Second-order differential-delay equation to describe a hybrid bistable device

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We discuss the problem of a dynamical system with delayed feedback, a hybrid bistable device, characterized by n response times and described by an nth-order differential-delay equation (DDE). Starting from a linear-stability analysis of the DDE we show the effects of the second-order differential terms on the position of the first bifurcation and on the frequency of the resulting self-oscillation. We also consider the effects of the third-order differential terms on the first bifurcation. Experimental results are shown to support the linear analysis. A numerical solution of the second-order DDE is also presented and a comparison is established with the first-order case.

I. INTRODUCTION

Most of the experimental results obtained recently from hybrid bistable devices are directly concerned with the questions of self-oscillation and turbulent behavior appearing in such systems. Among the hybrid bistable devices (HBD) the electro-optic[1–6] and acousto-optic[7–10] devices have been thoroughly studied and analyzed with regard to these aspects. Relatively little work, however, has been done to study the effect of a distributed response time on the behavior of the system. Indeed, in all the preceding papers the HBD appeared to be well described by a first-order differential-delay equation (DDE) involving one time delay and one global response time. Supporting such a first-order model was the fact that most of the preceding results were obtained within the regime where the delay τ_D is significantly larger than the response time τ. However, it has been pointed out[11] that a hybrid bistable device involving a Pockels cell could be exactly described by a second-order differential equation (DE). It has also been shown that a piezoelectrically driven Fabry-Perot interferometer is conveniently described by a second-order DDE and is therefore self-oscillating under certain conditions.[12] The theoretical analysis in the latter paper was restricted, however, to the case where the delay is significantly smaller than the response time. More recently, Chrostowski *et al.*[13] have shown that the acousto-optic HBD could be described by a second-order DE and that self-oscillating behavior could result when a variable damping force was provided to the system. In the present paper we first discuss the problem of a HBD characterized by a number n of individual response times and describe it in terms of an nth-order DDE. We analyze the effects of the second-order term on the period of the self-oscillation (which is due here to retarded action) and the effects of the second- and third-order terms on the position of the first bifurcation. Experimental and numerical results are presented to support the theoretical analysis.

II. ANALYSIS

A. The nth-order DDE

One can generally associate a characteristic response time to each of the components of a hybrid bistable device. In the acousto-optic HBD, for instance (see Ref. 7 for a detailed description of the setup), the global response time is the sum of the contributions arising from the modulator, the detector, and the amplifier. In many practical situations, however, one of these individual response times is much larger than the others so that they can be neglected. Moreover, as will be shown below, the fact that the response time τ of a system is associated with one or many components loses its relevance as the ratio τ_D/τ becomes larger, provided naturally that τ is understood as the sum of the individual response times. Nevertheless, when the delay in the feedback loop is comparable to or less than the global response time, the various individual response times appear to play an important role in the stability of the solution. To analyze the relevance of the distribution of the various response times we introduce, following the approach of Ref. 13, a general model for our system which consists in representing its various components by RC circuits each separated by amplification (or isolation) stages of unitary gain (Fig. 1). It is then readily shown using Kirchhoff laws that such a system is described by the nth-order DDE

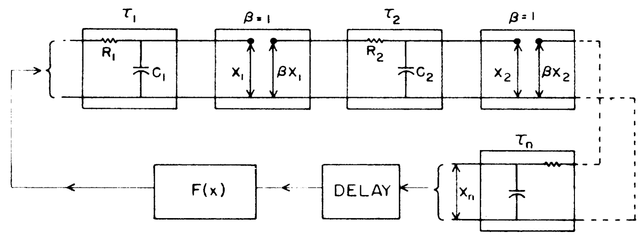

FIG. 1. Schematic diagram of a dynamical system with a delayed feedback having multiple response times. $F(X)$ is a single maximum nonlinear function.

