Electron-spin polarization in high-energy storage rings. II. Evaluation of the equilibrium polarization

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(Received 10 December 1986)

A new algorithm is presented to evaluate the equilibrium degree of polarization in a high-energy electron storage ring (the Derbenev-Kondratenko formula). The algorithm includes all modes of orbital motion, to arbitrary orders in principle, thus facilitating the calculation of so-called "spin resonances," especially higher-order resonances. The algorithm is applicable to storage rings of arbitrary geometry and energy, and, in particular, is able to deal with overlapping resonances. Precautions are described to ensure stability of the algorithm. In the approximation of linear orbital dynamics, a computer program has been written to implement this algorithm, and sample results are presented.

I. INTRODUCTION

It has been realized for many years that electrons and positrons in high-energy storage rings become polarized by the emission of synchrotron radiation.[1,2] This phenomenon can be useful to high-energy physicists. As well as permitting the study of polarization dependencies of basic reactions, measurement of the polarization in the vicinity of so-called spin resonances is currently the most accurate method of calibrating the energy of an electron beam in a storage ring. This technique has been used to determine the masses of a number of hadrons produced as resonances in e^+e^- collisions.[2] Furthermore, a storage ring called HERA, a 30-GeV electron–820-GeV proton collider is presently under construction at DESY (Deutsches Elektronen Synchrotron), where it is planned to control the polarization of the electron beam so as to be longitudinal at the interaction point: this would provide valuable new tests of the electroweak interactions. It is therefore desirable to be able to calculate the polarization. The companion paper to this work[3] presented a detailed exposition on the origin and buildup of the polarization, in particular rederiving the so-called Derbenev-Kondratenko formula[4] for the equilibrium degree of polarization. Many efforts have been made to evaluate this formula, which suffer from various limitations.[2] This paper presents a new algorithm, which does not suffer from these restrictions, to evaluate the above formula for a given accelerator.

To calculate the polarization of a high-energy storage ring one must calculate not only the spin motion but also the orbital motion of the electrons, because there is spin-orbit coupling. The orbital trajectories consist of oscillations around a central trajectory, and calculations of the polarization have so far been restricted to first order in the orbital oscillations,[5] or else have required restrictive approximations, e.g., only longitudinal, but not transverse, oscillations are treated, and some terms in the spin equation of motion are neglected.[6] The algorithm below includes all modes of orbital motion to, in principle, arbitrary orders, and also retains all terms in the spin equation of motion. This facilitates the calculation of so-called "spin resonances," in particular the higher-order resonances. Higher-order resonances have been observed experimentally,[7] but have been hard to calculate theoretically. A computer program has been written to implement this algorithm, based on the program described in Ref. 5. For numerical work, I consider only linear orbital dynamics, although the formal theory can accommodate nonlinear, but integrable, orbital dynamics as well. Preliminary results are displayed, of the polarization as a function of accelerator energy, showing spin resonances of various orders, including also some overlapping resonances, to illustrate the type of effects that can be calculated.

I begin with some general remarks in Sec. II. The algorithm proper is given in Sec. III. A description of the computer program, and numerical results, are presented in Sec. IV, and Sec. V contains my conclusions. Some subsidiary calculations are presented in two Appendixes.

II. GENERAL REMARKS

In a high-energy storage ring, the orbital particle trajectories consist of oscillations around a central trajectory called the equilibrium closed orbit.[8] The equilibrium closed orbit is periodic around the ring, but, in general, the oscillations are not. The unperturbed spin motion consists of classical precessions described by the Thomas-Bargmann-Michel-Telegdi (Thomas-BMT) equation[9] $d\mathbf{s}/dt=\boldsymbol{\Omega}\times\mathbf{s}$, where

$$\boldsymbol{\Omega}=-\frac{e}{mc}\left[\left[a+\frac{1}{\gamma}\right]\mathbf{B}-\frac{a\gamma}{\gamma+1}\boldsymbol{\beta}\cdot\mathbf{B}\boldsymbol{\beta}-\left[a+\frac{1}{\gamma+1}\right]\boldsymbol{\beta}\times\mathbf{E}\right] \tag{1}$$

is the spin precession vector. Here $a=(g-2)/2$, $\boldsymbol{\beta}$ is the particle velocity in units of c, $\gamma=(1-|\boldsymbol{\beta}|^2)^{-1/2}$, and $\mathbf{E}$ and $\mathbf{B}$ are the accelerator electromagnetic fields. Since $\boldsymbol{\Omega}$ depends on the orbital trajectory, so does the spin motion, hence a fixed spin quantization axis is inadequate to diagonalize the unperturbed Hamiltonian. The axis I use,

called $\hat{\mathbf{n}}$, depends on the orbital trajectory, and by definition satisfies the Thomas-BMT equation on the trajectory[3,4,10] It is chosen so as to diagonalize the unperturbed Hamiltonian. I also need the quantity $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, which is the derivative of $\hat{\mathbf{n}}$ between two trajectories which differ only by energy at a given point in the accelerator.[3] In Sec. III, I shall solve for $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ in terms of the accelerator fields and other parameters, e.g., the accelerator energy.

The equilibrium degree of polarization $P_{\rm eq}$ in a high-energy electron storage ring is given by the Derbenev-Kondratenko formula[4]

$$P_{\rm eq}=-\frac{8}{5\sqrt{3}}\frac{\left\langle |\dot{\mathbf{v}}|^3\hat{\mathbf{b}}\cdot\left[\hat{\mathbf{n}}-\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\right\rangle}{\left\langle |\dot{\mathbf{v}}|^3\left[1-\frac{2}{9}(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2+\frac{11}{18}\left|\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right|^2\right]\right\rangle} . \quad (2)$$

Here $\mathbf{v}$ is the particle velocity, $\hat{\mathbf{b}}=\mathbf{v}\times\dot{\mathbf{v}}/|\mathbf{v}\times\dot{\mathbf{v}}|$, and the large angular brackets denote an average around the ring and over the distribution of particle orbits. This formula was rederived and extended to first order in $a=(g-2)/2$ in Ref. 3, and the coefficients in Eq. (2) changed slightly as a result. It was shown that the fact that $a\simeq 10^{-3}$ made little difference explicitly to the coefficients in Eq. (2), but a significant difference to the behavior of $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. Hence I shall use Eq. (2), *but* with $a\neq 0$ in the calculation of $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$.

III. ALGORITHM

In Refs. 3 and 4 no explicit expression was provided for either $\hat{\mathbf{n}}$ or $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. Only their formal properties were needed. In this section I describe an algorithm to calculate $\hat{\mathbf{n}}$, and then $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. I begin by introducing some notation. The azimuth θ is used as the independent variable ($\theta\equiv 2\pi x/C$, where x is the arc length and C is the circumference of the accelerator), hence the Thomas-BMT equation reads $d\mathbf{s}/d\theta=\boldsymbol{\Omega}\times\mathbf{s}$.[11] I shall write $\boldsymbol{\Omega}=\boldsymbol{\Omega}_0+\boldsymbol{\omega}$, where $\boldsymbol{\Omega}_0$ is the value of $\boldsymbol{\Omega}$ on the equilibrium closed orbit, and denote by $\hat{\mathbf{n}}_0$ the value of $\hat{\mathbf{n}}$ on that trajectory. I shall also need two other unit vectors $\hat{\boldsymbol{l}}_0$ and $\hat{\mathbf{m}}_0$, defined to be solutions of the Thomas-BMT equation on the equilibrium closed orbit, such that $\{\hat{\boldsymbol{l}}_0,\hat{\mathbf{m}}_0,\hat{\mathbf{n}}_0\}$ is a right-handed orthonormal triad,[5] hence $d\hat{\boldsymbol{l}}_0/d\theta=\boldsymbol{\Omega}_0\times\hat{\boldsymbol{l}}_0$, etc. Since the Thomas-BMT equation describes a precession, the transformation of $\{\hat{\boldsymbol{l}}_0,\hat{\mathbf{m}}_0,\hat{\mathbf{n}}_0\}$ from θ to $\theta+2\pi$ is a rotation which can be calculated from a knowledge of $\boldsymbol{\Omega}_0$. The vector $\hat{\mathbf{n}}_0$ is obtained by finding the axis of this rotation, i.e., $\hat{\mathbf{n}}_0(\theta+2\pi)=\hat{\mathbf{n}}_0(\theta)$.[12] The vectors $\hat{\boldsymbol{l}}_0$ and $\hat{\mathbf{m}}_0$ suffer a rotation, viz.,

$$\begin{bmatrix}\hat{\boldsymbol{l}}_0\\ \hat{\mathbf{m}}_0\end{bmatrix}_{\theta+2\pi}=\begin{bmatrix}\cos(2\pi\nu) & -\sin(2\pi\nu)\\ \sin(2\pi\nu) & \cos(2\pi\nu)\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{l}}_0\\ \hat{\mathbf{m}}_0\end{bmatrix}_{\theta}, \quad (3a)$$

or

$$(\hat{\boldsymbol{l}}_0\pm i\hat{\mathbf{m}}_0)_{\theta+2\pi}=e^{\pm i2\pi\nu}(\hat{\boldsymbol{l}}_0\pm i\hat{\mathbf{m}}_0)_{\theta} . \quad (3b)$$

The quantity ν is a constant, called the "spin tune."[13] We see that $\hat{\mathbf{n}}_0$ is ambiguous if ν is an integer. Numerical programs are available to calculate $\{\hat{\boldsymbol{l}}_0,\hat{\mathbf{m}}_0,\hat{\mathbf{n}}_0\}$, e.g., see Ref. 5, and I shall take such information as given. In this chapter I describe an algorithm to express $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ in terms of $\{\hat{\boldsymbol{l}}_0,\hat{\mathbf{m}}_0,\hat{\mathbf{n}}_0\}$. To do so, I write $\hat{\mathbf{n}}=n_1\hat{\boldsymbol{l}}_0+n_2\hat{\mathbf{m}}_0+n_3\hat{\mathbf{n}}_0$. Then

$$\frac{d}{d\theta}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}=\begin{bmatrix}0 & -\hat{\mathbf{n}}_0\cdot\boldsymbol{\omega} & \hat{\mathbf{m}}_0\cdot\boldsymbol{\omega}\\ \hat{\mathbf{n}}_0\cdot\boldsymbol{\omega} & 0 & -\hat{\boldsymbol{l}}_0\cdot\boldsymbol{\omega}\\ -\hat{\mathbf{m}}_0\cdot\boldsymbol{\omega} & \hat{\boldsymbol{l}}_0\cdot\boldsymbol{\omega} & 0\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix} . \quad (4)$$

