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Far-zone behavior of electromagnetic fields generated by fluctuating current distributions

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General expressions are derived for various quantities that characterize the far field generated by any fluctuating localized current distribution that is statistically stationary, at least in the wide sense. In particular, expressions are derived for the cross-spectral density tensors of the electromagnetic field, for the coherence matrix and the Stokes parameters, for the degree of polarization, and for the degree of coherence of the far field in terms of the cross-spectral density of the transverse part of the source current. The analysis is illustrated by considering radiation from a fluctuating linear current source.

I. INTRODUCTION

Past investigations relating to radiation generated by fluctuating light sources have ignored, as a rule, the vector nature of the source distribution, and usually only planar sources were considered.[1] Radiation from fluctuating three-dimensional scalar sources was studied by Carter and Wolf[2] and by LaHaie.[3] For some purposes it is desirable to go beyond the scalar model and take into account the vectorial characteristics of the source. This is, of course, essential when one is interested not only in the angular distribution of the radiated energy but also in the polarization properties of the far field. Moreover, the far-zone properties of a field generated by fluctuating sources has become recently of special interest because of the discovery that source correlations can produce red shifts and blue shifts of lines in the spectrum of the emitted radiation.[4]

Expressions for the cross-spectral density tensors of the electromagnetic field generated by a localized three-dimensional, fluctuating, statistically stationary charge-current distribution in terms of the cross-spectral density tensor of the source current were derived by Carter.[5] Now radiation from deterministic sources of electromagnetic radiation is well known to depend only on the transverse part of the source current and hence it seems appropriate to reexamine the problem considered by Carter in order to elucidate the role of the transverse part in the context of radiation from stochastic sources. This problem is treated in the present paper. We derive expressions for the cross-spectral density tensors of the electromagnetic field in the far zone in terms of the cross-spectral density tensors of the transverse part of the source current. We also obtain expressions for the (spectral) coherence matrix, the Stokes parameters, the degree of polarization, and the degree of coherence of the far field, and we illustrate some of the results with reference to radiation from a fluctuating linear current source.

II. SUMMARY OF FORMULAS RELATING TO THE ELECTROMAGNETIC FIELD IN THE FAR ZONE PRODUCED BY A MONOCHROMATIC CURRENT DISTRIBUTION

It will be useful to begin by summarizing some formulas that we will need later, relating to radiation in free space from a monochromatic current density distribution

$$\mathbf{J}(\mathbf{r},t)=\mathbf{j}(\mathbf{r},\omega)e^{-i\omega t}\ , \tag{2.1}$$

localized in some finite domain D. In Eq. (2.1) $\mathbf{r}$ denotes the position vector of a typical point, t denotes the time, and ω denotes the frequency. The time-independent part of the electric field $\mathbf{E}$ and of the magnetic field $\mathbf{H}$ at a point $\mathbf{r}$ in the far zone, in a direction specified by unit vector $\mathbf{s}$, is given by[6]

$$\mathbf{E}(r\mathbf{s},\omega)\sim\mathbf{F}(\mathbf{s},\omega)\frac{e^{ikr}}{r}\quad\text{as } kr\to\infty\ , \tag{2.2a}$$

$$\mathbf{H}(r\mathbf{s},\omega)\sim\mathbf{G}(\mathbf{s},\omega)\frac{e^{ikr}}{r}\quad\text{as } kr\to\infty\ , \tag{2.2b}$$

where (in Gaussian system of units)

$$\mathbf{F}(\mathbf{s},\omega)=-(2\pi)^3\frac{ik}{c}\mathbf{s}\times[\mathbf{s}\times\tilde{\mathbf{j}}(k\mathbf{s},\omega)]\ , \tag{2.3a}$$

$$\mathbf{G}(\mathbf{s},\omega)=(2\pi)^3\frac{ik}{c}\mathbf{s}\times\tilde{\mathbf{j}}(k\mathbf{s},\omega)\ . \tag{2.3b}$$

In these formulas

$$k=\frac{\omega}{c} \tag{2.4}$$

is the wave number associated with the frequency ω, c is the speed of light *in vacuo,* and $\tilde{\mathbf{j}}(\mathbf{K},\omega)$ is the Fourier transform of the current density, viz.,

$$\tilde{\mathbf{j}}(\mathbf{K}\omega)=\frac{1}{(2\pi)^3}\int_D\mathbf{j}(\mathbf{r}',\omega)e^{i\mathbf{K}\cdot\mathbf{r}'}d^3r'\ . \tag{2.5}$$

The right-hand side of Eq. (2.3) may readily be expressed in terms of the Fourier transform of the transverse part of the current

$$\tilde{\mathbf{j}}^T(\mathbf{K},\omega)=\frac{[\mathbf{K}\times\tilde{\mathbf{j}}(\mathbf{K},\omega)]\times\mathbf{K}}{K^2}\ , \tag{2.6}$$

in the simpler form

$$\mathbf{F}(\mathbf{s},\omega)=(2\pi)^3\frac{ik}{c}\tilde{\mathbf{j}}^T(k\mathbf{s},\omega)\ , \tag{2.7a}$$

$$\mathbf{G}(\mathbf{s},\omega)=(2\pi)^3\frac{ik}{c}\mathbf{s}\times\tilde{\mathbf{j}}^T(k\mathbf{s},\omega)\ . \tag{2.7b}$$

From Eqs. (2.2) and (2.3) we readily deduce that

$$\mathbf{H}(r\mathbf{s},\omega)=\mathbf{s}\times\mathbf{E}(r\mathbf{s},\omega)\ , \tag{2.8}$$

$$\mathbf{s}\cdot\mathbf{E}(r\mathbf{s},\omega)=\mathbf{s}\cdot\mathbf{H}(r\mathbf{s},\omega)=0\ . \tag{2.9}$$

The formulas (2.2), together with the relations (2.8) and (2.9), express the well-known fact that the electromagnetic field in the far zone behaves globally as a divergent spherical wave whereas it behaves locally as a plane electromagnetic wave.

III. CROSS-SPECTRAL DENSITY TENSORS OF THE FAR FIELD PRODUCED BY FLUCTUATING CURRENTS

According to the Wiener-Khintchine theorem (in an obvious generalization), the cross-spectral density $W(\mathbf{r}_1,\mathbf{r}_2,\omega)$ of a fluctuating scalar source $Q(\mathbf{r},t)$, characterized by a stationary ensemble, is equal to the Fourier transform of its cross-correlation function,

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\int_{-\infty}^{\infty}\langle Q^*(\mathbf{r}_1,t)Q(\mathbf{r}_2,t+\tau)\rangle e^{i\omega\tau}d\tau\ . \tag{3.1}$$

Here the asterisk denotes the complex conjugate and the angular brackets denote the ensemble average.

It has been shown not long ago[7] that there is always an ensemble $\{q(\mathbf{r},\omega)\exp(-i\omega t)\}$ of monochromatic oscillations such that the cross-spectral density can be expressed in the form

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle \mathbf{q}^*(\mathbf{r}_1,\omega)\mathbf{q}(\mathbf{r}_2,\omega)\rangle_\omega\ , \tag{3.2}$$

where the symbol $\langle\ \rangle_\omega$ represents the average over this ensemble of "frequency-dependent" realizations. A similar representation also exists for the cross-spectral density of the field generated by the source.[8]

There are similar representations, of course, for the cross-spectral density tensors of various fluctuating vector quantities, e.g., of current densities or of the field vectors, that are characterized by stationary ensembles. In particular, the cross-spectral density tensor of the current density and of the transverse current density are expressible, in dyadic notation, in the form

$$\overleftrightarrow{\mathbf{W}}_{jj}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle \mathbf{j}^*(\mathbf{r}_1,\omega)\mathbf{j}(\mathbf{r}_2,\omega)\rangle_\omega\ , \tag{3.3a}$$

and

$$\overleftrightarrow{\mathbf{W}}_{jj}^T(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle \mathbf{j}^{T*}(\mathbf{r}_1,\omega)\mathbf{j}^T(\mathbf{r}_2,\omega)\rangle_\omega\ , \tag{3.3b}$$

respectively; and the cross-spectral density tensors involving the field vectors are expressible as

$$\overleftrightarrow{\mathbf{W}}_{ee}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle \mathbf{E}^*(\mathbf{r}_1,\omega)\mathbf{E}(\mathbf{r}_2,\omega)\rangle_\omega\ , \tag{3.4a}$$

$$\overleftrightarrow{\mathbf{W}}_{eh}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle \mathbf{E}^*(\mathbf{r}_1,\omega)\mathbf{H}(\mathbf{r}_2,\omega)\rangle_\omega\ , \tag{3.4b}$$

etc.