$$\left[\prod_{i=1}^{n}\tau_i\right]\frac{d^nX(t)}{dt^n}+\cdots+\left[\sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{k>j}^{n}\tau_i\tau_j\tau_k\right]\frac{d^3X(t)}{dt^3}+\left[\sum_{i=1}^{n}\sum_{j>i}^{n}\tau_i\tau_j\right]\frac{d^2X(t)}{dt^2}+\left[\sum_{i=1}^{n}\tau_i\right]\frac{dX(t)}{dt}+X(t)=F(X(t-\tau_D)) , \tag{1}$$

where $F(X(t-\tau_D))$ is the nonlinear transfer function of the system. In the acousto-optic device the transfer function is

$$F(X(t-\tau_D))=\pi\{A-\mu\sin^2[X(t-\tau_D)-X_B]\} , \tag{2}$$

where A and X_B are fixed parameters while μ allows one to control the amplitude of the nonlinearity. However, for the purpose of our analysis, we will consider $F(X)$ as a general single maximum nonlinear function. In the previous analysis[1–10] of the HBD, only the first two terms on the left-hand side of (1) were considered and the solutions of the resulting first-order DDE were shown to undergo a period-doubling sequence leading to chaotic behavior as the control parameter μ was increased.[1] We will analyze here the effects of the higher-order terms on this scenario. Let us first point out that all the coefficients of the higher-order terms are maximum when all the τ_i are equal, that is, when the response time is uniformly distributed. In this particular case we are led to the equation

$$\left[\frac{\tau}{n}\right]^n\frac{d^nX(t)}{dt^n}+C_{n-1}^n\left[\frac{\tau}{n}\right]^{n-1}\frac{d^{n-1}X(t)}{dt^{n-1}}+\cdots+C_2^n\left[\frac{\tau}{n}\right]^2\frac{d^2X(t)}{dt^2}+\tau\frac{dX(t)}{dt}+X(t)=F(X(t-\tau_D)) , \tag{3}$$

where $\tau\equiv\sum_{i=1}^{n}\tau_i=n\tau_i$ (for $i=1,n$) and C_m^n represents the number of combinations of n quantities taken m at a time. The linear stability analysis of the nth-order DDE (1) rapidly becomes tedious as n increases so we will present the details only for the second-order DDE, the procedure for the higher orders being essentially the same. Moreover, a comparison between the second- and the first-order DDE will allow us to establish the principal effects of the higher-order terms.

B. The second-order DDE: Linear stability analysis

Let us recall the second-order DDE that we want to analyze:

$$\tau_1\tau_2\frac{d^2X(t)}{dt^2}+(\tau_1+\tau_2)\frac{dX(t)}{dt}+X(t)=F(X(t-\tau_D)) . \tag{4}$$

Substituting for $X(t)$ the shifted variable

$$Z(t)=X(t)-X^* , \tag{5}$$

where X^* is the fixed point, and keeping the first two terms of the Taylor-series expansion of $F(X(t-\tau_D))$ around X^* one obtains

$$\tau_1\tau_2\frac{d^2Z(t)}{dt^2}+(\tau_1+\tau_2)\frac{dZ(t)}{dt}+Z(t)=-BZ(t-\tau_D) , \tag{6}$$

where

$$B\equiv\frac{-dF}{dX}\bigg|_{X=X^*}=\pi\mu\sin[2(X^*-X_B)] . \tag{7}$$

Equation (6) is a linear second-order DDE. Its stability can be analyzed locally around $z=0$. The characteristic equation[14] associated with equation (6) is the following:

$$\tau_1\tau_2S^2+(\tau_1+\tau_2)S+1+Be^{-S\tau_D}=0 , \tag{8}$$

where S is a complex parameter. Substituting for S its real (α) and imaginary (β) parts one obtains

$$\tau_1\tau_2(\alpha^2-\beta^2)+(\tau_1+\tau_2)\alpha+1+Be^{-\alpha t_D}\cos(\beta\tau_D)=0 , \tag{9a}$$

$$2\alpha\beta\tau_1\tau_2+(\tau_1+\tau_2)\beta-Be^{-\alpha t_D}\sin(\beta\tau_D)=0 . \tag{9b}$$

In this analysis α represents a real attenuation (or amplification) factor depending on its sign while β is the corresponding angular frequency. Since we are primarily concerned with the threshold of instability (bifurcation) of the solution we can set $\alpha=0$ so that Eqs. (9) become

$$(\tau_1+\tau_2)\beta-B\sin(\beta\tau_D)=0 , \tag{10a}$$

$$\tau_1\tau_2\beta^2-1-B\cos(\beta\tau_D)=0 . \tag{10b}$$

Now for the sake of simplicity we will also define the normalized product

$$P\equiv\frac{\tau_1\tau_2}{\tau^2}=\frac{R}{(1+R)^2} , \tag{11}$$

where R is the ratio τ_1/τ_2. From this relation between P and R one can easily show that P reaches its maximum value of $\frac{1}{4}$ when $\tau_1=\tau_2$.