It is useful to introduce spherical components (V_1,V_0,V_{-1}), defined by

$$V_{\pm1}=\mp\frac{1}{\sqrt{2}}(n_1\pm in_2),\quad V_0=n_3 \quad (5)$$

and to write

$$\omega_{\pm}=(\hat{\boldsymbol{l}}_0\pm i\hat{\mathbf{m}}_0)\cdot\boldsymbol{\omega},\quad \omega_3=\hat{\mathbf{n}}_0\cdot\boldsymbol{\omega} \quad (6)$$

and to introduce spin-1 angular momentum matrices $\{J_1,J_2,J_3\}$, and $J_{\pm}=J_1\pm iJ_2$, where

$$J_3=\begin{bmatrix}1 & & \\ & 0 & \\ & & -1\end{bmatrix},\quad J_+=\begin{bmatrix}0 & \sqrt{2} & \\ & 0 & \sqrt{2}\\ & & 0\end{bmatrix}, \quad (7)$$

and $J_-=J_+^{\dagger}$. Blank spaces indicate zeros in the above matrices. I also write

$$\boldsymbol{\omega}\cdot\mathbf{J}=\omega_3J_3+\tfrac{1}{2}(\omega_+J_-+\omega_-J_+) . \quad (8)$$

Then

$$\frac{d}{d\theta}\begin{bmatrix}V_1\\ V_0\\ V_{-1}\end{bmatrix}=i(\boldsymbol{\omega}\cdot\mathbf{J})^T\begin{bmatrix}V_1\\ V_0\\ V_{-1}\end{bmatrix}=i\begin{bmatrix}\omega_3 & \omega_+/\sqrt{2} & 0\\ \omega_-/\sqrt{2} & 0 & \omega_+/\sqrt{2}\\ 0 & \omega_-/\sqrt{2} & -\omega_3\end{bmatrix}\begin{bmatrix}V_1\\ V_0\\ V_{-1}\end{bmatrix}, \quad (9)$$

and the solution for $\hat{\mathbf{n}}$ is

$$\begin{bmatrix}V_1\\ V_0\\ V_{-1}\end{bmatrix}=T\left[\exp\left[i\int_{-\infty}^{\theta}(\boldsymbol{\omega}\cdot\mathbf{J})^Td\theta'\right]\right]\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\equiv\lim_{\epsilon\to 0+}\left[T\left[\exp\left[i\int_{-\infty}^{\theta}(\boldsymbol{\omega}\cdot\mathbf{J})^Te^{\epsilon\theta'}d\theta'\right]\right]\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\right] . \quad (10)$$

Here $T(\cdots)$ denotes a θ-ordered product, and a convergence factor is used to render the integrals well defined as $\theta'\to-\infty$.[14,15]

To proceed further, I expand the exponential in a power series, and evaluate the resulting integrals term by term, i.e., a perturbative expansion in powers of $\boldsymbol{\omega}$. For-

mally, there is no constraint on the orbital dynamics, other than it be stable. For practical purposes, however, I assume from now on that the orbital dynamics is linear. In that case, an orbital oscillation can be decomposed into a sum of six normal modes $E_\lambda, \lambda = \pm 1, \pm 2, \pm 3$, viz.,

$$y(\theta) = \sum_\lambda a_\lambda E_\lambda(\theta) \ . \tag{11}$$

Here $y(\theta) = (r_1, p_1, r_2, p_2, r_3, p_3)^T$,[5,16] where $\{r_1(\theta), r_2(\theta), r_3(\theta)\}$ are the longitudinal and transverse coordinate offsets from the equilibrium closed orbit and $\{p_1(\theta), p_2(\theta), p_3(\theta)\}$ are their conjugate momenta, hence $y = 0$ corresponds to the equilibrium closed orbit. The a_λ are constants. The E_λ are six-component column vectors. They have the property

$$E_\lambda(\theta + 2\pi) = e^{i2\pi Q_\lambda} E_\lambda(\theta), \quad E_{-\lambda} = E_\lambda^* \tag{12}$$

where the Q_λ are real constants, whence $Q_{-\lambda} = -Q_\lambda$.[17] Since $y(\theta)$ is real, it follows that $a_{-\lambda} = a_\lambda^*$. In terms of action-angle variables, $I_\lambda = |a_\lambda|^2$ and $d\psi_\lambda / d\theta = Q_\lambda$ $(\lambda > 0)$, and

$$y(\theta) = \mathrm{Re}\left[\sum_{\lambda > 0} \sqrt{I_\lambda} e^{i\psi_\lambda} E_\lambda(\theta) e^{-iQ_\lambda \theta} \right] . \tag{13}$$

The E_λ are normalized so that $E_\lambda^\dagger S E_\lambda = i$ for $\lambda > 0$ and $E_\lambda^\dagger S E_\lambda = -i$ for $\lambda < 0$, where

$$S = \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{bmatrix} . \tag{14}$$

Blank spaces denotes zeros in the above matrix. For sufficiently small $|a_\lambda|$, one can write $\omega = \sum_\lambda a_\lambda \omega_\lambda$, where ω_λ is the value of ω on the trajectory E_λ. Then, expanding Eq. (10) to first order in ω, and neglecting explicit mention of the convergence factor, one has

$$\begin{bmatrix} V_1 \\ V_0 \\ V_{-1} \end{bmatrix} \simeq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{i}{\sqrt{2}} \int_{-\infty}^{\theta} \begin{bmatrix} \omega_+ \\ 0 \\ \omega_- \end{bmatrix} d\theta' \ , \tag{15a}$$

whence $V_0 \simeq 1$ and

$$V_1 \simeq \frac{i}{\sqrt{2}} \int_{-\infty}^{\theta} \omega_+(\theta') d\theta' = \frac{i}{\sqrt{2}} \sum_\lambda a_\lambda \int_{-\infty}^{\theta} \omega_{\lambda +}(\theta') d\theta' \ , \tag{15b}$$

$$V_{-1} \simeq \frac{i}{\sqrt{2}} \int_{-\infty}^{\theta} \omega_-(\theta') d\theta' = \frac{i}{\sqrt{2}} \sum_\lambda a_\lambda \int_{-\infty}^{\theta} \omega_{\lambda -}(\theta') d\theta' \ ,$$

where $\omega_{\lambda \pm} = (\hat{I}_0 \pm i \hat{\mathbf{m}}_0) \cdot \omega_\lambda$. I shall also need $\omega_{\lambda 3} = \hat{\mathbf{n}}_0 \cdot \omega_\lambda$. Then the solution, to first order, for n_1 and n_2 is

$$n_1 \simeq \int_{-\infty}^{\theta} \hat{\mathbf{m}}_0 \cdot \omega d\theta', \quad n_2 \simeq -\int_{-\infty}^{\theta} \hat{I}_0 \cdot \omega d\theta' \ . \tag{16}$$

Also $\omega_\lambda(\theta + 2\pi) = \exp(i2\pi Q_\lambda)\omega_\lambda(\theta)$, whence

$$\begin{aligned} \omega_{\lambda \pm}(\theta + 2\pi) &= e^{i2\pi(Q_\lambda \pm \nu)} \omega_{\lambda \pm}(\theta) \ , \\ \omega_{\lambda 3}(\theta + 2\pi) &= e^{i2\pi Q_\lambda} \omega_{\lambda 3}(\theta) \ , \end{aligned} \tag{17}$$

and so, using Eqs. (3b) and (17),

$$\begin{aligned} V_{\pm 1} \simeq \frac{i}{\sqrt{2}} \sum_\lambda a_\lambda \int_{-\infty}^{\theta} \omega_{\lambda \pm}(\theta') d\theta' &= \frac{i}{\sqrt{2}} \sum_\lambda a_\lambda \left[\int_{\theta - 2\pi}^{\theta} \omega_{\lambda \pm} d\theta' + \int_{\theta - 4\pi}^{\theta - 2\pi} \omega_{\lambda \pm} d\theta' + \int_{\theta - 6\pi}^{\theta - 4\pi} \omega_{\lambda \pm} d\theta' + \cdots \right] \\ &= \frac{i}{\sqrt{2}} \sum_\lambda a_\lambda \int_{\theta - 2\pi}^{\theta} \omega_{\lambda \pm} d\theta' (1 + e^{-i2\pi(Q_\lambda \pm \nu)} + e^{-i4\pi(Q_\lambda \pm \nu)} + \cdots) \\ &= \frac{i}{\sqrt{2}} \sum_\lambda a_\lambda \frac{1}{\exp[i2\pi(Q_\lambda \pm \nu)] - 1} \int_{\theta}^{\theta + 2\pi} \omega_{\lambda \pm}(\theta') d\theta' \ , \end{aligned} \tag{18}$$

where I have used the convergence factor to justify the summation of the phase factors. The integrals $\int_\theta^{\theta + 2\pi} \hat{I}_0 \cdot \omega_\lambda d\theta'$ and $\int_\theta^{\theta + 2\pi} \hat{\mathbf{m}}_0 \cdot \omega_\lambda d\theta$ are known as the Chao integrals; the above derivation places them in a new light and offers new insights into their properties.[5,18] For example, let us calculate $\gamma(\partial \hat{\mathbf{n}} / \partial \gamma)$ to zeroth order.[19] To do so, we find n_1, n_2, and n_3 on two trajectories that differ only by energy, say $mc^2 \Delta\gamma$, at a given azimuth, and divide the difference by $\Delta\gamma / \gamma$. Let the trajectories be $\sum_\lambda a_\lambda E_\lambda$ and $\sum_\lambda (a_\lambda + \delta a_\lambda) E_\lambda$, then we have