It follows at once from Eqs. (3.4) and (2.2) that in the far zone (superscript ∞)

$$\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)=\overleftrightarrow{\mathbf{W}}_{\mathrm{FF}}(\mathbf{s}_1,\mathbf{s}_2,\omega)\frac{e^{ik(r_2-r_1)}}{r_1r_2}\ , \tag{3.5}$$

where

$$\overleftrightarrow{\mathbf{W}}_{\mathrm{FF}}(\mathbf{s}_1,\mathbf{s}_2,\omega)=\langle \mathbf{F}^*(\mathbf{s}_1,\omega)\mathbf{F}(\mathbf{s}_2,\omega)\rangle_\omega\ , \tag{3.6}$$

with strictly similar expressions for the other three cross-spectral density tensors $\overleftrightarrow{\mathbf{W}}_{hh}^{(\infty)}$, $\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}$, and $\overleftrightarrow{\mathbf{W}}_{he}^{(\infty)}$ of the far field. Now it follows from Eq. (2.7a) that

$$\overleftrightarrow{\mathbf{W}}_{\mathrm{FF}}(\mathbf{s}_1,\mathbf{s}_2,\omega)=(2\pi)^6\left[\frac{k}{c}\right]^2\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}^T(k\mathbf{s}_1,k\mathbf{s}_2,\omega)\ , \tag{3.7}$$

where

$$\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}^T(k\mathbf{s}_1,k\mathbf{s}_2,\omega)=\langle \tilde{\mathbf{j}}^{T*}(k\mathbf{s}_1,\omega)\tilde{\mathbf{j}}^T(k\mathbf{s}_2,\omega)\rangle_\omega\ . \tag{3.8}$$

On substituting from Eq. (3.7) into Eq. (3.5), we obtain the following expression for $\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}$:

$$\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}(r\mathbf{s}_1,r\mathbf{s}_2,\omega)=(2\pi)^6\left[\frac{k}{c}\right]^2\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}^T(k\mathbf{s}_1,k\mathbf{s}_2,\omega)\frac{e^{ik(r_2-r_1)}}{r_1r_2}\ . \tag{3.9}$$

The cross-spectral density tensor of $\tilde{\mathbf{j}}^T$ on the right of Eq. (3.9) may readily be expressed in terms of the Fourier inverse of the cross-spectral density tensor of the transverse part $\mathbf{j}^T$ of the current density, as we will now show. On substituting in Eq. (3.8) for $\tilde{\mathbf{j}}^T$ in terms of $\mathbf{j}^T$ viz.,

$$\tilde{\mathbf{j}}^T(\mathbf{K},\omega)=\frac{1}{(2\pi)^3}\int_D \mathbf{j}^T(\mathbf{r}',\omega)e^{-i\mathbf{K}\cdot\mathbf{r}'}d^3r'\ , \tag{3.10}$$

we obtain the formula

$$\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}^T(\mathbf{K}_1,\mathbf{K}_2,\omega)=\frac{1}{(2\pi)^6}\int_D\int_D\langle \mathbf{j}^{T*}(\mathbf{r}_1',\omega)\mathbf{j}^T(\mathbf{r}_2',\omega)\rangle \times e^{-i(\mathbf{K}_2\cdot\mathbf{r}_2'-\mathbf{K}_1\cdot\mathbf{r}_1')}d^3r_1'd^3r_2'$$
$$=\frac{1}{(2\pi)^6}\int_D\int_D\overleftrightarrow{\mathbf{W}}_{jj}^T(\mathbf{r}_1',\mathbf{r}_2',\omega)\times e^{-i(\mathbf{K}_2\cdot\mathbf{r}_2'-\mathbf{K}_1\cdot\mathbf{r}_1')}d^3r_1'd^3r_2'\ ,$$

i.e.,

$$\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}^T(\mathbf{K}_1,\mathbf{K}_2,\omega)=\tilde{\overleftrightarrow{\mathbf{W}}}{}_{jj}^T(-\mathbf{K}_1,\mathbf{K}_2,\omega)\ , \tag{3.11}$$

where

$$\tilde{\overleftrightarrow{\mathbf{W}}}{}_{jj}^T(\mathbf{K}_1,\mathbf{K}_2,\omega)=\frac{1}{(2\pi)^6}\int_D\int_D\overleftrightarrow{\mathbf{W}}_{jj}^T(\mathbf{r}_1',\mathbf{r}_2',\omega)\times e^{-i(\mathbf{K}_1\cdot\mathbf{r}_1'+\mathbf{K}_2\cdot\mathbf{r}_2')}d^3r_1'd^3r_2' \tag{3.12}$$

is the Fourier inverse of the cross-spectral density tensor of the transverse current.

We note some transversality conditions that the above tensors satisfy. It follows from the definitions (3.8) and (2.6) that

$$\mathbf{K}_1\cdot\overleftrightarrow{\mathbf{W}}_{\bar{j}\bar{j}}^{T}(\mathbf{K}_1,\mathbf{K}_2,\omega)=\overleftrightarrow{\mathbf{W}}_{\bar{j}\bar{j}}^{T}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{K}_2=0 , \quad (3.13a)$$

which implies, in view of the relation (3.11), that

$$\mathbf{K}_1\cdot\overleftrightarrow{\mathbf{W}}_{jj}^{T}(-\mathbf{K}_1,\mathbf{K}_2,\omega)=\overleftrightarrow{\mathbf{W}}_{jj}^{T}(-\mathbf{K}_1,\mathbf{K}_2\cdot\omega)\cdot\mathbf{K}_2=0 . \quad (3.13b)$$

On substituting Eq. (3.11) into Eq. (3.9), we obtain the following expression for the cross-spectral density tensor (in dyadic notation) of the electric field in the far zone:

$$\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)=(2\pi)^6\left[\frac{k}{c}\right]^2\overleftrightarrow{\mathbf{W}}_{jj}^{T}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)\frac{e^{ik(r_2-r_1)}}{r_1r_2} . \quad (3.14a)$$

In a strictly similar manner one can obtain the following expressions for the three other cross-spectral density tensors of the far field:

$$\overleftrightarrow{\mathbf{W}}_{hh}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)=-(2\pi)^6\left[\frac{k}{c}\right]^2\mathbf{s}_1\times\overleftrightarrow{\mathbf{W}}_{jj}^{T}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)\times\mathbf{s}_2\frac{e^{ik(r_2-r_1)}}{r_1r_2} , \quad (3.14b)$$

$$\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)=-(2\pi)^6\left[\frac{k}{c}\right]^2\overleftrightarrow{\mathbf{W}}_{jj}^{T}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)\times\mathbf{s}_2\frac{e^{ik(r_2-r_1)}}{r_1r_2} , \quad (3.14c)$$

$$\overleftrightarrow{\mathbf{W}}_{he}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)=(2\pi)^6\left[\frac{k}{c}\right]^2\mathbf{s}_1\times\overleftrightarrow{\mathbf{W}}_{jj}^{T}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)\frac{e^{ik(r_2-r_1)}}{r_1r_2} . \quad (3.14d)$$

It is not difficult to express the tensor $\overleftrightarrow{\mathbf{W}}_{jj}^{T}$ associated with the transverse current density which appears in these formulas in terms of the tensor $\overleftrightarrow{W}_{jj}$ associated with the total current density. The calculation is carried out in Appendix A. The result is

$$\begin{aligned}\overleftrightarrow{\mathbf{W}}_{jj}^{T}(\mathbf{K}_1,\mathbf{K}_2,\omega)=&\overleftrightarrow{\mathbf{W}}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)-\mathbf{s}_1\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)\\&-\overleftrightarrow{\mathbf{W}}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2\\&+\mathbf{s}_1\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2 ,\end{aligned} \quad (3.15)$$

where

$$\mathbf{s}_1=\frac{\mathbf{K}_1}{|\mathbf{K}_1|},\quad \mathbf{s}_2=\frac{\mathbf{K}_2}{|\mathbf{K}_2|} \quad (3.16)$$

are unit vectors in the directions of $\mathbf{K}_1$ and $\mathbf{K}_2$, respectively.

We see from Eqs. (3.14) that the tensors $\overleftrightarrow{\mathbf{W}}_{hh}^{(\infty)}$, $\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}$, and $\overleftrightarrow{\mathbf{W}}_{he}^{(\infty)}$ are expressible in terms of the tensor $\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}$ by the following formulas:

$$\overleftrightarrow{\mathbf{W}}_{hh}^{(\infty)}=-\mathbf{s}_1\times\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}\times\mathbf{s}_2 , \quad (3.17a)$$

$$\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}=-\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}\times\mathbf{s}_2 , \quad (3.17b)$$

$$\overleftrightarrow{\mathbf{W}}_{he}^{(\infty)}=\mathbf{s}_1\times\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)} . \quad (3.17c)$$

Moreover, it is also readily seen from Eqs. (3.14), if use is made of the relations (3.13b), that

$$\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{ab}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)=0 , \quad (3.18a)$$

and

$$\overleftrightarrow{\mathbf{W}}_{ab}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega)\cdot\mathbf{s}_2=0 , \quad (3.18b)$$

where a and b each stands for either e or h.