1. *Frequency of the self-oscillation*

Let us now establish the relationship between the ratio τ_D/τ and the phase factor $\beta\tau_D$. From Eqs. (10) one has

$$\left[\frac{P\phi^2}{(\tau_D/\tau)^2}-1\right](\tau_D/\tau)=\frac{\phi}{\tan\phi} , \tag{12}$$

where

$$\phi\equiv\beta\tau_D . \tag{13}$$

Equation (12) leads to the following quadratic equation for τ_D/τ:

$$(\tau_D/\tau)^2+\frac{\phi}{\tan\phi}(\tau_D/\tau)-P\phi^2=0 , \tag{14}$$

which must be solved for a positive value of τ_D/τ

$$\tau_D/\tau=\frac{\phi}{2}\left[-\frac{1}{\tan\phi}+\frac{1}{|\tan\phi|}(1+4P\tan^2\phi)^{1/2}\right], \quad (15)$$

for $0<\phi<\pi$ (Ref. 15) which reduces to

$$\tau_D/\tau=\frac{\phi(1-\cos\phi)}{2\sin\phi} \quad (16)$$

when $\tau_1=\tau_2$. Equation (15) allows us to relate the frequency of the self-oscillation appearing in the system to the normalized delay (τ_D/τ). Such a relationship is readily obtained for the first-order DDE and one can establish[16]

$$\tau_D/\tau=\frac{-\phi}{\tan\phi} \quad (\pi/2<\phi<\pi) . \quad (17)$$

It is very interesting to compare Eqs. (15) and (17). One first sees that when P is strictly equal to zero (i.e., one single response time) Eq. (15) reduces to Eq. (17) for $\pi/2<\phi<\pi$ and is zero for $0<\phi<\pi/2$. However, the asymptotic behavior of Eq. (15) as the parameter R becomes infinite ($P\rightarrow0$) is different from the asymptotic behavior of Eq. (17) when the ratio τ_D/τ also tends toward zero. The situation is depicted in Fig. 2 where the solution of Eq. (15) is shown for three values of R and compared to the first-order curve. For values of τ_D/τ larger than unity all curves coincide quite well. However, as τ_D/τ is made smaller, the second-order curves eventually diverge from the first-order one to reach their own lower bound value zero, while the first-order curve tends towards $\pi/2$. This is a major discrepancy between the first- and second-order DDE. As a matter of fact one can show that for all the higher-order DDE the values for ϕ corresponding to the first mode solution are found within the interval $(0,\pi)$.

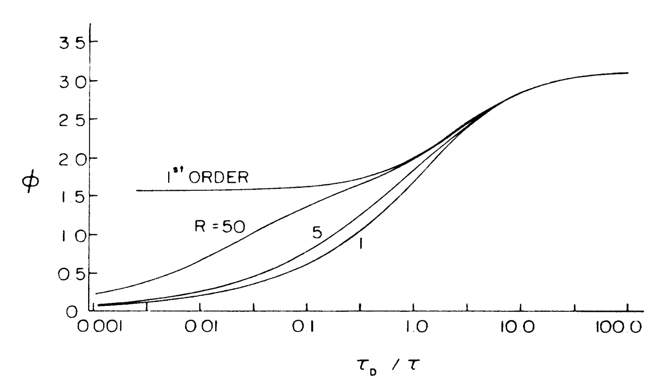

FIG. 2. Comparison between the value of the phase factor ϕ calculated from the linear analysis of the first-order DDE and the one obtained from the second-order DDE for three values of $R=\tau_1/\tau_2=50$, 5, and 1 as a function of τ_D/τ. The first-order curve tends toward $\pi/2$ while the second-order curve tends toward zero independently of the value R for decreasing τ_D/τ.

