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Covariant phase-space representation for localized light waves

Y. S. Kim
Department of Physics, University of Maryland, College Park, Maryland 20742

E. P. Wigner
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
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It is suggested that the light-cone coordinate system is the natural language for the Lorentz-covariant phase-space representation of quantum mechanics. The localized light wave is discussed as an illustrative example. It is shown that the unitary transformation of a localized light wave from one Lorentz frame to another can be achieved through its covariant phase-space representation.

I. INTRODUCTION

Since photons are relativistic particles, the quantum mechanics of photons occupies an important place in relativistic quantum mechanics. The difficulty in formulating the theory of photons is that there is no position operator which is covariant and Hermitian.[1] This is known as the photon localization problem.[1–3]

However, when we discuss photons, we always think of localized light waves in a given Lorentz frame.[4] The question then is whether someone in a different Lorentz frame will think in the same way. If the localized light wave represents a photon, it cannot be covariant. If, on the other hand, the localized wave is covariant, it cannot describe the photon.

With this point in mind, Han *et al.*[5] considered the covariance of localized light waves. They concluded that their covariant model of localized light waves cannot represent photons. Their conclusion is summarized in Table I. Han *et al.* pointed out further that, if the momentum distribution is sufficiently narrow, the light-wave distribution can numerically be close to that of the photon. For this reason, it is still useful to study the covariance of localized light waves.

The covariance of the model discussed in Ref. 5 is derived from an analogy with the relativistic quark model.[6] However, the analogy is less than perfect, because the relativistic quark model contains one of the space-time dimensions which does not exist in the case of light waves. It is therefore of interest to look into the possibility of establishing the covariance of localized light waves without resorting to the quark model.

TABLE I. Covariant light waves and photons. They are different. The light waves are localizable but cannot have a particle interpretation. The particle interpretation of free photons is given in terms of the Fock space representation in quantum electrodynamics. However, the photons in field theory are not localizable.

	Localizable	Particle interpretation
Light waves	Yes	No
Photons	No	Yes

There is another research line of current interest. Since its introduction in 1932,[7,8] the phase-space representation has been proven useful in many branches of modern physics, including statistical mechanics,[9] nuclear physics,[10] elementary particle physics,[11] quantum mechanics,[12] condensed matter physics,[13] atomic and molecular physics,[14] semiclassical dynamics,[15] and in modern optics.[16] However, we do not know how the phase-space distribution in one Lorentz frame would appear to an observer in a different frame.

The purpose of the present paper is to solve the covariance problem for light waves using its phase-space representation. We shall construct a covariant phase-space representation for localized light waves. The light-cone coordinate system turns out to be the natural language for the covariant phase-space representation.

In Sec. II, we extend the little group of photons to include Lorentz boosts of the plane-wave solutions of Maxwell's equations. This extended little group can accommodate superpositions of light waves propagating in the same direction with different frequencies. Section III deals with localized light waves having the symmetry of the extended little group. It is pointed out that the problem of unitarity is an outstanding problem for the covariance of localized light waves. In Sec. IV, a localized phase-space representation is constructed. This representation is unitary and produces the multiplier needed in the spatial or momentum representation of covariant light waves.

II. EXTENDED LITTLE GROUP FOR PHOTONS

The little group is the maximal subgroup of the Lorentz group which leaves the four momentum of a given particle invariant.[17] For a massless particle moving along the z direction, the little group is generated by

$$J_3, N_1, N_2 \; , \tag{1}$$

with

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1 \; ,$$

where J_i and K_i are the generators of rotations and boosts, respectively. The above generators satisfy the commutation relations:

$$[N_1,N_2]=0\ ,\quad [J_3,N_1]=iN_2\ ,\quad [J_3,N_2]=-iN_1\ . \tag{2}$$

These commutation relations are identical to those of the two-dimensional Euclidean group.

In addition, we can consider K_3 which generates boosts along the z direction. This operator satisfies the following commutation relations with the above generators of the little group:

$$[K_3,J_3]=0\ ,\quad [K_3,N_1]=-iN_1\ ,\quad [K_3,N_2]=-iN_2\ . \tag{3}$$

Since the operators N_1, N_2, J_3, and K_3 form a closed Lie algebra,[18] we shall call the group generated by these four operators the "extended little group."