IV. RADIATED POWER

We will now derive expressions for the average values of the energy density of the far field and for the angular distribution of the power radiated by the source. The averaged energy density $\langle U(\mathbf{r},\omega)\rangle_\omega$ and the averaged Poynting vector $\langle\mathbf{S}(\mathbf{r},\omega)\rangle_\omega$ at a point $\mathbf{r}$ in a stationary field, at frequency ω, are given by an obvious generalization of the corresponding formulas relating to a monochromatic field,[9] viz.,

$$\langle U(\mathbf{r},\omega)\rangle_\omega=\frac{1}{16\pi}[\langle\mathbf{E}^*(\mathbf{r},\omega)\cdot\mathbf{E}(\mathbf{r},\omega)\rangle_\omega+\langle\mathbf{H}^*(\mathbf{r},\omega)\cdot\mathbf{H}(\mathbf{r},\omega)\rangle_\omega] , \quad (4.1)$$

$$\langle\mathbf{S}(\mathbf{r},\omega)\rangle_\omega=\frac{c}{8\pi}\,\mathrm{Re}\langle\mathbf{E}^*(\mathbf{r},\omega)\times\mathbf{H}(\mathbf{r},\omega)\rangle_\omega , \quad (4.2)$$

where Re denotes the real part and the Gaussian system of units is used. If we recall the definitions (3.4) of the cross-spectral density tensors $\overleftrightarrow{\mathbf{W}}_{ee}$ and $\overleftrightarrow{\mathbf{W}}_{eh}$, the formulas (4.1) and (4.2) may be rewritten in the form

$$\langle U(\mathbf{r},\omega)\rangle_\omega=\frac{1}{16\pi}[\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{ee}(\mathbf{r},\mathbf{r},\omega)+\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{hh}(\mathbf{r},\mathbf{r},\omega)] , \quad (4.3)$$

and

$$\langle\mathbf{S}(\mathbf{r},\omega)\rangle_\omega=\frac{c}{8\pi}\,\mathrm{Re}[\mathcal{V}(\overleftrightarrow{\mathbf{W}}_{eh}(\mathbf{r},\mathbf{r},\omega))] . \quad (4.4)$$

In Eq. (4.3) Tr denotes the trace. In Eq. (4.4) $\mathcal{V}$ denotes the vector of the dyadic, i.e., $\mathcal{V}(\overleftrightarrow{\mathbf{W}}_{eh})$ is the vector with components $(W_{eh})_{23}-(W_{eh})_{32}$, $(W_{eh})_{31}-(W_{eh})_{13}$, and $(W_{eh})_{12}-(W_{eh})_{21}$, where $(W_{eh})_{jk}$ $(j=1,2,3,\ k=1,2,3)$

are the components of the Cartesian tensor represented by the dyadic $\overleftrightarrow{\mathbf{W}}_{eh}$.

Now we have from Eqs. (3.14a)–(3.14c), with the choice $\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}$, $\mathbf{s}_1=\mathbf{s}_2=\mathbf{s}$,

$$\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=(2\pi)^6\left[\frac{k}{cr}\right]^2\tilde{W}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)\ , \tag{4.5a}$$

$$\overleftrightarrow{\mathbf{W}}_{hh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=-(2\pi)^6\left[\frac{k}{cr}\right]^2\mathbf{s}\times\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)\times\mathbf{s}\ , \tag{4.5b}$$

$$\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=-(2\pi)^6\left[\frac{k}{cr}\right]^2\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)\times\mathbf{s}\ . \tag{4.5c}$$

We also have, according to Appendix B,

$$\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{hh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)\ . \tag{4.6}$$

Let us now set $\mathbf{r}=r\mathbf{s}$ in the expression (4.3) and let us proceed to the asymptotic limit as $kr\rightarrow\infty$, with the direction $\mathbf{s}$ being fixed. If we make use of the expression (4.5a) and of the relation (4.6) we obtain the following formula for the average energy density of the far field:

$$\langle U^{(\infty)}(r\mathbf{s},\omega)\rangle_\omega=8\pi^5\left[\frac{k}{cr}\right]^2\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)\ . \tag{4.7}$$

Next let us consider the average Poynting vector of the far field. We show in Appendix C that

$$\mathcal{V}(\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega))=(2\pi)^6\left[\frac{k}{cr}\right]^2\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)\mathbf{s}\ . \tag{4.8}$$

Now according to Eq. (4.7) the trace that appears on the right-hand side of Eq. (4.8) is proportional to the averaged energy density and is, therefore, necessarily real. This fact may also be established more directly by making use of the fact that the cross-spectral density tensor of the transverse current is necessarily non-negative definite. Hence the expression (4.8) represents a real vector. Using this result in the formula (4.4), specialized to the far field, we find that

$$\langle \mathbf{S}^{(\infty)}(r\mathbf{s},\omega)\rangle_\omega=\frac{8\pi^5k^2}{cr^2}\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)\mathbf{s}\ . \tag{4.9}$$

On comparing Eqs. (4.7) and (4.9) we see at once that the average energy density and the average Poynting vector of the far field generated by the fluctuating current distribution are related by the formula

$$\langle \mathbf{S}^{(\infty)}(r\mathbf{s},\omega)\rangle_\omega=c\langle U^{(\infty)}(r\mathbf{s},\omega)\rangle_\omega\mathbf{s}\ . \tag{4.10}$$

This relation is of exactly the same form as the corresponding formula pertaining to the far field generated by a localized monochromatic current distribution. It implies that at each point in the far zone the average energy density may be regarded as propagating in the outward radial direction with the vacuum speed of light c.

Formulas (4.7) and (4.9) express the average energy density and the averaged Poynting vector of the far field in terms of the trace of the Fourier transform of the cross-spectral density tensor of the *transverse* part of the current density $\mathbf{j}^T(\mathbf{r},\omega)$. We may readily express this trace in terms of the trace of the Fourier transform of the cross-spectral density tensor of the *total* current density $\mathbf{j}(\mathbf{r},\omega)$. The relationship between these two traces is derived in Appendix D and is found to be

$$\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)=\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)-\mathbf{s}\cdot\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)\cdot\mathbf{s}\ . \tag{4.11}$$

It follows at once from Eq. (4.9) and the significance of the Poynting vector that the radiant intensity $J(\mathbf{s},\omega)$, i.e., the rate at which the source radiates energy at frequency ω per unit solid angle around a direction specified by a unit vector $\mathbf{s}$ is given by

$$J(\mathbf{s},\omega)=\lim_{r\rightarrow\infty}r^2\mathbf{s}\cdot\langle \mathbf{S}^{(\infty)}(r\mathbf{s},\omega)\rangle_\omega=\frac{8\pi^5k^2}{c}\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)\ . \tag{4.12}$$

This formula is the electromagnetic analogue of an expression for the radiant intensity well known in the theory of radiation from fluctuating three-dimensional scalar sources.[10]

Using Eq. (4.12) we see at once that the total power radiated by the fluctuating current at frequency ω is given by the formula

$$\mathcal{P}(\omega)\equiv\int J(\mathbf{s},\omega)d\Omega=\frac{8\pi^5k^2}{c}\int\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)d\Omega\ , \tag{4.13}$$

where the integration extends over the whole 4π solid angle generated by the unit vector $\mathbf{s}$.

If we make use of the relations (4.11), formulas (4.12) and (4.13) may be expressed at once in terms of dyadic $\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}$ associated with the total current density rather than in terms of the dyadic $\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}^{T}$ associated with its transverse part.

V. THE 2×2 ELECTRIC COHERENCE MATRIX, THE STOKES PARAMETERS, AND THE DEGREE OF POLARIZATION OF THE FAR FIELD

We noted in Sec. II that any realization of the electromagnetic field at each point $\mathbf{r}=r\mathbf{s}$ in the far zone of a fluctuating localized source has the structure of a plane electromagnetic wave that propagates in the $\mathbf{s}$ direction. Now the polarization properties of such a wave are most conveniently analyzed in terms of a 2×2 coherence matrix or in terms of (four) Stokes parameters. Hence we can use either of these two descriptions to investigate the state of polarization of the far field generated by our fluctuating source. In this section we will derive formulas for the coherence matrix and for the Stokes parameters of the radiated field.

Let us set $\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}$, $\mathbf{s}_1=\mathbf{s}_2=\mathbf{s}$ in the formula (3.14a). This gives

$$\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=(2\pi)^6\left[\frac{k}{cr}\right]^2\overleftrightarrow{\tilde{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega) . \quad (5.1)$$

It will be convenient to simplify the notation by setting

$$\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)=\overleftrightarrow{\mathbf{W}}_{ee}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega) , \quad (5.2a)$$

$$A=(2\pi)^6\left[\frac{k}{cr}\right]^2 . \quad (5.2b)$$

The formula (5.1) then becomes

$$\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)=A\overleftrightarrow{\tilde{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega) . \quad (5.3)$$

With the dyadic $\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)$ we may introduce a 3×3 electric coherence matrix with elements

$$\overleftrightarrow{\mathscr{E}}_{pq}(r\mathbf{s},\omega)=\hat{\mathbf{p}}\cdot\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)\cdot\hat{\mathbf{q}}=\langle E_p^*(r\mathbf{s},\omega)E_q(r\mathbf{s},\omega)\rangle_\omega , \quad (5.4)$$

where p,q ($p=x,y,z$, $q=x,y,z$) label Cartesian components, taken with respect to some fixed rectangular coordinate systems, and $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ represent unit vectors along the corresponding coordinate axes.