$$\gamma \frac{\partial n_1}{\partial \gamma} \bigg|_\theta \simeq \sum_\lambda \left[\gamma \frac{\partial a_\lambda}{\partial \gamma} \bigg|_\theta \int_{-\infty}^{\theta} \hat{\mathbf{m}}_0 \cdot \omega_\lambda d\theta' \right] , \tag{19}$$

with a similar expression for $\gamma(\partial n_2 / \partial \gamma)$, while $\gamma(\partial n_3 / \partial \gamma)$ vanishes, to this order. Note that the derivative affects only the coefficient a_λ, not the normal modes E_λ, or ω_λ. Then

$$\sum_\lambda \delta a_\lambda E_\lambda = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Delta\gamma / \gamma_0 \end{bmatrix} , \tag{20}$$

where $\gamma_0 mc^2$ is the average electron energy. Using the orthonormality relations for the E_λ,

$$\delta a_\lambda = \begin{cases} -iE_{5\lambda}^*(\theta) \dfrac{\Delta\gamma}{\gamma_0}, & \lambda > 0 \\ iE_{5\lambda}^*(\theta) \dfrac{\Delta\gamma}{\gamma_0}, & \lambda < 0 \end{cases} \tag{21a}$$

or

$$\gamma\frac{\partial a_\lambda}{\partial\gamma}\bigg|_\theta \simeq \gamma_0\frac{\partial a_\lambda}{\partial\gamma}\bigg|_\theta = \begin{cases} -iE^*_{5\lambda}(\theta), & \lambda>0 \\ iE^*_{5\lambda}(\theta), & \lambda<0 \end{cases} \tag{21b}$$

where $E_{5\lambda}$ is the fifth component of E_λ. Then

$$\gamma\frac{\partial n_1}{\partial\gamma}\bigg|_\theta \simeq 2\,\mathrm{Im}\left[\sum_{\lambda(>0)} E^*_{5\lambda}(\theta)\int_{-\infty}^{\theta}\hat{\mathbf{m}}_0\cdot\boldsymbol{\omega}_\lambda d\theta'\right], \tag{22}$$

and similarly for $\gamma(\partial n_2/\partial\gamma)$, which is Chao's expression for $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$,[5] and we see now that it *is* a derivative after all.[18] As in Eq. (18), the range of integration in Eq. (22) can be reduced to $(\theta,\theta+2\pi)$, and the above expressions also diverge when either $Q_\lambda+\nu$ or $Q_\lambda-\nu$ equals an integer: a "first-order spin resonance" occurs and $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ are ill defined.[20] Expansion of the θ-ordered product to various powers in $\boldsymbol{\omega}$ yields successively higher-order contributions to the expression for $\hat{\mathbf{n}}$, thence $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$.

The foregoing elucidates a property of spin resonances which does not seem to have been generally appreciated: it is generally felt that zeroth-order resonances are caused by the behavior of $\hat{\mathbf{n}}_0$, while first-order resonances are caused by the behavior of $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. From above, it is clear that the resonances, at all orders, are caused by *the behavior of* $\hat{\mathbf{n}}$.

Let us now consider the calculation of the higher-order terms. To do so, I write

$$\begin{bmatrix} V_1 \\ V_0 \\ V_{-1} \end{bmatrix} = \lim_{\epsilon\to 0+}\left[\sum_{N=0}^{\infty}\sum_{\substack{\text{distinct}\\ \text{combinations}\\ \lambda_1,\ldots,\lambda_N}} a_{\lambda_1}\cdots a_{\lambda_N}\begin{bmatrix} V_1^\Lambda \\ V_0^\Lambda \\ V_{-1}^\Lambda \end{bmatrix}\right], \tag{23}$$

where $\Lambda\equiv(\Lambda_1,\Lambda_{-1},\Lambda_2,\Lambda_{-2},\Lambda_3,\Lambda_{-3})$ is a six-component index and Λ_λ is the number of indexes $\lambda_1,\ldots,\lambda_N$ which take the value $\lambda(=\pm1,\pm2,\pm3)$. Thus the permutations $\lambda_1=1,\lambda_2=3,\lambda_3=-1$, and $\lambda_1=-1,\lambda_2=1,\lambda_3=3$ both contribute to $\Lambda=(1,1,0,0,1,0)$. Then, using Eqs. (9) or (10),

$$\begin{bmatrix} V_1^\Lambda \\ V_0^\Lambda \\ V_{-1}^\Lambda \end{bmatrix} = i\sum_\lambda\int_{-\infty}^{\theta} e^{\epsilon\theta'}\begin{bmatrix} \omega_{\lambda3} & \omega_{\lambda+}/\sqrt{2} & 0 \\ \omega_{\lambda-}/\sqrt{2} & 0 & \omega_{\lambda+}/\sqrt{2} \\ 0 & \omega_{\lambda-}/\sqrt{2} & -\omega_{\lambda3} \end{bmatrix}\times\begin{bmatrix} V_1^{\Lambda'} \\ V_0^{\Lambda'} \\ V_{-1}^{\Lambda'} \end{bmatrix}d\theta', \tag{24}$$

where Λ' equals Λ when incremented by unity in the λth slot. Equation (24) is now applied recursively: given V_i^Λ, $i=1,0,-1$, as a function of θ, at some order, multiply by $e^{\epsilon\theta}(\boldsymbol{\omega}_\lambda\cdot\mathbf{J})^T$ and integrate over θ to get V_i^Λ at the next order. We have already seen the zeroth- and first-order solutions. As with the first-order integrals, I exploit the periodicities of $\boldsymbol{\omega}_\lambda$ and V_i^Λ to reduce the range of integration to one circumference, i.e., $(\theta,\theta+2\pi)$, and obtain so-called "resonance denominators" $\exp[i2\pi(\nu+\sum_\lambda m_\lambda Q_\lambda)]-1$. Then, at Nth order,

$$\begin{aligned} V_1^\Lambda(\theta+2\pi) &= \exp\left\{2\pi\left[N\epsilon+i\left[\nu+\sum_\lambda m_\lambda Q_\lambda\right]\right]\right\}V_1^\Lambda(\theta), \\ V_0^\Lambda(\theta+2\pi) &= \exp\left[2\pi\left[N\epsilon+i\sum_\lambda m_\lambda Q_\lambda\right]\right]V_0^\Lambda(\theta), \\ V_{-1}^\Lambda(\theta+2\pi) &= \exp\left\{2\pi\left[N\epsilon+i\left[-\nu+\sum_\lambda m_\lambda Q_\lambda\right]\right]\right\}V_{-1}^\Lambda(\theta), \end{aligned} \tag{25}$$

where $\Lambda=(m_1,m_{-1},m_2,m_{-2},m_3,m_{-3})$, whence

$$V_1^\Lambda(\theta) = \frac{i}{\exp\left\{2\pi\left[N\epsilon+i\left[\nu+\sum_\lambda m_\lambda Q_\lambda\right]\right]\right\}-1}\times\sum_\lambda\int_\theta^{\theta+2\pi} e^{\epsilon\theta'}\left[\omega_{\lambda3}V_1^{\Lambda'}+\frac{\omega_{\lambda+}}{\sqrt{2}}V_0^{\Lambda'}\right]d\theta', \tag{26}$$

with similar relations for V_0^Λ and V_{-1}^Λ.[21] For example, for $\Lambda=(1,0,1,0,0,0)$,

$$\begin{aligned} \lim_{\epsilon\to0+}[V_1^\Lambda(\theta)] = &-\frac{1}{\sqrt{2}}\frac{1}{\exp[i2\pi(\nu+Q_1+Q_2)]-1} \\ &\times\left[\frac{1}{\exp[i2\pi(\nu+Q_2)-1}\int_\theta^{\theta+2\pi}d\theta'\left[\omega_{13}(\theta')\int_{\theta'}^{\theta'+2\pi}d\theta''\omega_{2+}(\theta'')\right]\right. \\ &\left.+\frac{1}{\exp[i2\pi(\nu+Q_1)]-1}\int_\theta^{\theta+2\pi}d\theta'\left[\omega_{23}(\theta')\int_{\theta'}^{\theta'+2\pi}d\theta''\omega_{1+}(\theta'')\right]\right]. \end{aligned} \tag{27}$$

To use Eq. (26) in a numerical program, however, still involves a lot of work, viz., an integral for each value of θ. However,

$$V_1^\Lambda(\theta+\Delta\theta)\simeq V_1^\Lambda(\theta)+\Delta\theta\frac{dV_1^\Lambda}{d\theta'}\bigg|_\theta = V_1^\Lambda(\theta)+\Delta\theta\sum_\lambda\left[e^{\epsilon\theta}\left[\omega_{\lambda3}V_1^{\Lambda'}+\frac{\omega_{\lambda+}}{\sqrt{2}}V_0^{\Lambda'}\right]\right], \tag{28}$$

similarly for $V_0^\Lambda(\theta+\Delta\theta)$ and $V_{-1}^\Lambda(\theta+\Delta\theta)$. Hence separate integrals for each value of θ are unnecessary: one calculates only $V_i^\Lambda(0)$ by integration; other values of $V_i^\Lambda(\theta)$ are found by proceeding in small steps $\Delta\theta$ around the ring.