The boost generated by K_3 has no effect on J_3, while changing the scale of N_1 and N_2. In particular, if we start with a monochromatic light wave whose four potential is

$$A^{\mu}(x)=(A,0,0,0)e^{i(kz-\omega t)}\ , \tag{4}$$

in the metric convention: $x^{\mu}=(x,y,z,t)$, the Lorentz boost generated by K_3 leaves the above expression invariant. Since N_1 and N_2 generate gauge transformations on $A^{\mu}(x)$ which do not lead to observable consequences, we can stick to the above expression, and ignore the effect of N_1 and N_2.

J_3 generates rotations around the z axis. In this case, the rotation leads to a linear combination of the x and y components. This operation is consistent with the fact that the photon has two independent components, which is thoroughly familiar to us. Therefore, for all practical purposes, $A^{\mu}(x)$ has just one component which remains invariant under transformations of the extended little group. We can thus write $A^{\mu}(x)$ as

$$A^{\mu}(x)=Ae^{i(kz-\omega t)}\ . \tag{5}$$

While the group of Lorentz transformations has six generators, the extended little group has only four. This means that the extended little group is a subgroup of the Lorentz group. How can we then generalize the above reasoning to take into account the most general case? The choice of the z axis is purely for convenience, and it was chosen to be the direction of the wave propagation. If this direction is rotated, it is not difficult to deal with the problem. If the boost is made along the direction different from that of propagation, then the operation is equivalent to a gauge transformation followed by a rotation.[5] Therefore, the extended little group, while being simpler than the six-parameter Lorentz group, can take care of all possible Lorentz transformations of the monochromatic wave.

The above reasoning remains valid for the case of the superposition of several waves with different frequencies propagating in the same direction:

$$A^{\mu}(x)=\sum_i A_i e^{i(k_i z-\omega_i t)}\ , \tag{6}$$

and the norm

$$N=\sum_i |A_i|^2 \tag{7}$$

remains invariant under transformations of the extended little group. We shall study in Sec. III how the summations given by Eqs. (6) and (7) can be written as integrals.

III. LOCALIZED LIGHT WAVES

For light waves, we are familiar with the expression[19]

$$f(z,t)=(1/2\pi)^{1/2}\int g(k)e^{i(kz-\omega t)}dk\ . \tag{8}$$

However, the expression commonly used in quantum electrodynamics is

$$A(z,t)=\int a(k)e^{i(kz-\omega t)}(1/\sqrt{\omega})dk\ . \tag{9}$$

Equation (9) is a covariant expression in the sense that the norm

$$\int |a(k)|^2(1/\omega)dk \tag{10}$$

is invariant under the Lorentz boost, because the integration measure $(1/\omega)dk$ is Lorentz invariant. On the other hand, the expression given in Eq. (8) is not covariant if $g(k)$ is a scalar function, because the measure dk is not invariant.

It is possible to give a particle interpretation to Eq. (9) after second quantization. However, $A(z,t)$ cannot be used for the localization of photons. On the other hand, it is possible to give a localized probability interpretation to $f(z,t)$ of Eq. (8), while it does not accept the particle interpretation of quantum field theory. This situation is summarized in Table I.

If $g(k)$ is not a scalar function, what is its transformation property? We shall approach this problem using the light-cone coordinate system. We define the light-cone variables as

$$s=(z+t)/2,\quad u=(z-t)\ . \tag{11}$$

The Fourier-conjugate momentum variables are

$$k_s=(k-\omega),\quad k_u(k+\omega)/2\ . \tag{12}$$

If we boost the light wave (or move against the wave with velocity parameter β), the new coordinate variables become

$$s'=\alpha_+ s,\quad u=\alpha_- u\ ,\qquad k_s'=\alpha_- k_s,\quad k_u'=\alpha_+ k_u\ , \tag{13}$$

where

$$\alpha_{\pm}=\left[\frac{1\pm\beta}{1\mp\beta}\right]^{1/2}.$$

If we construct a phase-space consisting of s and k_s or u and k_u, the effect of the Lorentz boost will simply be the elongation and contraction of the coordinate axes. If the coordinate s is elongated by α_+, then k_s is contracted by α_- with $\alpha_+\alpha_-=1$. The orthogonality of the coordinate system is preserved. It appears therefore that the light-cone coordinate system is the natural language for the covariance of the phase-space representation of quantum mechanics.