The matrix $\mathscr{E}_{pq}(r\mathbf{s},\omega)$ is a 3×3 correlation matrix of the electric field in the far zone. Similar correlation matrices may, of course, be introduced for the magnetic field or for the mixed combinations containing both the electric and the magnetic fields.

Since the far field is transverse, we may introduce at each point $P(r\mathbf{s})$ in the far zone a local orthogonal coordinate system of axes, such that only four of the nine elements of the electric coherence matrix are, in general, nonvanishing. A natural such coordinate system is provided by the directions defined by the coordinate lines of a spherical polar system (see Fig. 1). In that system of the axes the components s_x,s_y,s_z of the unit vector $\mathbf{s}$ are given by

$$s_x=\sin\theta\cos\phi,\quad s_y=\sin\theta\sin\phi,\quad s_z=\cos\theta . \quad (5.5)$$

Let $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ denote the unit vectors along the Cartesian rectangular axes and $\hat{\mathbf{e}}_r,\hat{\mathbf{e}}_\theta,\hat{\mathbf{e}}_\phi$ the unit vectors along the coordinate lines of the spherical polar system. Then[11,12]

$$\hat{\mathbf{e}}_r=\sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\,\hat{\mathbf{y}}+\cos\theta\,\hat{\mathbf{z}} ,$$
$$\hat{\mathbf{e}}_\theta=\cos\theta\cos\phi\,\hat{\mathbf{x}}+\cos\theta\sin\phi\,\hat{\mathbf{y}}-\sin\theta\,\hat{\mathbf{z}} , \quad (5.6)$$
$$\hat{\mathbf{e}}_\phi=-\sin\phi\,\hat{\mathbf{x}}+\cos\phi\,\hat{\mathbf{y}} .$$

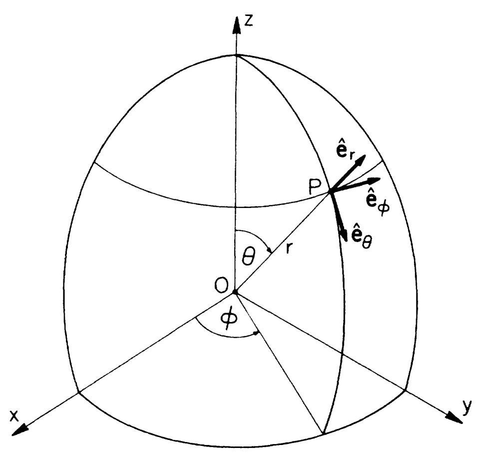

FIG. 1. Illustrating the significance of the unit vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_\phi$ pointing along the coordinate lines of a spherical polar system.

Referred to the spherical polar coordinate system, the electric coherence matrix at each point in the far zone will have only four (generally nonvanishing) elements, given by

$$\mathscr{E}_{\alpha\beta}(r\mathbf{s},\omega)=\hat{\mathbf{e}}_\alpha\cdot\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)\cdot\hat{\mathbf{e}}_\beta$$
$$=\langle E_\alpha^*(r\mathbf{s},\omega)E_\beta(r\mathbf{s},\omega)\rangle_\omega , \quad (5.7)$$

where α and β each stands for θ and ϕ, which now label "angular" components. Finally, on substituting for $\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)$ from Eq. (5.3) into the second expression in Eq. (5.7), we obtain the following expression for the four elements of the 2×2 electric coherence matrix of the far field:

$$\mathscr{E}_{\alpha\beta}(r\mathbf{s},\omega)=A\hat{\mathbf{e}}_\alpha\cdot\overleftrightarrow{\tilde{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega)\cdot\hat{\mathbf{e}}_\beta . \quad (5.8)$$

To avoid any possible misunderstanding we stress that in this formula the subscripts α and β label components taken along the curvilinear θ, ϕ coordinate lines at the point $P(r\mathbf{s})$ in the far zone, whereas the subscript jj on the right indicates that $\overleftrightarrow{\tilde{\mathbf{W}}}_{jj}^{T}$ is the Fourier transform of the cross-spectral density of the transverse current $\mathbf{j}^T$.

From the knowledge of the 2×2 electric coherence matrix (5.8), one can completely answer all questions relating to the state of polarization of the far field. The appropriate formulas are obtained, with trivial modifications,[13] from formulas given in Sec. 10.8 of Ref. 9.

In terms of the elements of the 2×2 electric coherence matrix (5.8), one may readily obtain expressions for the four Stokes parameters s_0,s_1,s_2,s_3 that are also frequently used to analyze the polarization properties of a fluctuating electromagnetic wave. They are given by (cf. Ref. 9, Sec. 10.8.3)

$$s_0=\mathscr{E}_{\alpha\alpha}+\mathscr{E}_{\beta\beta} ,$$
$$s_1=\mathscr{E}_{\alpha\alpha}-\mathscr{E}_{\beta\beta} ,$$
$$s_2=\mathscr{E}_{\alpha\beta}+\mathscr{E}_{\beta\alpha} , \quad (5.9)$$
$$s_3=i(\mathscr{E}_{\beta\alpha}-\mathscr{E}_{\alpha\beta})$$

(no summation over repeated indexes).

Finally, let us consider the degree of polarization $P_\omega(\mathbf{s})$ of the far field, in a direction specified by the unit vector $\mathbf{s}$. It is given by [cf. Ref. 9, Sec. 10.8, Eq. (52)]

$$P_\omega(\mathbf{s})=\left[1-\frac{4\det\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)}{[\mathrm{Tr}\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)]^2}\right]^{1/2} . \quad (5.10)$$

Since both the determinant and the trace of a dyadic (tensor) are invariant with respect to the rotation of axes, we have from Eq. (5.8)

$$\mathrm{Det}\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)=A^2\det\overleftrightarrow{\tilde{\mathbf{W}}}_{jj}^{T}(-k\mathbf{s},k\mathbf{s},\omega) , \quad (5.11a)$$

$$\mathrm{Tr}\overleftrightarrow{\mathscr{E}}(r\mathbf{s},\omega)=A\,\mathrm{Tr}\widetilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)\ . \tag{5.11b}$$

On substituting from Eqs (5.11) into Eq. (5.10) we obtain the following expression for the degree of polarization of the electric field in the far zone:

$$P_\omega(\mathbf{s})=\left[1-\frac{4\det\widetilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)}{[\mathrm{Tr}\widetilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)]^2}\right]^{1/2}\ . \tag{5.12}$$

VI. AN EXAMPLE: THE FAR FIELD FROM A FLUCTUATING LINEAR CURRENT SOURCE

We will now illustrate some of our formulas by considering the far field generated by a volume distribution of current density that oscillates in a fixed direction but is subject to some random perturbations, which will be assumed to be characterized by a stationary ensemble. For simplicity we will also assume that the perturbations affect only the magnitude but not the direction of the oscillating current.

Let us choose a rectangular Cartesian system of axes, with the z direction along the direction of the oscillations. The cross-spectral density tensor $\overleftrightarrow{\mathbf{W}}_{jj}$ of the current may then by expressed in terms of an ensemble of frequency-dependent realizations $j(\mathbf{r},\omega)\hat{\mathbf{z}}$ of the current density, in the form [cf. Eq. (3.3a)]

$$\overleftrightarrow{\mathbf{W}}_{jj}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\mathscr{J}(\mathbf{r}_1,\mathbf{r}_2,\omega)\hat{\mathbf{z}}\hat{\mathbf{z}}\ , \tag{6.1}$$

where

$$\mathscr{J}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle j^*(\mathbf{r}_1,\omega)j(\mathbf{r}_2,\omega)\rangle_\omega\ . \tag{6.2}$$

Let us first determine the radiant intensity $J(\mathbf{s},\omega)$ produced by the source. For this purpose we must determine, according to the general formula (4.12), the Fourier transform of the cross-spectral density tensor $\widetilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}$ of the transverse current. It follows from Eqs. (3.15) and (6.1) that it is given by

$$\widetilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)=\widetilde{\mathscr{J}}(\mathbf{K}_1,\mathbf{K}_2,\omega)[\hat{\mathbf{z}}\hat{\mathbf{z}}-(\mathbf{s}_1\cdot\hat{\mathbf{z}})\mathbf{s}_1\hat{\mathbf{z}}-(\hat{\mathbf{z}}\cdot\mathbf{s}_2)\hat{\mathbf{z}}\mathbf{s}_2+(\mathbf{s}_1\cdot\hat{\mathbf{z}})(\hat{\mathbf{z}}\cdot\mathbf{s}_2)\mathbf{s}_1\mathbf{s}_2]\ , \tag{6.3}$$

where

$$\widetilde{\mathscr{J}}(\mathbf{K}_1,\mathbf{K}_2,\omega)=\frac{1}{(2\pi)^6}\int_D\int_D\mathscr{J}(\mathbf{r}'_1,\mathbf{r}'_2,\omega)\times e^{-i(\mathbf{K}_1\cdot\mathbf{r}'_1+\mathbf{K}_2\cdot\mathbf{r}'_2)}d^3r'_1d^3r'_2\ , \tag{6.4}$$

D denotes the domain occupied by the current, and

$$\mathbf{s}_1=\frac{\mathbf{K}_1}{|\mathbf{K}_1|},\quad \mathbf{s}_2=\frac{\mathbf{K}_2}{|\mathbf{K}_2|}\ . \tag{6.5}$$

With $\mathbf{K}_1=-k\mathbf{s}_1$, $\mathbf{K}_2=k\mathbf{s}_2$ $(k=\omega/c)$, the formula (6.3) reduces to

$$\widetilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)=\widetilde{\mathscr{J}}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)\times[\hat{\mathbf{z}}\hat{\mathbf{z}}-(\cos\theta_1)\mathbf{s}_1\hat{\mathbf{z}}-(\cos\theta_2)\hat{\mathbf{z}}\mathbf{s}_2+(\cos\theta_1)(\cos\theta_2)\mathbf{s}_1\mathbf{s}_2]\ , \tag{6.6}$$

where θ_1 and θ_2 are the angles that the unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ make with the z direction (the direction of the current), as shown in Fig. 2.