There is one further problem with the evaluation of the above integrals, associated with the limit $\epsilon\to0+$. In most cases one can safely set $\epsilon=0$ in the above expressions, but problems arise with V_0^Λ whenever $m_1=m_{-1}$, $m_2=m_{-2}$, and

$m_3 = m_{-3}$, in which case $\sum_\lambda m_\lambda Q_\lambda = 0$. Such terms are always present in the perturbation series, even when the motion is nonresonant. In these cases the denominator is

$$\exp\left[2\pi\left[N\epsilon + i\sum_\lambda m_\lambda Q_\lambda\right]\right] - 1 = \exp(2\pi N\epsilon) - 1 \to 0 \quad \text{as } \epsilon \to 0^+ .$$

These terms are called "periodic terms" because their integrands are periodic (neglecting the $e^{2\pi N\epsilon}$ factor).[22] It is shown in Appendix A that they are finite. However, for any given periodic term, V_0^Λ is a sum of several integrals, and they are individually divergent. At second order, for $\Lambda = (1,1,0,0,0,0)$,

$$\begin{aligned}
V_0^\Lambda = -\frac{1}{2}\Bigg[& \int_{-\infty}^{\theta} d\theta' \left[e^{\epsilon\theta'}\omega_{1-}(\theta') \int_{-\infty}^{\theta'} d\theta'' e^{\epsilon\theta''}\omega_{-1+}(\theta'') \right] \\
& + \int_{-\infty}^{\theta} d\theta' \left[e^{\epsilon\theta'}\omega_{-1+}(\theta') \int_{-\infty}^{\theta'} d\theta'' e^{\epsilon\theta''}\omega_{1-}(\theta'') \right] + \int_{-\infty}^{\theta} d\theta' \left[e^{\epsilon\theta'}\omega_{1+}(\theta') \int_{-\infty}^{\theta'} d\theta'' e^{\epsilon\theta''}\omega_{-1-}(\theta'') \right] \\
& + \int_{-\infty}^{\theta} d\theta' \left[e^{\epsilon\theta'}\omega_{-1-}(\theta') \int_{-\infty}^{\theta'} d\theta'' e^{\epsilon\theta''}\omega_{1+}(\theta'') \right] \Bigg] \\
= -\frac{1}{2}\Bigg[& \left[\int_{-\infty}^{\theta} e^{\epsilon\theta'}\omega_{1-}(\theta')d\theta' \right] \left[\int_{-\infty}^{\theta} e^{\epsilon\theta''}\omega_{-1+}(\theta'')d\theta'' \right] \\
& + \left[\int_{-\infty}^{\theta} e^{\epsilon\theta'}\omega_{1+}(\theta')d\theta' \right] \left[\int_{-\infty}^{\theta} e^{\epsilon\theta''}\omega_{-1-}(\theta'')d\theta'' \right] \Bigg] \\
= -\frac{1}{2}\Bigg[& \frac{1}{\exp\{2\pi[\epsilon - i(\nu - Q_1)]\} - 1} \int_\theta^{\theta+2\pi} e^{\epsilon\theta'}\omega_{1-}(\theta')d\theta' \\
& \times \frac{1}{\exp\{2\pi[\epsilon + i(\nu - Q_1)]\} - 1} \int_\theta^{\theta+2\pi} e^{\epsilon\theta''}\omega_{-1+}(\theta'')d\theta'' \\
& + \frac{1}{\exp\{2\pi[\epsilon + i(\nu + Q_1)]\} - 1} \int_\theta^{\theta+2\pi} e^{\epsilon\theta'}\omega_{1+}(\theta')d\theta' \frac{1}{\exp\{2\pi[\epsilon - i(\nu + Q_1)]\} - 1} \int_\theta^{\theta+2\pi} e^{\epsilon\theta''}\omega_{-1-}(\theta'')d\theta'' \Bigg] ,
\end{aligned} \tag{29}$$

which is clearly finite as $\epsilon \to 0+$. However, application of the algorithm yields

$$\begin{aligned}
V_0^\Lambda = -\frac{1}{2}\frac{1}{e^{4\pi\epsilon} - 1}\Bigg[& \frac{1}{e^{2\pi[\epsilon + i(\nu - Q_1)]} - 1} \int_\theta^{\theta+2\pi} d\theta' \left[e^{\epsilon\theta'}\omega_{1-}(\theta') \int_{\theta'}^{\theta'+2\pi} d\theta'' e^{\epsilon\theta''}\omega_{-1+}(\theta'') \right] \\
& + \frac{1}{e^{2\pi[\epsilon - i(\nu - Q_1)]} - 1} \int_\theta^{\theta+2\pi} d\theta' \left[e^{\epsilon\theta'}\omega_{-1+}(\theta') \int_{\theta'}^{\theta'+2\pi} d\theta'' e^{\epsilon\theta''}\omega_{1-}(\theta'') \right] \\
& + \frac{1}{e^{2\pi[\epsilon - i(\nu + Q_1)]} - 1} \int_\theta^{\theta+2\pi} d\theta' \left[e^{\epsilon\theta'}\omega_{1+}(\theta') \int_{\theta'}^{\theta'+2\pi} d\theta'' e^{\epsilon\theta''}\omega_{-1-}(\theta'') \right] \\
& + \frac{1}{e^{2\pi[\epsilon + i(\nu + Q_1)]} - 1} \int_\theta^{\theta+2\pi} d\theta' \left[e^{\epsilon\theta'}\omega_{-1-}(\theta') \int_{\theta'}^{\theta'+2\pi} d\theta'' e^{\epsilon\theta''}\omega_{1+}(\theta'') \right] \Bigg] .
\end{aligned} \tag{30}$$

Delicate cancellations therefore occur, and a way must be found to avoid a spurious infinity as $\epsilon \to 0+$. In general, the integrand of a contribution to a periodic term is of the form $e^{\epsilon\theta'}(\omega_{\lambda-} V_1^{\Lambda'} + \omega_{\lambda+} V_{-1}^{\Lambda'})$, and is of $O(1)$, for each value of λ, but the sum is small, of $O(\epsilon)$. Numerically this is a serious problem. I deal with it as follows: I calculate *two* sets of functions V_i^Λ, one with $\epsilon = 0$ and the other not, say $V_i^\Lambda(\theta,0)$ and $V_i^\Lambda(\theta,\epsilon)$. Here ϵ is set to a constant value; in numerical work I do not take the limit $\epsilon \to 0+$. Only the former terms are used to evaluate Eq. (2). At a periodic term, I use not Eq. (24), but set

$$\begin{aligned}
V_0^\Lambda(\theta,0) = V_0^\Lambda(\theta,\epsilon) &= \frac{i}{2\pi N\epsilon} \sum_\lambda \int_\theta^{\theta+2\pi} e^{\epsilon\theta'} \left[\frac{\omega_{\lambda-}}{\sqrt{2}} V_1^{\Lambda'}(\theta',\epsilon) + \frac{\omega_{\lambda+}}{\sqrt{2}} V_{-1}^{\Lambda'}(\theta',\epsilon) \right] d\theta' \\
&= \frac{i}{2\pi N\epsilon}\frac{1}{\sqrt{2}} \sum_\lambda \int_\theta^{\theta+2\pi} e^{\epsilon\theta'}[\omega_{\lambda-} V_1^{\Lambda'}(\theta',\epsilon) + \omega_{\lambda+} V_{-1}^{\Lambda'}(\theta,\epsilon)]d\theta' \\
&\quad - \frac{i}{2\pi N\epsilon}\frac{1}{\sqrt{2}} \sum_\lambda \int_\theta^{\theta+2\pi} [\omega_{\lambda-} V_1^{\Lambda'}(\theta',0) + \omega_{\lambda+} V_{-1}^{\Lambda'}(\theta',0)]d\theta' \\
&= \frac{i}{2\pi N\epsilon}\frac{1}{\sqrt{2}} \sum_\lambda \int_\theta^{\theta+2\pi} \{\omega_{\lambda-}[e^{\epsilon\theta'} V_1^{\Lambda'}(\theta',\epsilon) - V_1^{\Lambda'}(\theta',0)] + \omega_{\lambda+}[e^{\epsilon\theta'} V_{-1}^{\Lambda'}(\theta',\epsilon) - V_{-1}^{\Lambda'}(\theta',0)]\}d\theta' .
\end{aligned} \tag{31}$$

Now, by integrating the *difference,* not the individual integrands themselves, I obtain $O(\epsilon)$ terms throughout, and the numerical accuracy of the final result is much improved. The introduction of $V_{\pm 1}^{\Lambda'}(\theta',0)$ in Eq. (31) is permissible because

$$\sum_{\lambda} \int_{\theta}^{\theta+2\pi} [\omega_{\lambda-} V_1^{\Lambda'}(\theta',0)+\omega_{\lambda+} V_{-1}^{\Lambda'}(\theta',0)]d\theta'=0 , \quad (32)$$

since it is the $\epsilon \rightarrow 0+$ value of the numerator of a periodic term, and by definition vanishes. In numerical work ϵ is set to a fixed user-specified value, typically $\epsilon \simeq 10^{-4}-10^{-6}$. This procedure does introduce "contamination" into $V_0^{\Lambda}(\theta,0)$ from $V_0^{\Lambda}(\theta,\epsilon)$, because I do not take the limit $\epsilon \rightarrow 0+$, but this only happens in the periodic terms and is not large.[23] It is shown in Appendix B that the final result does not depend strongly on the value of ϵ.

There are two items left before one can evaluate the equilibrium degree of polarization, viz., the calculation of $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ and the ensemble averages. To obtain $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ to Nth order, I follow the procedure described above for zeroth order: I calculate $\hat{\mathbf{n}}$ to $(N+1)$th order on two orbital trajectories that differ only by energy $mc^2\Delta\gamma$ at a given azimuth, and divide the difference by $\Delta\gamma/\gamma$. At zeroth order, we saw that this amounted simply to replacing a_λ in the first-order expression for $\hat{\mathbf{n}}$ by $\mp iE_{5\lambda}^*$. Similarly, to obtain $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ to higher order, I replace, in each term in the series for $\hat{\mathbf{n}}$, one of the coefficients a_λ by $-iE_{5\lambda}^*(\theta)$ if $\lambda>0$, or $iE_{5\lambda}^*(\theta)$ if $\lambda<0$, doing so for all possible replacements in each term. For example, the term $a_1a_2V_1^{\Lambda}(\theta)$ in $\hat{\mathbf{n}}$, i.e., $\Lambda=(1,0,0,0,0,0)$, yields the terms $-i[a_1E_{52}^*(\theta)+a_2E_{51}^*(\theta)]V_1^{\Lambda}(\theta)$ in $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, and the term $a_1a_{-1}V_0^{\Lambda}(\theta)$, i.e., $\Lambda=(1,1,0,0,0,0)$, yields $i[a_1E_{5-1}^*(\theta)-a_{-1}E_{51}^*(\theta)]V_0^{\Lambda}(\theta)$. Notice that this is merely the rule for differentiating a product.