2. Threshold of the self-oscillation

An important parameter to consider at the threshold of a self-oscillation is the slope of the nonlinear function at the fixed point since it is directly related to the parameter controlling the amplitude of the nonlinearity. In the case of a one-dimensional (1D) map (difference equation), the self-oscillation appears when the slope becomes less than -1. In the case of a DDE of the type considered here, the solution remains stable down to a critical value of the slope which is less than -1, as if the nonlinearity had to compensate for the presence of a differential term. For the second-order DDE, the parameter B [previously defined at Eq. (7) as the additive inverse of the slope at the fixed point] can be simply related to the ratio τ_D/τ. Indeed, starting from Eqs. (10) and (11) one has

$$(\beta\tau)^2+[P(\beta t)^2-1]^2=B^2 . \quad (18)$$

Now from Eq. (10b) one obtains

$$\tau_D/\tau=\frac{\arccos\{[P(\beta\tau)^2-1]/B\}}{(\beta t)} , \quad (19)$$

where the value of (βt) can be obtained from (18),

$$(\beta\tau)^2=\frac{(2P-1)+(1-4P+4P^2B^2)^{1/2}}{2P^2} . \quad (20)$$

In the special case where $\tau_1=\tau_2$ ($P=\frac{1}{4}$) Eqs. (19) and (20) reduce to the simple form

$$\tau_D/\tau=\frac{\arccos(1-2/B)}{2\sqrt{B-1}} . \quad (21)$$

It is also interesting to note that the corresponding result for the first-order DDE (Ref. 16),

$$\tau_D/\tau=\frac{\arccos(-1/B)}{(B^2-1)^{1/2}} , \quad (22)$$

can be deduced from Eqs. (19) and (20) in the limit of $P\rightarrow0$.

In the case of the acousto-optic HBD the value of the parameter B at the first bifurcation is simply given by

$$B_{\text{1st bif}}=\pi\mu_1\sin2[(X^*-X_B)] , \quad (23)$$

so that Eqs. (19), (20), and (23) allow us to relate μ_1 (the value of μ for which the first bifurcation occurs) to the ratio τ_D/τ for different values of R.

III. RESULTS

A. Analysis at the first bifurcation: Experimental

The experiment was performed with an acousto-optic bistable device, a detailed description of which can be found in Ref. 7. This device usually involves three components having their own inherent and non-negligible response time. However, these intrinsic response times cannot be conveniently varied to allow for a complete analysis of the various cases. Therefore we artificially introduced RC circuits of time constants smoothly variable and large enough so that the intrinsic response times of the device could be neglected by comparison with them. The RC circuits were separated by isolation steps of unitary gain and negligible response time.

Originally, the fact of taking into consideration the higher-order differential terms in the DDE describing our device was intended to explain a discrepancy that we had previously observed (for small values of the ratio τ_D/τ) between the frequency of the self-oscillation measured experimentally and the one predicted by the linear stability analysis of the first-order DDE. Therefore our first concern was to analyze the dependence of this frequency on the parameter R. Figure 3 shows experimental values of $\beta\tau_D$ measured for four values of τ_D/τ and three different values of the ratio R (1, 5, and 50). Special care was taken to measure the frequency of the self-oscillation just after it appeared in the system (i.e., $\mu \gtrsim \mu_1$). This precaution was necessary to ensure a fair comparison with the results of the linear analysis which are expected to be valid at the bifurcation point only. As a matter of fact, the frequency of the self-oscillation appears to shift as μ is further increased (see Ref. 8). The excellent agreement between the experimental and the theoretical results is obvious from Fig. 3. Moreover, these results show the influence of the second-order term even for values of τ_D/τ slightly greater than unity.

The relevance of the third-order differential term was also analyzed by introducing a third response-time component in the device. For this analysis we made the individual response times equal in order to obtain the maximum effect in every order. The results are shown in Fig. 4. Obviously the corrective effect of the third order is not as striking as that of the second order and this is mainly due to the fact previously pointed out that, contrary to the first order, the second, third, and higher orders all have the same lower bound for ϕ, that is to say zero. Therefore one can reasonably argue that in most practical cases the second-order DDE can be used as a fair model to predict the self-oscillation frequency of systems with feedback loops involving more than one individual response time.