From Eq. (6.6) it follows at once that

$$\mathrm{Tr}\widetilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)=\widetilde{\mathscr{J}}(-k\mathbf{s},k\mathbf{s},\omega)\sin^2\theta\ . \tag{6.7}$$

On substituting from Eq. (6.7) into the formula (4.12) we obtain the following expression for the radiant intensity generated by our fluctuating linear current source:

$$J(\mathbf{s},\omega)=\frac{8\pi^5k^2}{c}\widetilde{\mathscr{J}}(-k\mathbf{s},k\mathbf{s},\omega)\sin^2\theta\ . \tag{6.8}$$

This expression has a similar mathematical structure as the corresponding formula for the radiant intensity of a fluctuating scalar source [cf. Ref. 2(a), Eq. (3.9)], but it differs from it mainly in that it does not include the factor $\sin^2\theta$. The factor $\sin^2\theta$ in the formula (6.8) is, of course, familiar from the theory of linear antennas.[14] However, unlike in the case of a sinusoidally oscillating linear antenna, there is now an additional contribution to the angular distribution of the radiation arising from the $\mathbf{s}$ dependence of the Fourier transform of the correlation function $\mathscr{J}$.

Next let us consider the state of polarization of the far field. Because of the rather simple form of the tensor (6.1), it will be advantageous to make use, in the present case, of a certain simple property of $\widetilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}$ which we will now establish.

It follows from Eq. (2.6) with $\mathbf{K}=k\mathbf{s}$ and $\tilde{\mathbf{j}}(k\mathbf{s},\omega)=\tilde{j}(k\mathbf{s},\omega)\hat{\mathbf{z}}$ that

$$\tilde{\mathbf{j}}^T(k\mathbf{s},\omega)=\tilde{j}(k\mathbf{s},\omega)\mathbf{t}\ , \tag{6.9}$$

where

$$\mathbf{t}=\mathbf{s}\times(\hat{\mathbf{z}}\times\mathbf{s})\ . \tag{6.10}$$

Let us introduce a unit vector $\hat{\mathbf{t}}$ along $\mathbf{t}$, i.e.,

$$\hat{\mathbf{t}}=\frac{\mathbf{s}\times(\hat{\mathbf{z}}\times\mathbf{s})}{|\hat{\mathbf{s}}\times(\hat{\mathbf{z}}\times\mathbf{s})|}\ . \tag{6.11}$$

By straightforward calculation one can show that $\mathbf{s}\times(\hat{\mathbf{z}}\times\mathbf{s})=\hat{\mathbf{t}}\sin\theta$ and using this result in Eq. (6.11) it follows that

$$\mathbf{t}=\hat{\mathbf{t}}\sin\theta\ . \tag{6.12}$$

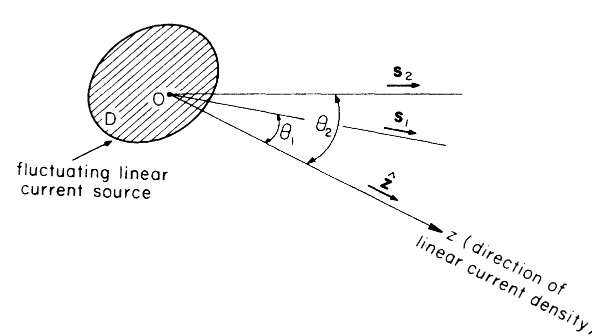

FIG. 2. Illustrating the notation pertaining to calculation of the far-zone behavior of a field generated by a fluctuating linear current source.

It is clear from Eqs. (6.10) and (6.12) that the unit vector $\hat{\mathbf{t}}$ is perpendicular to s and lies in the plane specified by s and $\hat{\mathbf{z}}$ (see Fig. 3). On substituting from Eq. (6.9) into Eq. (3.8), specialized to the case when $\mathbf{s}_1=\mathbf{s}_2=\mathbf{s}$, and on using the relations (3.11) and (6.12) we obtain the following expression for $\overleftrightarrow{\mathbf{W}}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)$:

$$\overleftrightarrow{\mathbf{W}}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)=\tilde{\mathcal{J}}(-k\mathbf{s},k\mathbf{s},\omega)(\sin^2\theta)\hat{\mathbf{t}}\hat{\mathbf{t}} \ . \tag{6.13}$$

In deriving this formula we also made use of Eqs. (6.2) and (6.4).

Since in the representation (6.13) the dyadic $\overleftrightarrow{\mathbf{W}}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)$ has only one component, namely the $\hat{\mathbf{t}}\hat{\mathbf{t}}$ component, it is clear that

$$\mathrm{Det}\overleftrightarrow{\mathbf{W}}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)=0 \ . \tag{6.14}$$

It then follows from Eq. (5.12) that the degree of polarization

$$P_\omega(\mathbf{s})=1 \quad \text{for all } \mathbf{s} \ , \tag{6.15}$$

i.e., the electric field is completely polarized at every point in the far zone. Clearly the polarization is linear, along the direction of the unit vector $\hat{\mathbf{t}}$, as is evident from Eqs. (6.13) and (5.1).

Although the far field is completely polarized, it is not fully coherent, in general. To see this we introduce the degree of spectral coherence of the electric field by the formula[15–17]

$$\mu_{EE}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\frac{\langle \mathbf{E}^*(\mathbf{r}_1,\omega)\cdot\mathbf{E}(\mathbf{r}_2,\omega)\rangle}{[\langle \mathbf{E}^*(\mathbf{r}_1,\omega)\cdot\mathbf{E}(\mathbf{r}_1,\omega)\rangle]^{1/2}[\langle \mathbf{E}^*(\mathbf{r}_2,\omega)\cdot\mathbf{E}(\mathbf{r}_2,\omega)\rangle]^{1/2}} \ , \tag{6.16}$$

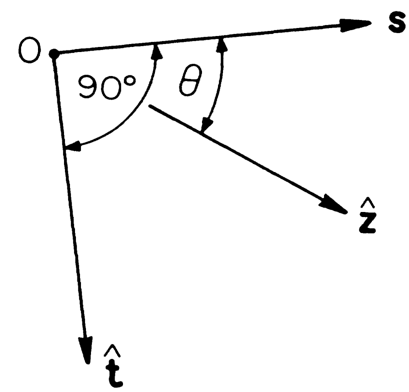

FIG. 3. Illustrating the relative orientation of the three unit vectors which represent the direction of the linear current density ($\hat{\mathbf{z}}$), the direction of observation ($\hat{\mathbf{s}}$), and the direction of polarization of the far field ($\hat{\mathbf{t}}$).

or, using Eq. (3.4a),

$$\mu_{EE}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\frac{\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{ee}(\mathbf{r}_1,\mathbf{r}_2,\omega)}{[\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{ee}(\mathbf{r}_1,\mathbf{r}_1,\omega)]^{1/2}[\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{ee}(\mathbf{r}_2,\mathbf{r}_2,\omega)]^{1/2}} \ . \tag{6.17}$$

Let us now specialize Eq. (6.17) to the case when $\mathbf{r}_1$ and $\mathbf{r}_2$ represent points in the far zone at the same distance r from the origin in directions specified by unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. Then $\mathbf{r}_1=r\mathbf{s}_1$, $\mathbf{r}_2=r\mathbf{s}_2$, and if we make this substitution in Eq. (6.17) and use the relation (3.14a) we find that

$$\mu^{(\infty)}_{EE}(r\mathbf{s}_1,r\mathbf{s}_2,\omega)=\frac{\mathrm{Tr}\overleftrightarrow{\mathbf{W}}^{T}_{jj}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)}{[\mathrm{Tr}\overleftrightarrow{\mathbf{W}}^{T}_{jj}(-k\mathbf{s}_1,k\mathbf{s}_1,\omega]^{1/2}[\mathrm{Tr}\overleftrightarrow{\mathbf{W}}^{T}_{jj}(-k\mathbf{s}_2,k\mathbf{s}_2,\omega)]} \ . \tag{6.18}$$

Now for a fluctuating linear current source we have from Eq. (6.6) that

$$\mathrm{Tr}\overleftrightarrow{\mathbf{W}}^{T}_{jj}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)=\tilde{\mathcal{J}}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)\Phi(\theta_1,\theta_2,\phi) \ , \tag{6.19}$$

where

$$\Phi(\theta_1,\theta_2,\phi)=1-\cos^2\theta_1-\cos^2\theta_2+\cos\theta_1\cos\theta_2\cos\phi \ , \tag{6.20}$$

ϕ denoting the angle between the unit vectors $\mathbf{s}_1$ and $\mathbf{s}_2$.