I have not so far specified the numerical values used for the coefficients a_λ. In fact, it is not necessary to select a large sample of values and to average over them explicitly. Note that one is not really interested in $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ on specific orbital trajectories in Eq. (2), one is only interested in ensemble averages of various functions of these quantities. Let us therefore consider the ensemble average at this point. It affects only the coefficients a_λ, not the functions V_i^{Λ}. Since $a_{-\lambda}=a_\lambda^*$, only $\lambda>0$ is needed. I assume, as is standard, that in equilibrium the phase of a_λ is uniformly distributed in the range $[0,2\pi)$, while the probability density of finding $|a_\lambda|$ in the range $(A_\lambda, A_\lambda+\delta A_\lambda)$ is proportional to $A_\lambda \exp(-A_\lambda^2/2\sigma_\lambda^2)$, where σ_λ^2 is one of the beam emittances.[24] This means that, in the perturbation series for $\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}$, $(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2$, etc. in Eq. (2), the ensemble averages are taken by replacing the coefficients $a_{\lambda_1}\cdots a_{\lambda_N}$ by their averages $\langle a_{\lambda_1}\cdots a_{\lambda_N}\rangle$. This means that I do not specify numerical values for the a_λ on individual trajectories at all; I only need the moments $\langle |a_\lambda|^{2N}\rangle=\langle (a_\lambda a_{-\lambda})^N\rangle$, $N=1,2,\ldots$, since $\langle |a_1|^2|a_2|^2\rangle=\langle |a_1|^2\rangle\langle |a_2|^2\rangle$, etc. Using the above equilibrium probability density function, it then suffices to know only σ_1^2, σ_2^2, and σ_3^2. Consequently, it is not necessary to calculate either $\hat{\mathbf{n}}$ or $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ *per se,* or to select a sample of orbital trajectories; one only needs the functions $V_i^{\Lambda}(\theta)$, $E_{5\lambda}(\theta)$, and the emittances σ_λ^2.

The only terms $a_{\lambda_1}\cdots a_{\lambda_N}V_i^{\Lambda}$ in the expression for $\langle |\dot{\mathbf{v}}|^3\hat{\mathbf{b}}\cdot\hat{\mathbf{n}}\rangle$ in the numerator of Eq. (2) which survive the ensemble average are those for which the coefficients a_λ come in complex conjugate pairs: Λ must be of the form $(\Lambda_1,\Lambda_1,\Lambda_2,\Lambda_2,\Lambda_3,\Lambda_3)$. As for $\langle |\dot{\mathbf{v}}|^3\hat{\mathbf{b}}\cdot\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)\rangle$ in Eq. (2), note that in calculating $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, one coefficient a_λ is replaced by $\mp iE_{5\lambda}^*$, hence Λ has almost the same form as above, but with one "unpaired" coefficient, e.g., $\Lambda=(1,0,0,0,0,0)$ or $\Lambda=(1,1,2,3,0,0)$, which contribute to the zeroth- and sixth-order terms in $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, respectively. The evaluation of $\langle |\dot{\mathbf{v}}|^3(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2\rangle$ and $\langle |\dot{\mathbf{v}}|^3|\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)|^2\rangle$, which appear in the denominator of Eq. (2), is more laborious, but similar. I first multiply two θ-ordered expansions, then take an ensemble average over the resulting products of coefficients a_λ.

IV. RESULTS

Some years ago, Chao[5] wrote a numerical program called SLIM to calculate the equilibrium degree of polarization in a high-energy electron storage ring. The polarization is evaluated using $\hat{\mathbf{n}}_0$ and Chao's expression for $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, which, as we saw in Sec. III, are the zeroth-order terms in the series for $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, respectively. This program has been extended to include higher-order terms in $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, using the formalism developed in Sec. III, and the new program is called SMILE. The calculation of the orbital motion is the same as in SLIM. The orbital trajectory is stored as a six-component column vector, say $y(\theta)$. Since only linear dynamics is treated, the evolution of this column vector around the ring is described by so-called "transfer matrices," i.e., $y(\theta')=M(\theta',\theta)y(\theta)$, where M is a 6×6 matrix. Arbitrary linear dynamical effects can be treated: it is not necessary to assume the equilibrium closed orbit lies in a plane, and the accelerator does not need to have any particular symmetry. Effects of closed orbit distortion due to magnet misalignments are also included. All of these features are retained in SMILE. In SLIM, these 6×6 matrices are extended in size to 8×8 and two extra components are added to the column vector of the trajectory. They are the first-order terms in the series for $\hat{\mathbf{n}}$, i.e., $y_{\text{SLIM}}(\theta)=(r_1,p_1,r_2,p_2,r_3,p_3,n_1,n_2)^T$, where n_1 and n_2 are calculated only to first order in the orbital motion. These extra components are used to calculate Chao's expression for $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. In SMILE I do not use 8×8 matrices; instead I return to 6×6 matrices which affect only the orbital motion. I store the spin information in separate arrays. I use V_1, V_0, and V_{-1} instead of n_1, n_2, and n_3 and since $V_{-1}=-V_1^*$, I in fact do not need V_{-1}. It then suffices to store four quantities for each index Λ: $\text{Re}(V_1^{\Lambda})$, $\text{Im}(V_1^{\Lambda})$, $\text{Re}(V_0^{\Lambda})$, and $\text{Im}(V_0^{\Lambda})$, hence ($S\equiv$spin array)

$$y(\theta)=\begin{bmatrix} r_1(\theta) \\ \vdots \\ p_3(\theta) \end{bmatrix}, \quad S=\begin{bmatrix} \cdots & \text{Re}(V_1^{\Lambda}) & \cdots \\ \cdots & \text{Im}(V_1^{\Lambda}) & \cdots \\ \cdots & \text{Re}(V_0^{\Lambda}) & \cdots \\ \cdots & \text{Im}(V_0^{\Lambda}) & \cdots \end{bmatrix} . \quad (33)$$

As explained in Sec. III, there are in fact two spin arrays, one for $\epsilon=0$ and the other for $\epsilon\neq 0$, where ϵ is the convergence factor. Various symmetries of the V_i^{Λ} are described in Appendix B, which reduce the number of independent indexes Λ. The expressions for the spin precession vectors Ω_0 and ω are the same as in SLIM.[5] First-order spin integrals are obtained by integrating around the ring, starting from a user-specified origin, using the zeroth-order expression for $\hat{\mathbf{n}}$, which is coded explicitly. Given these integrals, I integrate around the ring again to get the second-order integrals. Because the algorithm is recursive, repetition of this procedure yields all the higher-order spin integrals, up to a user-specified maximum order.

Figure 1 displays the result of calculation up to second order for a simple accelerator model.[25,26] The dashed and solid curves show the zeroth- and second-order results, respectively.[27] The orbital tunes are $Q_1=2.182$, $Q_2=1.718$, and $Q_3=0.046$, where the oscillations are horizontal, vertical, and longitudinal, respectively. We see one zeroth-order, three first-order, and eight second-order resonances. They are listed in Table I in order of increasing energy. It is seen that the resonances $\nu=2Q_3+8$ and $\nu=10-Q_1-Q_2$, at about 3.565 GeV, are rather close together, as are the resonances $\nu=Q_1+Q_3+6$ and $\nu=10-Q_2-Q_3$, near 3.63 GeV. One of the difficulties in this field has been the calculation of overlapping higher-order resonances: one usually considers only cases where the resonances are well separated. To illustrate that the present algorithm does not suffer from this limitation, Q_3 was increased to 0.050 and the calculation was repeated. The result is displayed in Fig. 2. The resonances $\nu=2Q_3+8$ and $\nu=10-Q_1-Q_2$ overlap and only one distinct resonance is seen at 3.57 GeV. Similarly, at 3.63 GeV, the resonances $\nu=Q_1+Q_3+6$ and $\nu=10-Q_2-Q_3$ also merge into one. No change in the algorithm is required to calculate the polarization in the vicinity of these overlapping resonances. Figure 3 shows an example of experimental data.[7] The orbital tunes are labeled ν_x,ν_y, and ν_s, denoting horizontal, vertical, and longitudinal oscillations with respect to the equilibrium closed orbit. The polarization is measured in units of $P_{\max}=8/(5\sqrt{3})\simeq 92.4\%$. [Strictly, the polarization can exceed this value, but usually $P\leq 8/(5\sqrt{3})$.] A detailed theoretical comparison with this experiment is not yet available, but it is seen that the graphs in Figs. 1 and 2 are qualitatively similar to the experimental result.

TABLE I. List of spin resonances in Fig. 1.

Energy (GeV)	Spin resonance
3.525	$\nu=8$
3.546	$\nu=Q_3+8$
3.565	$\nu=2Q_3+8$
3.570	$\nu=10-Q_1-Q_2$
3.585	$\nu=Q_1-Q_3+6$
3.605	$\nu=Q_1+6$
3.625	$\nu=Q_1+Q_3+6$
3.629	$\nu=10-Q_2-Q_3$
3.650	$\nu=10-Q_2$
3.670	$\nu=10-Q_2+Q_3$
3.685	$\nu=2Q_1+4$
3.730	$\nu=Q_1-Q_2+8$

V. CONCLUSIONS

I have developed an algorithm to evaluate the polarization in a high-energy electron storage ring to a higher degree of approximation than has hitherto been possible. This algorithm offers a number of attractive features: it is able to handle all modes of orbital motion simultaneously, in principle to arbitrary orders, it is not restricted to accelerators of any particular energy range or geometry, and is able to deal with overlapping resonances without requiring special precautions. Numerically, the algorithm treats only linear orbital dynamics and preliminary results have been presented, including a calculation of overlapping resonances.