The effect of R on the position of the first bifurcation μ_1 was also analyzed for four values of the ratio τ_D/τ ranging between 0.2 and 1.0 (Fig. 5). Once again very good agreement with the results of the linear stability analysis

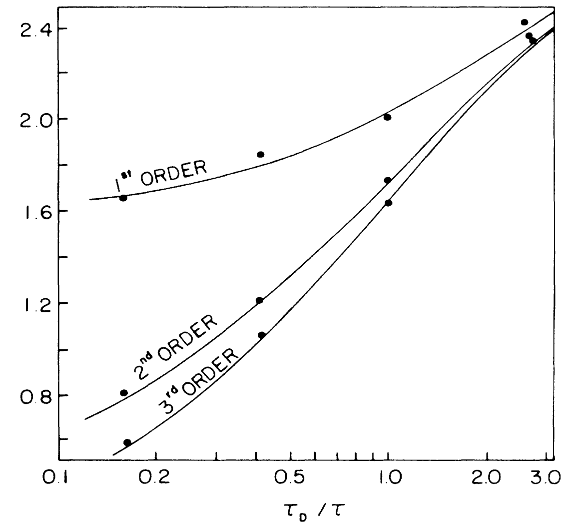

FIG. 4. Phase factor ϕ as a function of τ_D/τ for the first three orders. Both the theoretical and experimental results (●) of the second and third orders were obtained for equal response times.

is observed. It is also very interesting to note that the corrective effect of the second-order term is not to further shift the bifurcation point but to bring it back closer to the discrete case value. This tendency is more clearly illustrated in Fig. 6 where μ_1 is plotted as a function of τ_D/τ for the first three[17] orders and compared to the value ($\mu=0.324\ldots$) predicted by the corresponding difference equation (discrete model). These results show that for small values of τ_D/τ the first-order term significantly moves the bifurcation point away from the threshold value predicted by the discrete model, but that the higher-order terms gradually tend to reduce this discrepancy.

B. Shift of the period-doubling sequence: Numerical analysis

We solved numerically the second-order DDE defined by Eqs. (2) and (4) for $A=0.35$ and $X_B=0$. The results of this numerical analysis show very good agreement with

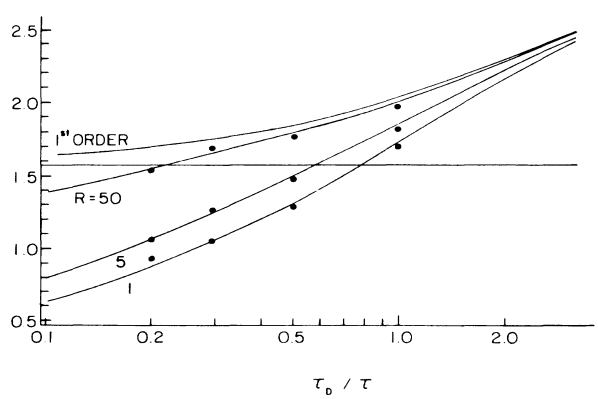

FIG. 3. Phase factor ϕ as a function of τ_D/τ for the first and second orders. The experimental points were obtained for $R=1$, 5, and 50. The $\pi/2$ asymptotic line for the first-order curve is also shown.

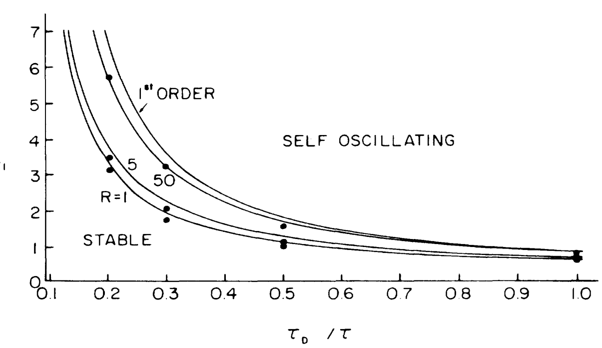

FIG. 5. Position of the first bifurcation μ_1 as a function of τ_D/τ for the first and the second ($R=1$, 5, and 50) orders. Experimental values are shown.

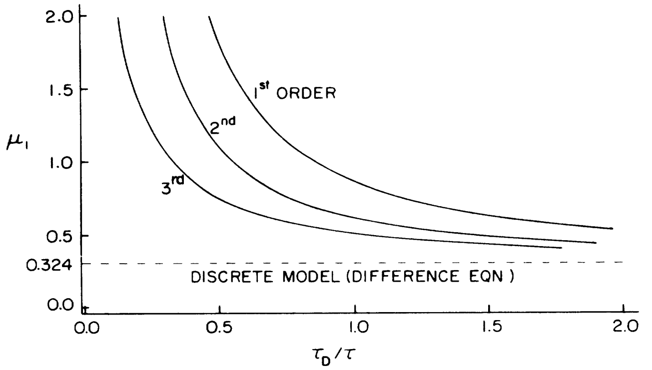

FIG. 6. Position of the first bifurcation μ_1 as a function of τ_D/τ for the first, second $\tau_1=\tau_2$, and third ($\tau_1=\tau_2=\tau_3$) orders.