On substituting from Eq. (6.19) into Eq. (6.18) we obtain the following expression for the degree of spectral coherence $\mu^{(\infty)}_{EE}$:

$$\mu^{(\infty)}_{EE}(r\mathbf{s}_1,r\mathbf{s}_2,\omega)=\frac{\tilde{\mathcal{J}}(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)}{[\tilde{\mathcal{J}}(-k\mathbf{s}_1,k\mathbf{s}_1,\omega)]^{1/2}[\tilde{\mathcal{J}}(-k\mathbf{s}_2,k\mathbf{s}_2,\omega)]^{1/2}}\,\frac{\Phi(\theta_1,\theta_2,\phi)}{[\Phi(\theta_1,\theta_1,0)]^{1/2}[\Phi(\theta_2,\theta_2,0)]^{1/2}} \ . \tag{6.21}$$

The factors in the denominator of the second term on the right have a simple form, as is seen at once from Eq. (6.20):

$$\Phi(\theta_1,\theta_1,0)=\sin^2\theta_1, \quad \Phi(\theta_2,\theta_2,0)=\sin^2\theta_2 \ . \tag{6.22}$$

Let us briefly consider some implications of the formula (6.21). It is clear from that equation that $|\mu^{(\infty)}_{EE}(r\mathbf{s}_1,r\mathbf{s}_2,\omega)|\neq 1$ in general and hence the far field is as a rule partially coherent. The first factor on the right of Eq. (6.21) is of the same form as the degree of spectral coherence of the far field V produced by a fluctuating scalar source distribution Q, viz. [cf. Ref. 2(a), Eq. (3.11)]

$$\mu_V^{(\infty)}(r\mathbf{s}_1,r\mathbf{s}_2,\omega) = \frac{\tilde{W}_Q(-k\mathbf{s}_1,k\mathbf{s}_2,\omega)}{[\tilde{W}_Q(-k\mathbf{s}_1,k\mathbf{s}_1,\omega)]^{1/2}[\tilde{W}_Q(-k\mathbf{s}_2,k\mathbf{s}_2,\omega)]^{1/2}}, \tag{6.23}$$

where $\tilde{W}_Q(\mathbf{K}_1,\mathbf{K}_2,\omega)$ is the spatial Fourier transform of the cross-spectral density function $W_Q(\mathbf{r}_1,\mathbf{r}_2,\omega)$ of the source distribution. It is therefore evident that the degree of spectral coherence of the electric field generated in the far zone by a fluctuating linear current source may be formally regarded as arising from an equivalent scalar source distribution [which contributes the first factor on the right of Eq. (6.21)], modified by a universal function of the two directions $\mathbf{s}_1,\mathbf{s}_2$, namely by the function

$$\Psi(\theta_1,\theta_2,\phi)=\frac{\Phi(\theta_1,\theta_2,\phi)}{[\Phi(\theta_1,\theta_1,0)]^{1/2}[\Phi(\theta_2,\theta_2,0)]^{1/2}}. \tag{6.24}$$

The behavior of this function for some selected values of its three arguments is shown in Figs. 4 and 5.

APPENDIX A: DERIVATION OF EXPRESSION (3.15) FOR $\overleftrightarrow{W}_{jj}^{T}$ IN TERMS OF $\overleftrightarrow{W}_{jj}$

The expression (2.6) for the Fourier transform $\tilde{\mathbf{j}}^{T}(\mathbf{K},\omega)$ of the transverse current may be rewritten in the form

$$\tilde{\mathbf{j}}^{T}(\mathbf{K},\omega)=\tilde{\mathbf{j}}(\mathbf{K},\omega)-\mathbf{s}\mathbf{s}\cdot\tilde{\mathbf{j}}(\mathbf{K},\omega), \tag{A1}$$

where $\mathbf{s}=\mathbf{K}/K$ ($K=|\mathbf{K}|$) is the unit vector along the $\mathbf{K}$ direction. Hence the correlation tensor, defined by Eq. (3.8) of the Fourier transform of the transverse current, is given by

$$\begin{aligned}\overleftrightarrow{W}_{\tilde{j}\tilde{j}}^{T}(\mathbf{K}_1,\mathbf{K}_2,\omega)&=\langle[\tilde{\mathbf{j}}(\mathbf{K}_1,\omega)-\mathbf{s}_1\mathbf{s}_1\cdot\tilde{\mathbf{j}}(\mathbf{K}_1,\omega)][\tilde{\mathbf{j}}(\mathbf{K}_2,\omega)-\mathbf{s}_2\mathbf{s}_2\cdot\tilde{\mathbf{j}}(\mathbf{K}_2,\omega)]\rangle_\omega\\&=\overleftrightarrow{W}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)-\mathbf{s}_1\mathbf{s}_1\cdot\overleftrightarrow{W}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)-\overleftrightarrow{W}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2+\mathbf{s}_1\mathbf{s}_1\cdot\overleftrightarrow{W}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2,\end{aligned} \tag{A2}$$

where $\mathbf{s}_1=\mathbf{K}_1/K_1$ and $\mathbf{s}_2=\mathbf{K}_2/K_2$ are unit vectors along the $\mathbf{K}_1$ and $\mathbf{K}_2$ directions, respectively. Now in a strictly similar manner as led to the formula (3.11), one may establish the analogous relation

$$\overleftrightarrow{W}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)=\tilde{\overleftrightarrow{W}}_{jj}(-\mathbf{K}_1,\mathbf{K}_2,\omega). \tag{A3}$$

Using this relation and also the relation (3.11) in Eq. (A2), we obtain the required expression (3.15) for $\tilde{\overleftrightarrow{W}}_{jj}^{T}$ in terms of $\tilde{\overleftrightarrow{W}}_{jj}$, viz.,

$$\begin{aligned}\tilde{\overleftrightarrow{W}}_{jj}^{T}(\mathbf{K}_1,\mathbf{K}_2,\omega)=&\tilde{\overleftrightarrow{W}}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)-\mathbf{s}_1\mathbf{s}_1\cdot\tilde{\overleftrightarrow{W}}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)\\&-\tilde{\overleftrightarrow{W}}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2\\&+\mathbf{s}_1\mathbf{s}_1\cdot\tilde{\overleftrightarrow{W}}_{jj}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2.\end{aligned} \tag{A4}$$

APPENDIX B: DERIVATION OF THE RELATION $\mathrm{Tr}\overleftrightarrow{W}_{hh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=\mathrm{Tr}\overleftrightarrow{W}_{ee}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)$ [EQ. (4.6)]

The cross-spectral density tensor of the magnetic field is defined by a formula of the form (3.4a), viz.,

$$\overleftrightarrow{W}_{hh}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle\mathbf{H}^*(\mathbf{r}_1,\omega)\mathbf{H}(\mathbf{r}_2,\omega)\rangle_\omega. \tag{B1}$$

Let us set $\mathbf{r}_1=r\mathbf{s}$, $\mathbf{r}_2=r\mathbf{s}$, where $\mathbf{s}$ is a unit vector and let us proceed to the asymptotic limit as $kr\to\infty$, with $\mathbf{s}$ being kept fixed. The formula (B1) then becomes

$$\overleftrightarrow{W}_{hh}(r\mathbf{s},r\mathbf{s},\omega)=\langle\mathbf{H}^*(r\mathbf{s},\omega)\mathbf{H}(r\mathbf{s},\omega)\rangle_\omega, \tag{B2}$$

where $\mathbf{H}(r\mathbf{s},\omega)$ is given by the asymptotic formula (2.2b).

Let us substitute in Eq. (B2) for $\mathbf{H}(r\mathbf{s},\omega)$ in terms of $\mathbf{E}(r\mathbf{s},\omega)$ from Eq. (2.8). This gives

$$\overleftrightarrow{W}_{hh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=\langle\mathbf{s}\times\mathbf{E}^*(r\mathbf{s},\omega)\mathbf{s}\times\mathbf{E}(r\mathbf{s},\omega)\rangle_\omega. \tag{B3}$$

Next let us take the trace of Eq. (B3). We then obtain the formula

$$\mathrm{Tr}\overleftrightarrow{W}_{hh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=\langle[\mathbf{s}\times\mathbf{E}^*(r\mathbf{s},\omega)]\cdot[\mathbf{s}\times\mathbf{E}(r\mathbf{s},\omega)]\rangle_\omega. \tag{B4}$$

Now we have the vector identities

$$\begin{aligned}(\mathbf{s}\times\mathbf{E}^*)\cdot(\mathbf{s}\times\mathbf{E})&=-[\mathbf{s}\times(\mathbf{s}\times\mathbf{E})]\cdot\mathbf{E}^*\\&=-[\mathbf{s}(\mathbf{s}\cdot\mathbf{E})-\mathbf{E}(\mathbf{s}\cdot\mathbf{s})]\cdot\mathbf{E}^*\\&=\mathbf{E}^*\cdot\mathbf{E}-(\mathbf{s}\cdot\mathbf{E}^*)(\mathbf{s}\cdot\mathbf{E}),\end{aligned} \tag{B5}$$

where, in going from the second to the third line, we assumed that $s^2=1$. Using this identity Eq. (B4) becomes

$$\mathrm{Tr}\overleftrightarrow{W}_{hh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=\langle\mathbf{E}^*(r\mathbf{s},\omega)\cdot\mathbf{E}(r\mathbf{s},\omega)\rangle_\omega-\langle[\mathbf{s}\cdot\mathbf{E}^*(r\mathbf{s},\omega)][\mathbf{s}\cdot\mathbf{E}(r\mathbf{s},\omega)]\rangle_\omega. \tag{B6}$$

The first term on the right of Eq. (B6) is just the trace of the cross-spectral density tensor of the electric field in the far zone; the second term on the right vanishes because of the transversality of the far field [Eq. (2.9)]. Hence Eq. (B6) reduces to

$$\mathrm{Tr}\overleftrightarrow{W}_{hh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega)=\mathrm{Tr}\overleftrightarrow{W}_{ee}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega), \tag{B7}$$

which is Eq. (4.6) of the text.