I began by presenting a formal solution for the spin quantization axis $\hat{\mathbf{n}}$, using a θ-ordered exponential, and then showed how to evaluate this solution to various orders. Naïvely, one might expect that $\hat{\mathbf{n}}$ must be calculated on a large sample of orbital trajectories to obtain the derivative $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ and the ensemble average, but I formulated the algorithm in such a way that this was unnecessary. Instead I found a more accurate method by employing the symmetries of the coefficients a_λ and the properties of the ensemble average, and also presented a simple rule to obtain $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ from $\hat{\mathbf{n}}$ without explicit numerical differentiation. I also used symmetry to classify the spin integrals V_i^{Λ}. There are many such integrals, at any order, and the combinatorics of classifying them is in general complicated. In particular, I showed that different permutations of modes, which yield the *same* in-

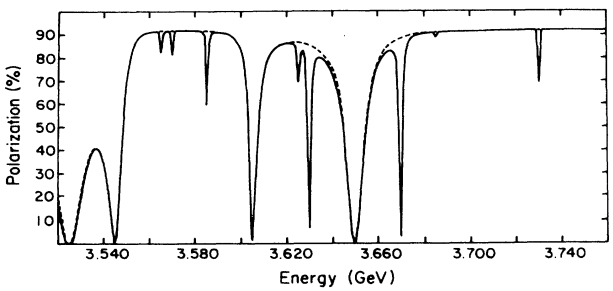

FIG. 1. Graph of polarization vs energy. See text for details.

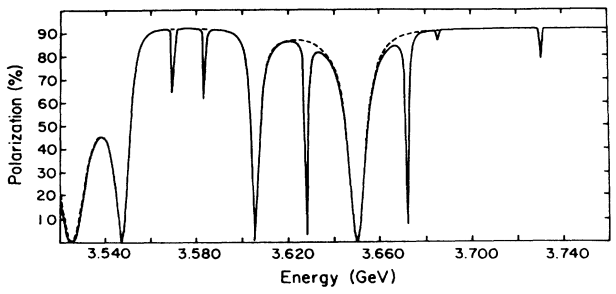

FIG. 2. Graph of polarization vs energy, with overlapping spin resonances at 3.57 and 3.63 GeV. See text for details.

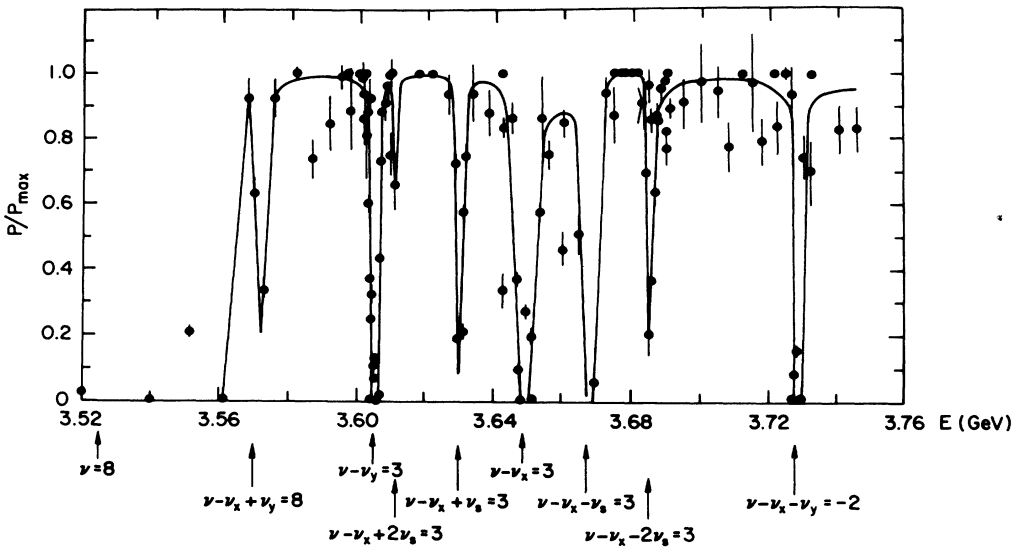

FIG. 3. Graph of experimental data of polarization vs energy (from Ref. 7). $P_{\max}=92.4\%$. The curve through the data is a guide for the eye, not a theoretical calculation.

dex Λ, need not be stored separately: only the sum total of modes is required, not the individual permutations. This reduces the number of integrals, hence the computer storage requirements. By using symmetry, I was able to develop a recursive technique for obtaining the higher-order integrals. Thus the same numerical code can be used to all orders: new subroutines do not have to be written for each new order of calculation. Some subtleties were encountered concerning the possible existence of secular terms in the series for $\hat{\mathbf{n}}$. I proved the absence of such terms and the finiteness of the periodic terms, to all orders. However, naïve evaluation of the θ-ordered exponential then led to numerically unstable calculations, but I was also able to avoid them.

This algorithm is applicable to all storage rings where the Derbenev-Kondratenko[3,4] formula is applicable, i.e., where synchrotron radiation is the dominant source of perturbation of the electron motion, in particular of spin flip. Numerically, it is applicable to the same systems for which SLIM (Ref. 5) can be used: it is not restricted, in principle, to any particular energy range, or to accelerators of any particular geometry. As in SLIM, symplectic orbital transfer matrices are used, and so one can deal with arbitrary linear coupling between horizontal, vertical, and longitudinal displacements from the equilibrium closed orbit.

ACKNOWLEDGMENTS

I am especially grateful to K. Yokoya for much help and encouragement in carrying out this research, and for checking early versions of the manuscript. I would also like to thank M. T. Bernius, R. G. Glasser, and R. Talman for comments on the manuscript, and M. E. Fisher, K. Gottfried, and the staff of the Laboratory of Nuclear Studies, Cornell University. This work was supported in part by the National Science Foundation, under Grant Nos. PHY84-12263 and PHY84-1531.

APPENDIX A: FINITENESS OF THE PERIODIC TERMS

In Sec. III I showed that the periodic terms are finite at second order. Here I prove that they are finite at all orders. The proof is inductive, hence I assume that finiteness has been established up to the $(N-2)$th order, where N is even. At second order, a periodic term is of the form

$$V_0^{\Lambda}=-\frac{1}{2}\left[\int_{-\infty}^{\theta}d\theta_1\left[e^{\epsilon\theta_1}\omega_{\lambda-}(\theta_1)\int_{-\infty}^{\theta_1}d\theta_2 e^{\epsilon\theta_2}\omega_{-\lambda+}(\theta_2)\right]+\int_{-\infty}^{\theta}d\theta_2\left[e^{\epsilon\theta_2}\omega_{-\lambda+}(\theta_2)\int_{-\infty}^{\theta_2}d\theta_1 e^{\epsilon\theta_1}\omega_{\lambda-}(\theta_1)\right]\right]+\cdots \tag{A1}$$

and we saw in Sec. III that

$$\int_{-\infty}^{\theta}d\theta_1\left[e^{\epsilon\theta_1}\omega_{\lambda-}(\theta_1)\int_{-\infty}^{\theta_1}d\theta_2 e^{\epsilon\theta_2}\omega_{-\lambda+}(\theta_2)\right]+\int_{-\infty}^{\theta}d\theta_2\left[e^{\epsilon\theta_2}\omega_{-\lambda+}(\theta_2)\int_{-\infty}^{\theta_2}d\theta_1 e^{\epsilon\theta_1}\omega_{\lambda-}(\theta_1)\right]$$
$$=\left[\int_{-\infty}^{\theta}d\theta_1 e^{\epsilon\theta_1}\omega_{\lambda-}(\theta_1)\right]\left[\int_{-\infty}^{\theta}d\theta_2 e^{\epsilon\theta_2}\omega_{-\lambda+}(\theta_2)\right], \tag{A2}$$

and that, although the left-hand-side terms are individually divergent, the right-hand side is finite as $\epsilon\to 0+$, which established the finiteness of the second-order periodic terms. The above result assumed linear orbital dynamics because I

decomposed ω into a linear combination $\sum_\lambda a_\lambda \omega_\lambda$. It is not necessary to do this, hence I shall not assume linear dynamics in the following proof. I need only to assume that the motion is nonresonant, i.e.,

$$v \neq \text{integer}, \quad \sum_{\lambda=1,2,3} m_\lambda Q_\lambda \neq \text{integer}, \quad \nu + \sum_{\lambda=1,2,3} m_\lambda Q_\lambda \neq \text{integer}, \quad (Q_\lambda \equiv d\psi_\lambda/d\theta) \tag{A3}$$

for any nonzero integers m_λ. Recall, from Sec. III, $\omega_\pm = \boldsymbol{\omega}\cdot(\hat{\boldsymbol{l}}_0 \pm i\hat{\mathbf{m}}_0)$ and $\omega_3 = \boldsymbol{\omega}\cdot\hat{\mathbf{n}}_0$. Then, at second order,

$$V_0 \propto \int_{-\infty}^{\theta} d\theta_1 \left[e^{\epsilon\theta_1}\omega_-(\theta_1) \int_{-\infty}^{\theta_1} d\theta_2 e^{\epsilon\theta_2}\omega_+(\theta_2) \right] + \int_{-\infty}^{\theta} d\theta_2 \left[e^{\epsilon\theta_2}\omega_+(\theta_2) \int_{-\infty}^{\theta_2} d\theta_1 e^{\epsilon\theta_1}\omega_-(\theta_1) \right]$$
$$= \left[\int_{-\infty}^{\theta} d\theta_1 e^{\epsilon\theta_1}\omega_-(\theta_1) \right] \left[\int_{-\infty}^{\theta} d\theta_2 e^{\epsilon\theta_2}\omega_+(\theta_2) \right], \tag{A4}$$

which is finite, by Eq. (A3). At Nth order, neglecting overall constants,

$$V_0 \propto \sum_{\substack{\text{allowed spin} \\ \text{combinations}}} \int_{-\infty}^{\theta} d\theta_1 \left[e^{\epsilon\theta_1}\omega_\pm(\theta_1) \int_{-\infty}^{\theta_1} d\theta_2 \left[e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \cdots \int_{-\infty}^{\theta_{N-1}} d\theta_N e^{\epsilon\theta_N}\omega_\pm(\theta_N) \right] \right]. \tag{A5}$$