In the case of light waves, k_s vanishes, and k_u becomes k or ω. In terms of the light-cone variables, the expression of Eq. (8) becomes

$$f(u)=(1/2\pi)^{1/2}\int g(k)e^{iku}dk \ . \tag{14}$$

We are interested in a unitary transformation of the above expression into another Lorentz frame. In order that the norm

$$\int |g(k)|^2 dk \tag{15}$$

be Lorentz invariant, $f(u)$ and $g(k)$ should be transformed like[5]

$$\begin{aligned} f(u)&\to(\alpha_+)^{1/2}f(\alpha_+u) \ , \\ g(k)&\to(\alpha_-)^{1/2}g(\alpha_-k) \ . \end{aligned} \tag{16}$$

Then Parseval's relation

$$\int_{-\infty}^{\infty}|f(u)|^2du=\int_{-\infty}^{\infty}|g(k)|^2dk \tag{17}$$

will remain Lorentz invariant.

It is not difficult to understand why u and k in Eq. (16) are multiplied by α_+ and α_-, respectively. However, we still have to give a physical reason for the existence of the multipliers $(\alpha_\pm)^{1/2}$ in front of $f(u)$ and $g(k)$. They are there because the integration measure in Eq. (15) is not Lorentz invariant.

Multiplier representations of transformation groups have been discussed in the literature.[20] The best known physical example is the Schrödinger wave function under the Galilei transformation.[21] In the case of Galilei boost, the multiplier has unit modulus, and its physical origin is well understood.[21] The purpose of the present paper is to determine the physical origin of the multipliers in Eq. (16).

In Ref. 5, Han *et al.* argued from their experience in the relativistic quark model that the integration measure can become Lorentz invariant if we take into account the remaining light-cone variables in Eqs. (11) and (12). Indeed, the measures $(du\,ds)$ and $(dk_u dk_s)$ are Lorentz invariant. However, this argument is not complete because the s and k_s variables do not exist in the case of light waves. In Sec. IV, we shall use the covariant phase-space representation to solve this problem.

Let us illustrate what we did above using a Gaussian form for $g(k)$:

$$g(k)=\left[\frac{1}{\pi b}\right]^{1/4}\exp\left[\frac{-1}{2b}(k-p)^2\right] , \tag{18}$$

where b is a constant and specifies the width of the distribution and p is the average momentum:

$$p=\int k\,|g(k)|^2dk \ . \tag{19}$$

$f(u)$ takes the form

$$f(u)=\left[\frac{b}{\pi}\right]^{1/4}\exp\left[ipu-\frac{b}{2}u^2\right] . \tag{20}$$

Under the Lorentz boost, $f(u)$ and $g(k)$ are transformed into

$$\begin{aligned} &(\alpha_+)^{1/2}(b/\pi)^{1/4}\exp\left[-i\alpha_+pu-\frac{b}{2}(\alpha_+u)^2\right] , \\ &(\alpha_-)^{1/2}(1/\pi b)^{1/4}\exp\left[\frac{-1}{2b}(\alpha_-)^2(k-\alpha_+p)^2\right] , \end{aligned} \tag{21}$$

respectively. The above expression will be useful in illustrating the phase-space representation in Sec. IV.

IV. COVARIANT PHASE-SPACE REPRESENTATION

Unlike the case of quantum electrodynamics, the Lorentz boost of $f(u)$ and $g(k)$ requires multiplicative factors, as is seen in Eq. (16). In order to trace the physical origin of these factors, let us study the phase-space representation for the light waves in the light-cone coordinate system. In this case, u and k act as the position and momentum variables, respectively.

We can now define the phase-space representation of the light wave as

$$P(u,k)=\left[\frac{1}{\pi}\right]\int f^*(u-y)f(u+y)e^{2iky}dy \ . \tag{22}$$

This form of phase-space representation has been studied extensively in the literature for the solution of the Schrödinger equation,[7–10,12–15] those of the Dirac equation,[11,22] and those of Maxwell's equations.[16] In this section, we are interested in how the above distribution in one Lorentz frame appears in different frames.[23–25]

Since $f(u)$ is the Fourier transform of $g(k)$, we can recover the distribution functions in position and momentum from the above phase-space representation by integrating over the k or u variable:[8]

$$\begin{aligned} \rho(u)&=|f(u)|^2=\int P(u,k)dk \ , \\ \sigma(k)&=|g(k)|^2=\int P(u,k)du \ . \end{aligned} \tag{23}$$

Under the Lorentz boost of Eq. (13), the phase-space distribution function becomes

$$P(u,k)\to P(\alpha_+u,\alpha_-k) \ . \tag{24}$$

Because the measure $(du\,dk)$ is Lorentz invariant, as is illustrated in Fig. 1, there is no need for a multiplicative factor in the above expression. The normalization integral

$$\int P(u,k)du\,dk \tag{25}$$

is Lorentz invariant.

