Electron-spin polarization in high-energy storage rings. I. Derivation of the equilibrium polarization

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(Received 10 December 1986)

A detailed exposition on the origin and buildup of polarization in high-energy electron storage rings is presented. Fundamental, but not clearly understood, theoretical results are rederived and clarified (Ya. S. Derbenev and A. M. Kondratenko, Zh. Eksp. Teor. Fiz. **64**, 1918 (1973) [Sov. Phys.—JETP **37**, 968 (1973)]). It is explained how to diagonalize the Hamiltonian of a storage ring, in particular the spin-dependent terms, to the first order in Planck's constant. Relevant perturbations, their time scales, and the various ensemble averages, are elucidated: the use of statistical concepts is shown to be essential to the calculation. Semiclassical techniques are used to derive, and extend to first order in $g-2$, the equilibrium degree of polarization (the Derbenev-Kondratenko formula). In so doing, some aspects of the polarization mechanism not previously recognized are uncovered. Because Derbenev and Kondratenko use a different mathematical approach, a proof is given of the equivalence between their formalism and the one used here.

I. INTRODUCTION

It was predicted many years ago that electrons and positrons in high-energy storage rings would become polarized by the emission of synchrotron radiation; this is now known as the Sokolov-Ternov effect.[1] Some time ago, Derbenev and Kondratenko[2] gave a detailed formula for the equilibrium degree of polarization. There has been much difficulty in understanding these classic papers, and in rederiving their results. This paper presents a detailed exposition on the origin and buildup of polarization in high-energy electron storage rings.[3] Schwinger's[4,5] semiclassical techniques are used to calculate the spin-flip synchrotron radiation power spectrum to rederive the Derbenev-Kondratenko formula. The formula is also extended to first order in $g-2$, and the previously uncertain consequences thereof are clarified. A number of points not evident in the work of the original authors are elucidated. In particular, an important point, which does not appear to have been generally appreciated, is that the polarization is a statistical mechanical phenomenon, and that the Derbenev-Kondratenko formula is fundamentally statistical in character. To keep this paper at a generally accessible level, I develop such accelerator physics as is needed in the text below. In a high-energy storage ring, the unperturbed particle motion can be satisfactorily described using classical mechanics. Further, a single-photon emission has relatively little effect on the orbital motion. For this reason, synchrotron radiation has generally been treated as a classical phenomenon. However, a single-photon emission can flip the spin of an electron completely, hence quantum mechanics must be used to describe the spin-dependent interactions. This leads us to a semiclassical description of the electron orbital and spin motion. This provides a fertile ground for the application of semiclassical quantum electrodynamics, statistical mechanics, and classical mechanics, and these techniques are employed to present a detailed exposition of high-energy electron spin polarization.

This paper is concerned only with the derivation of the formula for the equilibrium polarization. To be of practical value, however, one must also be able to evaluate the above formula for a given accelerator. The matter is discussed in the companion work to this paper.[6]

This introduction is followed by six sections. In Sec. II some general remarks are made about the polarization and the model used in this paper. The Hamiltonian is described in Sec. III, and is divided into unperturbed and interaction terms. The subtleties encountered in diagonalizing the unperturbed Hamiltonian are explained. A semiclassical approach, valid to the first order in $\hbar$, is used. In Sec. IV I consider the effect of perturbations, their time scales, and the various ensemble averages required. The use of statistical concepts is shown to be essential to the calculation. Section V contains the mathematical details of the calculation, some of which are relegated to two Appendixes. Since my mathematical approach does not follow that of Derbenev and Kondratenko, in Sec. VI, I present a proof of the equivalence of our formalisms. Section VII contains my conclusions.