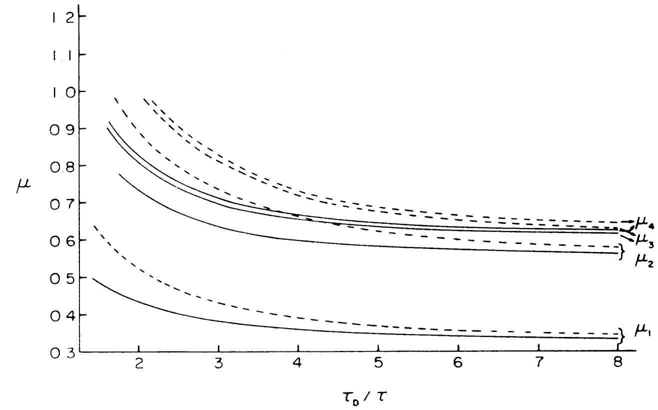

FIG. 7. Computed values of the first four bifurcations for the first- (dashed line) and second- ($\tau_1=\tau_2$) orders DDE as a function of τ_D/τ.

those of the linear stability analysis with regard to the first threshold of instability (μ_1). Moreover, the numerical analysis allows us to look at the effect of the second-order differential term on the period-doubling sequence following the first threshold of instability and leading the system to a chaotic regime. The results of this analysis are summarized in Fig. 7 where the positions of the first four bifurcations for the first- and second-order DDE are shown. These results clearly show that the main effect of the second-order term is to shift the whole bifurcation sequence downward without any modification to this sequence. For large values of τ_D/τ, the various bifurcation points tend toward their respective limits predicted by the discrete model. For instance for $\tau_D/\tau=8$ the $P8$ waveform appears for $\mu=0.630\ldots$ for the first order, and $\mu=0.615\ldots$ for the second order, while the discrete $P8$

