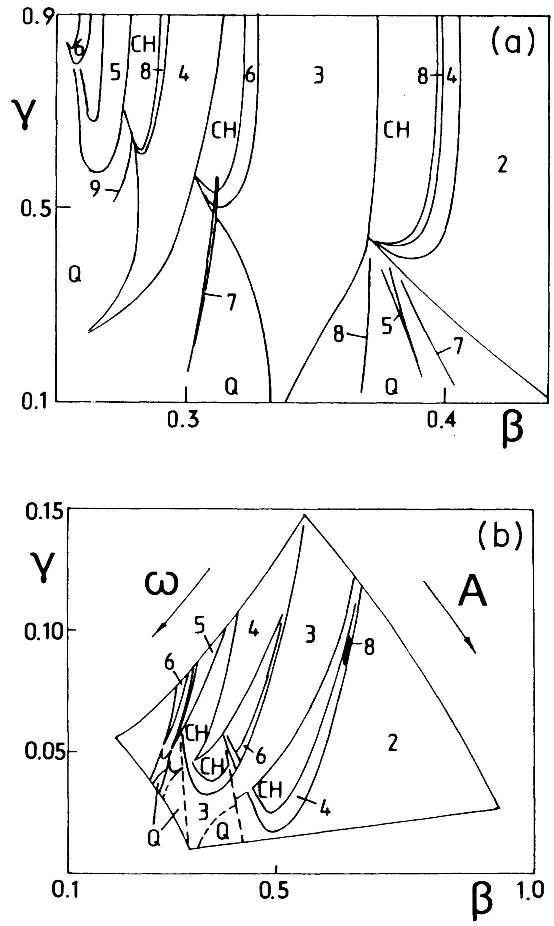

FIG. 4. (a) Structure of parameter space on the surface (10), parallel to the γ axis. The initial values are $x=0.1$ and $y=0.1$. The numbers in the figure indicate the periodicity which is stable in the corresponding zone, while Q indicates quasiperiodicity and CH indicates chaotic regions with embedded periodicities. (b) Reproduction of Fig. 1, using new parameters β and γ.

now consider the fast, but still finite, damping situation, say, $\gamma\geq 1$. In this situation we have $\exp(-4\pi\gamma)<10^{-5}$, so the return map (5) is almost one dimensional. Numerical results for this case are shown in Fig. 3. The parameter space can still be divided into three regions, as for the limiting case (i). In the weak-force region we have either mode locking with rational winding numbers, or quasiperiodic orbits. In the strong-force situation we may find a region $\alpha>\alpha(\beta)$, where the state of the system does *not* depend on the initial value and the U sequence of periodic orbits is found. In the intermediate region the state of the system may depend on the initial value. It will not be difficult to see that Fig. 2 is similar to Fig. 3, as it should be in the fast-damping situation.

In order to find the structure of parameter space such as that of Fig. 1 for the forced Brusselator, one should consider some appropriate cross section of the $(\alpha\beta\gamma)$ space. We take the surface

$$2\alpha\sin[(\tfrac{3}{4}-\beta)\pi]=1\ , \tag{10}$$

parallel to the γ axis. Note that this corresponds to a border line for the unimodal region when $\gamma\to\infty$, see Eq. (6). Results for this special selection are shown in Fig. 4(a). For slow relaxation (small γ) the system displays periodic or quasiperiodic behavior, while for fast relaxation (large γ), chaotic behavior is found. One sees that its structure is quite similar to Fig. 4(b), which is replotted from Fig. 1.

We conclude that these comparisons have substantiated the claim that the simple oscillator (4) is a prototype for the class of nonlinear oscillators with a limit cycle and a periodic drive. The forced van der Pol equation[11–13]

$$\ddot{x}-k(1-x^2)\dot{x}+x=\alpha\cos(\omega t) \tag{11}$$

also belongs to this class. It will be interesting to have a more complete discussion for models (1), (2), and (11), and others, in order to verify the similarity with the prototype model (4) in the whole parameter space.

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