After integrating over k of the transformed phase-space distribution, we can obtain the transformed distribution function in position. Then, $\rho(u)$ will be transformed as

$$\rho(u) \rightarrow \alpha_{+}\rho(\alpha_{+}u) . \tag{26}$$

Similarly, the transformation property of $\sigma(k)$ is

$$\sigma(k) \rightarrow \alpha_{-}\sigma(\alpha_{-}k) . \tag{27}$$

We can now safely conclude that the multiplicative factors for $f(u)$ and $g(k)$ in Eq. (16) are derivable from the covariance of the phase-space representation.

Let us illustrate what we did above using the Gaussian form of $g(k)$ given in Eq. (18). The phase-space distribution takes the form

$$P(u,k) = \left[\frac{1}{\pi}\right] \exp\{-[bu^2 + (k-p)^2/b]\} . \tag{28}$$

This means that the distribution is localized within an elliptic region:

$$[bu^2 + (k-p)^2/b] < 1 . \tag{29}$$

In Fig. 1, this region is illustrated by a circle whose center is at $u=0$, and $k=p$. Under the boost, the new phase-space distribution function becomes

$$\left[\frac{1}{\pi}\right] \exp\{-[b(\alpha_{+}u)^2 + (\alpha_{-})^2(k-\alpha_{+}p)^2/b]\} . \tag{30}$$

This will deform the ellipse (circle in Fig. 1) of Eq. (30). This deformation will not change the normalization integral of Eq. (26). By carrying out the integration over k or u of the above expression, we can obtain the spatial or momentum distribution function which can be derived from the expression of Eq. (21).

The elliptic deformation described in Fig. 1 is like the deformation of the phase-space representation of the squeezed coherent state of light. Indeed, in their recent work,[26] Schleich and Wheeler discussed the phase-space representation of the minimum-uncertainty wave packet in terms of the deformed Gaussian function. The physics of the squeezed state is quite different from the Lorentz boost. It is however interesting to note that these two different physical phenomena can be described by the same mathematics.

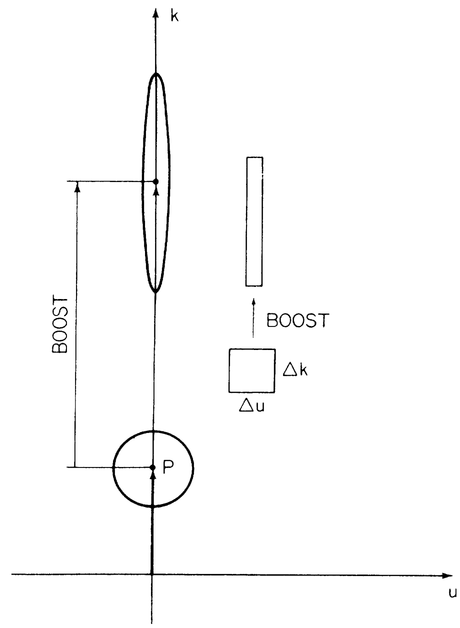

FIG. 1. Phase-space distribution of the localized light wave in the light-cone coordinate system. The Lorentz boost along the z direction contracts (elongates) the u axis while it elongates (contracts) the k axis in such a way that the area element in the phase space is conserved. This represents the Lorentz invariance of Planck's constant.

V. CONCLUDING REMARKS

It has been shown in this paper that the phase-space representation plays an essential role in the construction of the unitary representation for localized light waves. The result of the present paper indicates that the phase-space representation may prove useful in the covariant description of quantum systems. For instance, a covariant picture is urgently needed for relativistic hadronic models in which hadrons are bound states of quarks.

The concept of Planck's constant in terms of the volume element in phase space is quite familiar to us, and this is still being investigated.[27] We have learned in this paper that this procedure can be extended to the relativistic phase space. The light-cone coordinate system appears to be the natural language for this covariant description.

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