To avoid cumbersome notation, $\omega_{\pm,3}$ denotes ω_+, ω_-, or ω_3, I suppress the factors of $\sqrt{2}$ associated with $\omega_+/\sqrt{2}$ and $\omega_-/\sqrt{2}$, and indicate a sum over "allowed spin combinations." Integrating an individual term on the right-hand side of Eq. (A5) by parts, one finds

$$\int_{-\infty}^{\theta} d\theta_1 \left[e^{\epsilon\theta_1}\omega_\pm(\theta_1) \int_{-\infty}^{\theta_1} d\theta_2 \left[e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \cdots \right] \right]$$
$$= \left\{ \left[\int_{-\infty}^{\theta_1} d\theta' e^{\epsilon\theta'}\omega_\pm(\theta') \right] \left[\int_{-\infty}^{\theta_1} d\theta_2 \left[e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \cdots \right] \right] \right\}_{-\infty}^{\theta}$$
$$- \int_{-\infty}^{\theta} d\theta_2 \left[\left[\int_{-\infty}^{\theta_2} d\theta_1 e^{\epsilon\theta_1}\omega_\pm(\theta_1) \right] e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \int_{-\infty}^{\theta_2} d\theta_3 \left[e^{\epsilon\theta_3}\omega_{\pm,3}(\theta_3) \cdots \int_{-\infty}^{\theta_{N-1}} d\theta_N e^{\epsilon\theta_N}\omega_\pm(\theta_N) \right] \right]$$
$$= \left[\int_{-\infty}^{\theta} d\theta_1 e^{\epsilon\theta_1}\omega_\pm(\theta_1) \right] \left[\int_{-\infty}^{\theta} d\theta_2 \left[e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \cdots \right] \right]$$
$$- \left\{ \int_{-\infty}^{\theta} d\theta_2 \left[e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \left[\int_{-\infty}^{\theta_2} d\theta_1 e^{\epsilon\theta_1}\omega_\pm(\theta_1) \right] \right. \right.$$
$$\left. \left. \times \int_{-\infty}^{\theta_2} d\theta_3 \left[e^{\epsilon\theta_3}\omega_{\pm,3}(\theta_3) \cdots \int_{-\infty}^{\theta_{N-1}} d\theta_N e^{\epsilon\theta_N}\omega_\pm(\theta_N) \right] \right] \right\}. \tag{A6}$$

Repeating the integration by parts,

$$\int_{-\infty}^{\theta} d\theta_1 \left[e^{\epsilon\theta_1}\omega_\pm(\theta_1) \int_{-\infty}^{\theta_1} d\theta_2 \left[e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \cdots \int_{-\infty}^{\theta_{N-1}} d\theta_N e^{\epsilon\theta_N}\omega_\pm(\theta_N) \right] \right]$$
$$= \left[\int_{-\infty}^{\theta} d\theta_1 e^{\epsilon\theta_1}\omega_\pm(\theta_1) \right] \left[\int_{-\infty}^{\theta} d\theta_2 [e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \cdots] \right]$$
$$- \left[\int_{-\infty}^{\theta} d\theta_2 \left[e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \int_{-\infty}^{\theta_2} d\theta_1 e^{\epsilon\theta_1}\omega_\pm(\theta_1) \right] \right] \left[\int_{-\infty}^{\theta} d\theta_3 [e^{\epsilon\theta_3}\omega_{\pm,3}(\theta_3) \cdots] \right]$$
$$+ \cdots + \left\{ \int_{-\infty}^{\theta} d\theta_{N-1} \left[e^{\epsilon\theta_{N-1}}\omega_{\pm,3}(\theta_{N-1}) \int_{-\infty}^{\theta_{N-2}} d\theta_{N-2} \left[e^{\epsilon\theta_{N-2}}\omega_{\pm,3}(\theta_{N-2}) \cdots \int_{-\infty}^{\theta_2} d\theta_1 e^{\epsilon\theta_1}\omega_\pm(\theta_1) \right] \right] \right\}$$
$$\times \left[\int_{-\infty}^{\theta} d\theta_N e^{\epsilon\theta_N}\omega_\pm(\theta_N) \right]$$
$$- \left\{ \int_{-\infty}^{\theta} d\theta_N \left[e^{\epsilon\theta_N}\omega_\pm(\theta_N) \int_{-\infty}^{\theta_N} d\theta_{N-1} \left[e^{\epsilon\theta_{N-1}}\omega_{\pm,3}(\theta_{N-1}) \cdots \int_{-\infty}^{\theta_2} d\theta_1 e^{\epsilon\theta_1}\omega_\pm(\theta_1) \right] \right] \right\}. \tag{A7}$$

The last term on the rhs differs from the lhs merely by a relabeling of indexes, so, summing over all indexes, with appropriate factors of $\sqrt{2}$,

$$V_0 \propto \sum_{\substack{\text{allowed spin} \\ \text{combinations}}} + \int_{-\infty}^{\theta} d\theta_1 \left[e^{\epsilon\theta_1}\omega_\pm(\theta_1) \int_{-\infty}^{\theta_1} d\theta_2 \left[e^{\epsilon\theta_2}\omega_{\pm,3}(\theta_2) \cdots \int_{-\infty}^{\theta_{N-1}} d\theta_N e^{\epsilon\theta_N}\omega_\pm(\theta_N) \right] \right]$$
$$= \tfrac{1}{2} \sum_{\substack{\text{allowed spin} \\ \text{combinations}}} [\text{rhs of Eq. (A7) without the last term}]. \tag{A8}$$

The rhs of Eq. (A8) is finite as $\epsilon \to 0+$ because it consists of a finite sum of terms, each of which consists of products of integrals of lower than Nth order, hence assumed finite as $\epsilon \to 0+$. It then follows by induction on N that V_0 is finite to all orders. In particular, if the orbital dynamics is linear, all the periodic terms $\{V_0^{\Lambda}, \Lambda_1 = \Lambda_{-1}, \Lambda_2 = \Lambda_{-2}, \Lambda_3 = \Lambda_{-3}\}$ are finite.

To be more precise, it seems that I have proved only the finiteness of the sum of all terms which contribute to V_0 at any given order, hence the preceding statement does not follow, but note that

$$V_0 = \sum_{\lambda_1, \ldots, \lambda_N} \sum_{\substack{\text{allowed spin} \\ \text{combinations}}} a_{\lambda_1} a_{\lambda_2} \cdots a_{\lambda_N} V_0^{\Lambda} , \tag{A9}$$

and because the coefficients a_λ are arbitrary, I have in fact established the finiteness of each term V_0^{Λ} in the above sum, including, in particular, the periodic terms.

APPENDIX B: SYMMETRIES OF THE SPIN INTEGRALS

The integrals V_i^{Λ} have certain symmetries which limit the number of independent integrals. Here I describe these symmetries, and show in particular that all the periodic terms (see Sec. III) are real. Note, from the definitions given in Sec. III, that $V_{-1} = -V_1^*$, and that V_0 is real. Also recall that

$$V_i = \lim_{\epsilon \to 0+} \left[\sum_{N=0}^{\infty} \sum_{\substack{\text{distinct} \\ \text{combinations} \\ \lambda_1, \ldots, \lambda_N}} a_{\lambda_1} \cdots a_{\lambda_N} V_i^{\Lambda} \right] . \tag{B1}$$

I omit explicit mention of the convergence factor below. It is useful to introduce some notation: if $\Lambda = (\Lambda_1, \Lambda_{-1}, \Lambda_2, \Lambda_{-2}, \Lambda_3, \Lambda_{-3})$, then define $\overline{\Lambda} = (\Lambda_{-1}, \Lambda_1, \Lambda_{-2}, \Lambda_2, \Lambda_{-3}, \Lambda_3)$. It then follows that

$$\begin{aligned} V_{-1} = -V_1^* &= -\left[\sum_{N=0}^{\infty} \sum_{\Lambda} a_{\lambda_1} \cdots a_{\lambda_N} V_1^{\Lambda} \right]^* \\ &= -\sum_{N=0}^{\infty} \sum_{\Lambda} a_{\lambda_1}^* \cdots a_{\lambda_N}^* (V_1^{\Lambda})^* \\ &= -\sum_{N=0}^{\infty} \sum_{\Lambda} a_{\lambda_1} \cdots a_{\lambda_N} (V_1^{\overline{\Lambda}})^* , \end{aligned} \tag{B2}$$

where the last step is justified because I sum over all values of Λ. Since the coefficients are arbitrary, it follows that $V_{-1}^{\Lambda} = -(V_1^{\overline{\Lambda}})^*$, and by similar manipulations that $V_0^{\Lambda} = (V_0^{\overline{\Lambda}})^*$. Let us now consider the periodic terms as a special case. These are integrals V_0^{Λ} for which $\Lambda = \overline{\Lambda}$. From above, we immediately see that $V_0^{\Lambda} = (V_0^{\Lambda})^*$ for a periodic term, hence it is real at any order. This is demonstrated below, where I show the values of the second-order periodic terms for the model used in Sec. IV, obtained using the program SMILE described in Sec. IV, for the data point at 3.6 GeV. The program prints the values of the V_i^{Λ} at $\theta = 0$, the user-specified origin of azimuth around the accelerator. For $\epsilon = 10^{-4}$, the periodic terms are

$$\begin{aligned} V_0^{(1,1,0,0,0,0)} &= -0.067\,97 + i0.1618 \times 10^{-16} , \\ V_0^{(0,0,1,1,0,0)} &= -933.4 - i0.1811 \times 10^{-11} , \\ V_0^{(0,0,0,0,1,1)} &= -0.2535 \times 10^{-5} - i0.5266 \times 10^{-20} . \end{aligned} \tag{B3}$$

They are indeed real to high accuracy. Further, setting $\epsilon = 10^{-6}$, keeping all other parameters fixed, the values are

$$\begin{aligned} V_0^{(1,1,0,0,0,0)} &= -0.067\,95 - i0.3775 \times 10^{-14} , \\ V_0^{(0,0,1,1,0,0)} &= -932.8 - i0.2435 \times 10^{-9} , \\ V_0^{(0,0,0,0,1,1)} &= -0.2534 \times 10^{-5} - i0.5266 \times 10^{-18} . \end{aligned} \tag{B4}$$

The value of ϵ clearly does not significantly affect the values of the integrals, which shows that this method of calculating the periodic terms is numerically reliable.