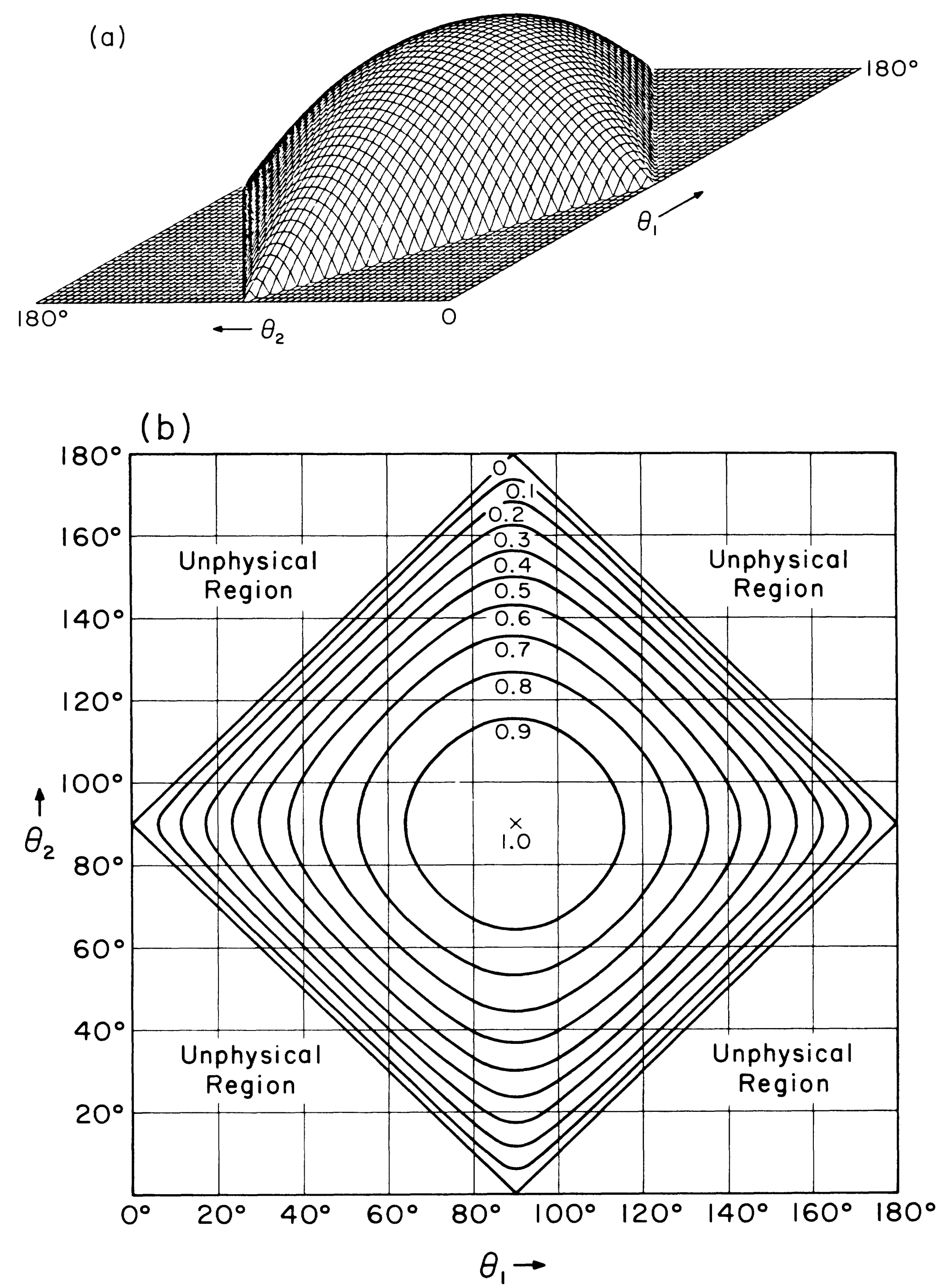

FIG. 4. Behavior of the function $\Psi(\theta_1,\theta_2,\phi)$ defined by Eqs. (6.24), (6.20), and (6.21) in the plane $\phi=\pi/2$. Three-dimensional plots (a) and contours of constant values of Ψ (b).

APPENDIX C: DERIVATION OF THE FORMULA (4.8)

According to Eqs. (4.5c) and (3.11)

$$\overleftrightarrow{\mathbf{W}}^{(\infty)}_{eh}(r\mathbf{s},r\mathbf{s},\omega)=-(2\pi)^6\left[\frac{k}{cr}\right]^2\overleftrightarrow{\mathbf{W}}^{T}_{\tilde{j}\tilde{j}}(k\mathbf{s},k\mathbf{s},\omega)\times\mathbf{s}\ . \tag{C1}$$

If on the right-hand side of Eq. (C1) we substitute from Eq. (3.8), we obtain the formula

$$\overleftrightarrow{\mathbf{W}}^{(\infty)}_{eh}(r\mathbf{s},r\mathbf{s},\omega)=-(2\pi)^6\left[\frac{k}{cr}\right]^2\langle\tilde{\mathbf{j}}^{T*}(k\mathbf{s},\omega)\tilde{\mathbf{j}}^{T}(k\mathbf{s},\omega)\rangle_{\omega}\times\mathbf{s}\ . \tag{C2}$$