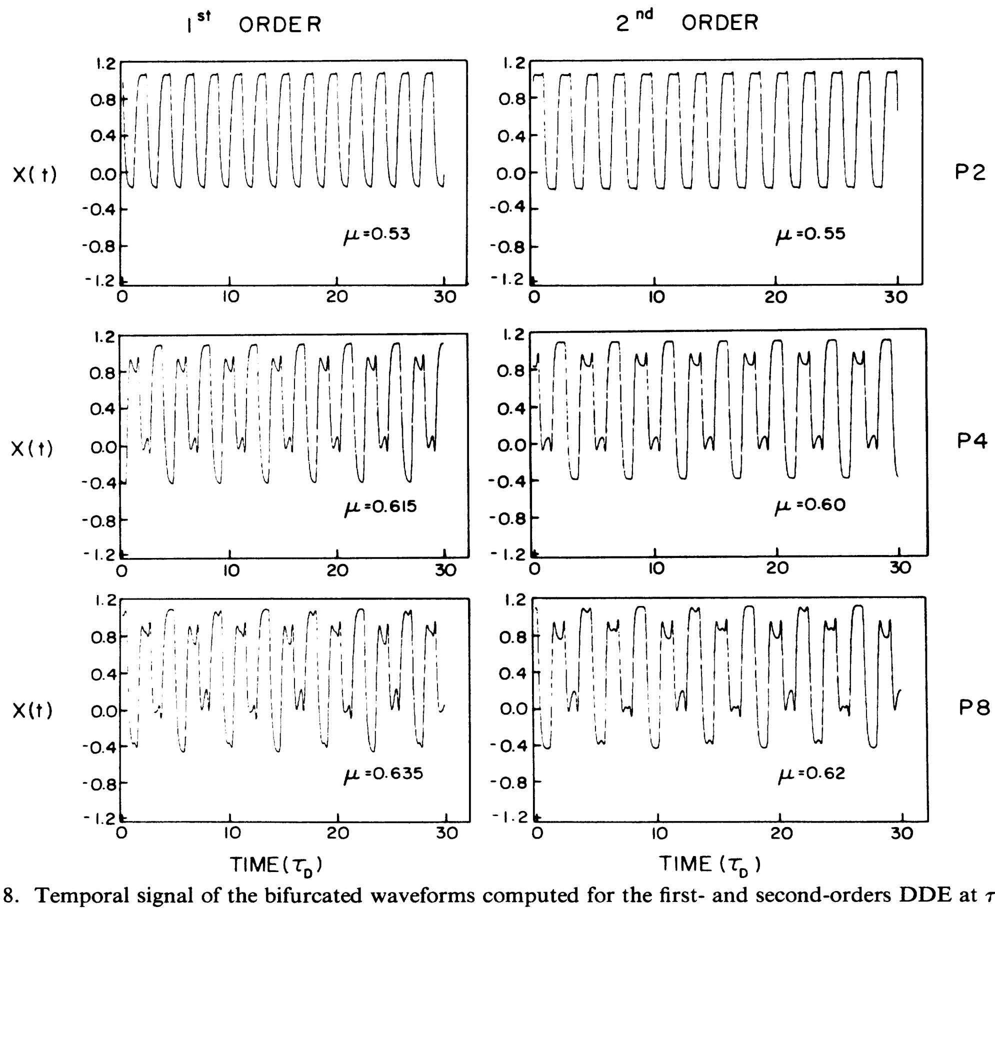

FIG. 8. Temporal signal of the bifurcated waveforms computed for the first- and second-orders DDE at $\tau_D/\tau=8$.

cycle appears for $\mu = 0.6013\ldots$. Moreover, the temporal shapes of the waveforms calculated from the second-order DDE become more and more similar to the corresponding first-order ones.

We show in Fig. 8 the shapes of the $P2$, $P4$, and $P8$ waveforms calculated numerically from the first- and second-order DDE. Generally speaking, the main effect of the second-order term appears to be a broadening of the plateaux in the signal. However, one can argue that as far as the period-doubling sequence is concerned the effect of a distributed response time is negligible within the region where the delay is sufficiently greater than the overall response time. As a matter of fact, the corrective effects of the second- and higher-order differential terms are not likely to be observed in a physical device for $\tau_D/\tau > 10$.

The rescaling of the nth-order DDE allows us to understand the behavior of the temporal solution as the ratio τ_D/τ increases. Expressing in Eq. (3) the time variable in τ_D unit leads us to the following equation:

$$\left[\frac{1}{n^n}\right]\epsilon^n\frac{d^nX(t)}{dt^n}+\left[\frac{C_{n-1}^n}{n^{n-1}}\right]\epsilon^{n-1}\frac{d^{n-1}X(t)}{dt^{n-1}}+\cdots+\left[\frac{C_2^n}{n^2}\right]\epsilon^2\frac{d^2X(t)}{dt^2}+\left[\frac{C_1^n}{n}\right]\epsilon\frac{dX(t)}{dt}+X(t)=F(X(t-1)) , \tag{24}$$

where $\epsilon \equiv \tau/\tau_D$. This equation clearly shows that for a given nth-order equation (corresponding to n response times) the differential terms can be considered as singular perturbations of increasing order of the discrete problem when $\epsilon \to 0$. On the other hand, in the regime where $\tau_D < \tau$ which implies $\epsilon > 1$, the influence of the higher differential terms become more significant and have to be accounted for when dealing with the various coefficients $(C_m^{(n)}/n^m)$.

IV. CONCLUSION

The number and relative values of the response times in a dynamical system with a delayed feedback appear to play an important role in the overall behavior of the solutions of the DDE describing the system as the value of τ_D/τ decreases. Furthermore, for a given normalized delay τ_D/τ the solution appears to become self-oscillating for a lower value of the control parameter μ when more than a single response time is considered. The lower bound for μ is the value predicted by the corresponding discrete model. Moreover, the period of the resulting self-oscillation increases when more than a single response time is considered. In fact, the discrepancy increases as τ_D/τ decreases to become very significant for $\tau_D/\tau < 1$ and decreases but is still observable up to $\tau_D/\tau = 10$. In many practical cases, however, the second-order DDE, associated with two non-negligible response times, should be a good approximation for most of the physical devices.

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