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[11]This means that $\boldsymbol{\Omega}$, as a function of θ, has different dimensions from $\boldsymbol{\Omega}$ as a function of t, and is in fact a new function. I shall henceforth treat $\boldsymbol{\Omega}$ as a function of θ.

[12]From Refs. 4 and 10, the axis $\hat{\mathbf{n}}$ displays the same periodicities as the orbital trajectory on which it is defined. It follows that $\hat{\mathbf{n}}_0$ is periodic, i.e., $\hat{\mathbf{n}}_0(\theta + 2\pi) = \hat{\mathbf{n}}_0(\theta)$. Note that $\hat{\boldsymbol{l}}_0$ and $\hat{\mathbf{m}}_0$ are arbitrary up to a rotation through an arbitray angle around $\hat{\mathbf{n}}_0$. It is shown below that this angle does not affect the expressions for $\hat{\mathbf{n}}$ and $\gamma(\partial \hat{\mathbf{n}}/\partial \gamma)$.

[13]For circular motion in a uniform static vertical magnetic field $\mathbf{B} = B_{\text{ext}} \hat{\mathbf{b}}$, the Lorentz equation is $d\boldsymbol{\beta}/dt = \omega_0 \hat{\mathbf{b}} \times \boldsymbol{\beta}$, and the Thomas-BMT equation is $d\mathbf{s}/dt = (1 + \gamma a)\omega_0 \hat{\mathbf{b}} \times \mathbf{s}$, where $\omega_0 = -eB_{\text{ext}}/\gamma mc$, whence $\hat{\mathbf{n}} = \hat{\mathbf{b}}$ and $\nu = \gamma a$. In practice, $\nu \simeq \gamma a$ to a good approximation in most modern storage rings.

We shall see below that the behavior of $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ depends strongly, in general, on the value of ν, hence on the value of a, as stated in Sec. II and Ref. 3. In practice the spin tune depends on the orbital trajectory, and so ν is only the spin tune on the equilibrium closed orbit.

[14]Calculations of $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ have been made with various approximations in the orbital and spin motion, e.g., Ref. 6, which presents a calculation of some of the higher-order resonances, taking into account only the longitudinal oscillations, and neglecting certain terms in the Thomas-BMT equation when solving for $\hat{\mathbf{n}}$.

[15]Using Eqs. (5), (6), and (9), I can now prove that the solution for $\hat{\mathbf{n}}$ is not affected by a rotation of $\hat{\boldsymbol{l}}_0$ and $\hat{\mathbf{m}}_0$ around $\hat{\mathbf{n}}_0$. Define a new basis $\{\hat{\boldsymbol{l}}'_0,\hat{\mathbf{m}}'_0,\hat{\mathbf{n}}_0\}$ such that $\hat{\boldsymbol{l}}'_0\pm i\hat{\mathbf{m}}'_0=e^{\pm i\phi}(\hat{\boldsymbol{l}}_0\pm i\hat{\mathbf{m}}_0)$, where ϕ is a constant. Then, using Eq. (6), $\omega'_\pm=e^{\pm i\phi}\omega_\pm$, $\omega'_3=\omega_3$, where primed quantities refer to the new basis, and, using Eq. (9), $V'_{\pm1}=e^{\pm i\phi}V_{\pm1}$ and $V'_0=V_0$. Since

$$\hat{\mathbf{n}}=V_0\hat{\mathbf{n}}_0-V_1(\hat{\boldsymbol{l}}_0-i\hat{\mathbf{m}}_0)/\sqrt{2}+V_{-1}(\hat{\boldsymbol{l}}_0+i\hat{\mathbf{m}}_0)/\sqrt{2}\ ,$$

it follows that $\hat{\mathbf{n}}'=\hat{\mathbf{n}}$.

[16]This is not the standard arrangement of coordinates and momenta in textbooks on Hamiltonian mechanics, but it is the usual practice in accelerator physics. It is also conventional in accelerator physics to replace the longitudinal momentum offset p_3 by $\Delta E/E_0$, the relative energy offset. I otherwise follow Golstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, New York, 1981).

[17]The transformation of y from azimuth θ to θ' can be described by a 6×6 matrix $M(\theta,\theta')$. The normal modes $E_\lambda(\theta)$ are defined to be the eigenvectors of the matrix $M(\theta+2\pi,\theta)$. It can be shown that when the motion is stable, the eigenvalues of this matrix have unit magnitude and come in complex conjugate pairs (see Ref. 16); $M(\theta+2\pi,\theta)E_\lambda(\theta)=\exp(i2\pi Q_\lambda)E_\lambda(\theta)$ and $Q_{-\lambda}=Q_\lambda$. The Q_λ are called the "tunes" of the normal modes.

[18]The Chao integrals are normally associated only with $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, not with $\hat{\mathbf{n}}$. It has generally been believed that $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ is not a derivative of $\hat{\mathbf{n}}$ because the properties of $\hat{\mathbf{n}}$ do not seem to have been clearly understood. The nature of the derivative is explained later on in this paper, and also in Ref. 3. We thus see the Chao integrals in a new context, and also how to obtain the generalizations of these integrals to higher orders.

[19]In the series for $\hat{\mathbf{n}}$, the number of vectors $\boldsymbol{\omega}_\lambda$ in any terms is the same as the number of coefficients a_λ, but for $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ a term with N coefficients a_λ contains $N+1$ vectors $\boldsymbol{\omega}_\lambda$.

[20]Strictly speaking, there are two possibilities here. One is that $\hat{\mathbf{n}}$ may be well defined but not expressible as a perturbation series. The other is, of course, that $\hat{\mathbf{n}}$ may be ill defined. It is shown that in Refs. 4 and 10 that $\hat{\mathbf{n}}$ is ill defined whenever integers m_λ exist such that $\nu+\sum_\lambda m_\lambda Q_\lambda$ is an integer. Since $\nu\pm Q_\lambda$ contains only one orbital tune, I call this a first-order spin resonance. The case ν is an integer, where $\hat{\mathbf{n}}_0$ itself is ill defined, is called a zeroth-order resonance. We may call $\sum_\lambda|m_\lambda|$ the order of a resonance. However, the various modes E_λ have very different effects on $\hat{\mathbf{n}}$, and so it is possible for third-order resonances to sometimes be stronger than second-order resonances, depending on the orbital modes involved, hence such a classification of spin resonances must be used with care. The zeroth-order resonances (ν is an integer) are often called "integer" resonances, and the first- and higher-order resonances are often called "linear" and "nonlinear" resonances, respectively. I shall not use these names; I shall reserve the term "nonlinear spin resonances" for cases when the orbital dynamics is nonlinear.

[21]Note that $V_0^\wedge$ apparently diverges as $\epsilon\to0+$ whenever $\sum_\lambda m_\lambda Q_\lambda$ is an integer, but this is not called a spin resonance because the spin tune is not involved. If $m_1=m_{-1}$, $m_2=m_{-2}$, and $m_3=m_{-3}$, the infinity is spurious and $V_0^\wedge$ is in fact finite. I shall deal with this problem below. Otherwise, this condition implies an orbital resonance and the behavior of $\hat{\mathbf{n}}$ is academic. In deriving the condition for a spin resonance, in Refs. 4 and 10, it is implicitly assumed the orbital motion is stable.

[22]The name periodic term is due to K. Yokoya (private communication), in response to a proof I sent him of the cancellation of the infinities in these terms. He warned that should the series for $\hat{\mathbf{n}}$ have a so-called "secular term," in the jargon of classical perturbation theory, i.e., a term *linear*, not oscillatory, in θ, the infinities would fail to cancel. Both $\hat{\boldsymbol{l}}$ and $\hat{\mathbf{m}}$ (the obvious generalizations of $\hat{\boldsymbol{l}}_0$ and $\hat{\mathbf{m}}_0$) have secular terms, but he was able to prove that $\hat{\mathbf{n}}$ does not, at second order. The above name is chosen to emphasize that $\hat{\mathbf{n}}$ has no secular terms.

[23]One might ask if two sets of $V_i^\wedge$ are really necessary. Initially, $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ were calculated using only $V_i^\wedge(\theta,\epsilon)$, by setting ϵ to a fixed nonzero value, relying on the computer to perform the cancellations, but K. Yokoya (private communication) showed this to be numerically unstable.

[24]This is equivalent to the statement that, in equilibrium, the coordinate and momentum offsets $\{\mathbf{r},\mathbf{p}\}$ have Gaussian distribution centered on the equilibrium closed orbit.

[25]The accelerator model consists of a tenfold symmetric lattice of magnets with a circumference of 200 m and an rms closed orbit distortion of 0.67 mm. When changing the accelerator energy, the magnetic fields are all increased in proportion to the energy so as to keep the same geometrical shape for all orbits. The orbital tunes are thereby kept constant.

[26]Results of calculation to higher order will be displayed elsewhere. A VAX 750 computer was used, which required approximately two minutes of cpu time per point. In Fig. 1 the energy was increased in steps of 1 MeV, hence there are 241 points per curve.

[27]We saw in Sec. III that the zeroth-order expression for $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ contains first-order resonances, hence such resonances appear even in a zeroth-order calculation of the polarization.