Hence

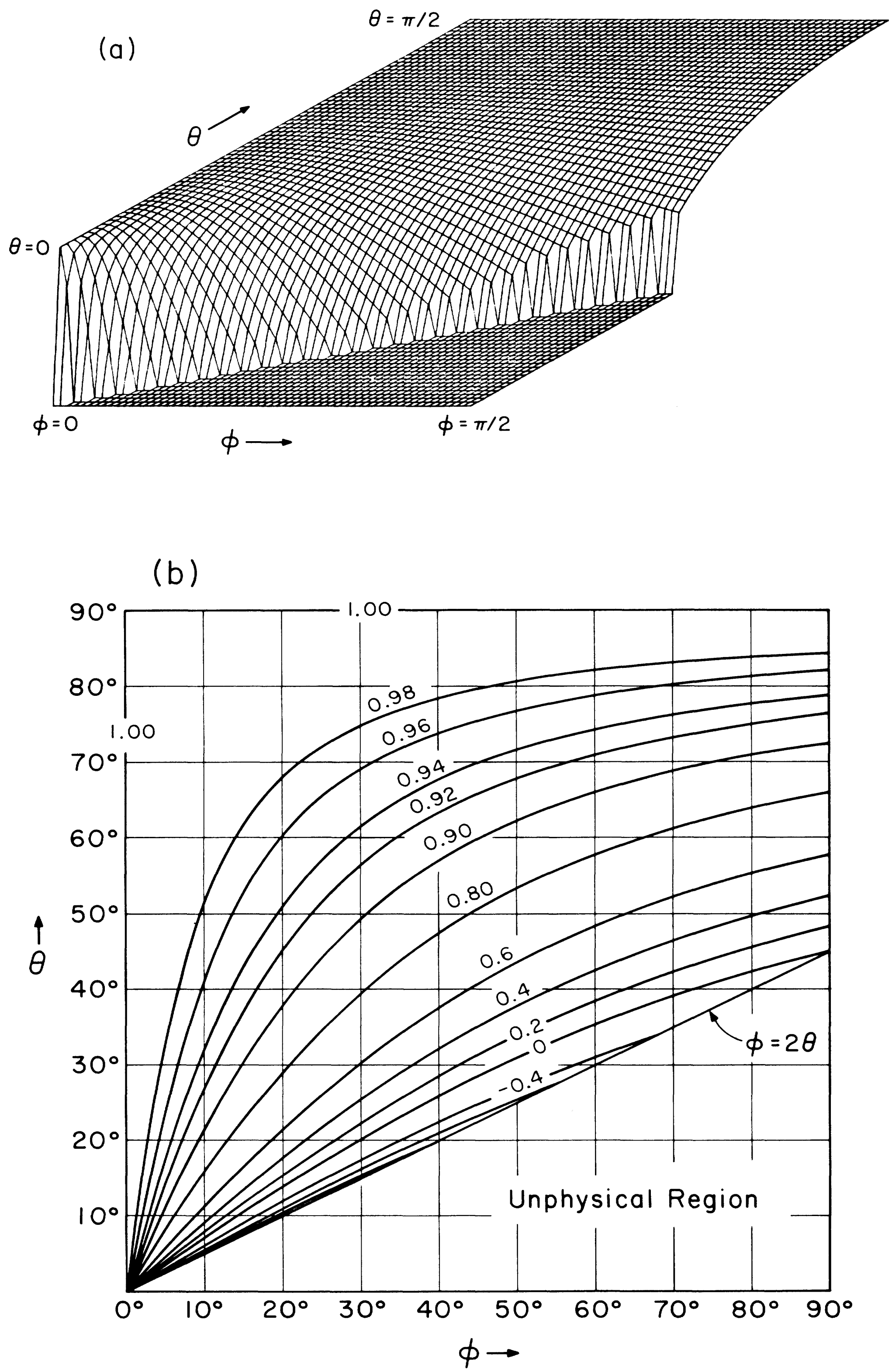

FIG. 5. The behavior of the function $\Psi(\theta_1,\theta_2,\phi)$ defined by Eqs. (6.24), (6.20), and (6.21) when $\theta_2=\theta_1=\theta$. Three-dimensional plots (a) and contours of constant values of Ψ (b).

$$\mathcal{V}(\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega))=-(2\pi)^6\left[\frac{k}{cr}\right]^2\langle\tilde{\mathbf{j}}^{T*}(k\mathbf{s},\omega)\times[\tilde{\mathbf{j}}^{T}(k\mathbf{s},\omega)\times\mathbf{s}]\rangle_\omega$$
$$=-(2\pi)^6\left[\frac{k}{cr}\right]^2\{\langle\tilde{\mathbf{j}}^{T*}(k\mathbf{s},\omega)[\tilde{\mathbf{j}}^{T}(k\mathbf{s},\omega)\cdot\mathbf{s}]\rangle_\omega-\mathbf{s}\langle\tilde{\mathbf{j}}^{T*}(k\mathbf{s},\omega)\cdot\tilde{\mathbf{j}}^{T}(k\mathbf{s},\omega)\rangle_\omega\}\ . \tag{C3}$$

The first term on the right-hand side of Eq. (C3) vanishes because, as follows at once from Eq. (2.6) $\tilde{\mathbf{j}}^{T*}(k\mathbf{s},\omega)\cdot\mathbf{s}=0$. The average that appears in the second term on the right is, according to Eq. (3.8), just the trace of the tensor $\overleftrightarrow{\mathbf{W}}^{T}_{\tilde{j}\tilde{j}}(k\mathbf{s},k\mathbf{s},\omega)$. Hence Eq. (C3) reduces to

$$\mathcal{V}(\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega))=(2\pi)^6\left[\frac{k}{cr}\right]^2\mathrm{Tr}\overleftrightarrow{\mathbf{W}}^{T}_{\tilde{j}\tilde{j}}(k\mathbf{s},k\mathbf{s},\omega)\mathbf{s}\ . \tag{C4}$$

Finally, if we again make use of Eq. (3.11), we obtain the formula (4.8) of the text, viz.,

$$\mathcal{V}(\overleftrightarrow{\mathbf{W}}_{eh}^{(\infty)}(r\mathbf{s},r\mathbf{s},\omega))=(2\pi)^6\left[\frac{k}{cr}\right]^2\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)\mathbf{s}\ . \tag{C5}$$

APPENDIX D: DERIVATION OF AN EXPRESSION FOR $\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}$ IN TERMS OF $\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}$

If we take the trace of Eq. (A2) we obtain the formula

$$\mathrm{Tr}\overleftrightarrow{\mathbf{W}}^{T}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)=\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)-\mathrm{Tr}[\mathbf{s}_1\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)]-\mathrm{Tr}[\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2]+\mathrm{Tr}[\mathbf{s}_1\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2]\ . \tag{D1}$$

Now

$$\mathrm{Tr}[\mathbf{s}_1\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)]=\mathrm{Tr}[\mathbf{s}_1\mathbf{s}_1\cdot\langle\tilde{\mathbf{j}}^*(\mathbf{K}_1,\omega)\tilde{\mathbf{j}}(\mathbf{K}_2,\omega)\rangle]=\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_1\ . \tag{D2}$$

Similarly

$$\mathrm{Tr}[\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2]=\mathbf{s}_2\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2 \tag{D3}$$

and

$$\mathrm{Tr}[\mathbf{s}_1\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2\mathbf{s}_2]=[\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2]\mathbf{s}_1\cdot\mathbf{s}_2\ . \tag{D4}$$

On substituting from Eqs. (D2)–(D4) into Eq. (D1) we find that

$$\mathrm{Tr}W^{T}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)=\mathrm{Tr}\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)-\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_1-\mathbf{s}_2\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2+[\mathbf{s}_1\cdot\overleftrightarrow{\mathbf{W}}_{\tilde{j}\tilde{j}}(\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2]\mathbf{s}_1\cdot\mathbf{s}_2\ . \tag{D5}$$

If we make use of the relations (3.11) and (A3), Eq. (D5) implies that

$$\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(-\mathbf{K}_1,\mathbf{K}_2,\omega)=\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}(-\mathbf{K}_1,\mathbf{K}_2,\omega)-\mathbf{s}_1\cdot\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}(-\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_1-\mathbf{s}_2\cdot\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}(-\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2+[\mathbf{s}_1\cdot\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}(-\mathbf{K}_1,\mathbf{K}_2,\omega)\cdot\mathbf{s}_2]\mathbf{s}_1\cdot\mathbf{s}_2\ . \tag{D6}$$

In particular, if we choose $\mathbf{K}_1=k\mathbf{s}$, $\mathbf{K}_2=k\mathbf{s}$, Eq. (D6) reduces to

$$\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}{}^{T}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)=\mathrm{Tr}\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)-\mathbf{s}\cdot\tilde{\overleftrightarrow{\mathbf{W}}}_{jj}(-k\mathbf{s},k\mathbf{s},\omega)\cdot\mathbf{s}\ , \tag{D7}$$

which is Eq. (4.11) of text.

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[1]For reviews of this research see, for example (a) E. Wolf, J. Opt. Soc. Am. **68**, 6 (1978); (b) A. T. Friberg, Opt. Eng. **21**, 362 (1982); and (c) W. H. Carter, Radio Sci. **18**, 149 (1983).

[2]W. H. Carter and E. Wolf, (a) Opt. Acta **28**, 227 (1981); (b) *ibid.* **28**, 245 (1981).

[3]I. J. LaHaie, J. Opt. Soc. Am. A **2**, 35 (1985).

[4]E. Wolf, (a) Nature (London) **326**, 363 (1987); (b) Opt. Commun. **62**, 12 (1987).

[5]W. H. Carter, J. Opt. Soc. Am. **70**, 1067 (1980).

[6]A. J. Devaney and E. Wolf, J. Math. Phys. **15**, 234 (1974), Sec. 4. Our formulas (2.2), together with the expressions (2.3), differ by a factor of $(2\pi)^3$ from Eq. (4.5) of that reference. The difference is due to the choice of the proportionality factor $1/(2\pi)^3$ in the definition (2.5) above of the Fourier transform.

[7]E. Wolf, J. Opt. Soc. Am. **72**, 343 (1982). There is a misprint in Eq. (5.5) of this reference. The factor $[\lambda_n(\omega)]^{2/3}$ should be replaced by $[\lambda_n(\omega)]^{1/2}$.

[8]E. Wolf, J. Opt. Soc. Am. A **3**, 76 (1986).

[9]M. Born and E. Wolf, *Principles of Optics,* 6th ed. (Pergamon, Oxford, 1980), Sec. 1.4.3.

[10]Reference 2(b), Eq. (2.5). It is to be noted that the definitions of cross-spectral densities in that reference and in the present paper differ by complex conjugation. Consequently, the first two arguments in their Fourier transforms differ in signs.

[11]These relations may readily be deduced, for example, from Eqs. (30) given on p. 419 of J. A. Stratton, *Electromagnetic*

Theory (McGraw-Hill, New York, 1941).

[12]G. S. Agarwal, Opt. Commun. **37**, 349 (1981).

[13]The modification consists in replacing equal-time correlations by spectral correlations and noting that the corresponding expressions differ by complex conjugation (cf. footnote 10 above).

[14]See, for example, J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 397.

[15]This formula may be regarded as the analog for the electromagnetic field of the degree of spectral coherence of a fluctuating scalar field, discussed in Ref. 16. It may also be considered as the "space-frequency" analog of the "space-time" degree of coherence of the electromagnetic field introduced in Refs. 17.

[16]L. Mandel and E. Wolf, J. Opt. Soc. Am. **66**, 529 (1976).

[17]B. Karczewski, Phys. Lett. **5**, 191 (1963); Nuovo Cimento **30**, 906 (1963).