II. GENERAL REMARKS

For spin-$\frac{1}{2}$ particles, the polarization density matrix is a 2×2 Hermitian matrix of unit trace, and can be specified completely by a real three-component vector

$$\mathbf{P}\equiv P\hat{\mathbf{P}}=\frac{N_\uparrow-N_\downarrow}{N_0}\hat{\mathbf{P}}\ , \qquad (1)$$

where $N_{\uparrow,\downarrow}$ denotes the number of electrons with spin projection $\pm\frac{1}{2}$ along the direction $\hat{\mathbf{P}}$, and $N_0=N_\uparrow+N_\downarrow$ is the total number of electrons which is constant. I shall calculate the equilibrium values of P and $\hat{\mathbf{P}}$, say P_{eq} and $\hat{\mathbf{P}}_{\text{eq}}$, respectively. If the beam is initially unpolarized, polarization builds up spontaneously along $\hat{\mathbf{P}}_{eq}$ according to

$$P(t)=P_{eq}[1-\exp(-t/\tau_{pol})] \ . \qquad (2)$$

The quantity τ_{pol} is called the polarization buildup time.[7] If $p_\downarrow$ and $p_\uparrow$ denote the probabilities per unit time for flipping spin from "up" to "down" along the direction $\hat{\mathbf{P}}_{eq}$ and vice versa, then in equilibrium one must have $p_\downarrow N_\uparrow = p_\uparrow N_\downarrow$, whence

$$P_{eq}=\frac{p_\uparrow - p_\downarrow}{p_\uparrow + p_\downarrow}, \quad \tau_{pol}=\frac{1}{p_\uparrow + p_\downarrow} \ . \qquad (3)$$

The above statements do not depend on the chosen axis of quantization, but the description, and calculation, of the equilibrium state of the ensemble is simplified by the use of certain preferred quantization axis. The axis I use is described below: its use simplifies the determination of the magnitude and direction of the equilibrium polarization, because it serves to diagonalize the unperturbed Hamiltonian of the system. The model I treat assumes that the individual electrons are independent, and that emission of distinct photons is uncorrelated. Hence a many-body Hamiltonian is unnecessary, and so a one-electron Hamiltonian will be used.

III. HAMILTONIAN AND UNPERTURBED MOTION

Let us consider a particle of mass m, charge e, spin $\mathbf{s}$, and $g \equiv 2(1+a)$, moving in prescribed external electromagnetic fields and interacting perturbatively with radiation fields. The Hamiltonian is $\mathcal{H}_{ext}+\mathcal{H}_{int}$, where

$$\mathcal{H}_{ext}=\left[\left[\mathbf{p}-\frac{e}{c}\mathbf{A}_{ext}\right]^2 c^2+m^2c^4\right]^{1/2} + e\Phi_{ext}+\mathbf{\Omega}_{ext}\cdot\mathbf{s} \ , \qquad (4a)$$

$$\mathcal{H}_{int}=e(\Phi_{rad}-\boldsymbol{\beta}\cdot\mathbf{A}_{rad})+\mathbf{\Omega}_{rad}\cdot\mathbf{s} \qquad (4b)$$

and

$$\mathbf{\Omega}_{ext}=-\frac{e}{mc}\left[\left[a+\frac{1}{\gamma}\right]\mathbf{B}_{ext}-\frac{a\gamma}{\gamma+1}\boldsymbol{\beta}\cdot\mathbf{B}_{ext}\boldsymbol{\beta} - \left[a+\frac{1}{\gamma+1}\right]\boldsymbol{\beta}\times\mathbf{E}_{ext}\right] \ , \qquad (5)$$

and $\mathbf{\Omega}_{rad}$ has the same form as $\mathbf{\Omega}_{ext}$, but with radiation fields in place of external fields.[8] The subscripts "ext" and "rad" denote external and radiation fields, $\mathbf{p}$ is the canonical momentum, $\boldsymbol{\beta}$ is the particle velocity in units of c, $\gamma=(1-|\boldsymbol{\beta}|^2)^{-1/2}$, and I shall also need $\mathbf{r}$, the laboratory-frame particle position. Throughout most of this paper, it will in fact be unnecessary to know the detailed form of the orbital part of $\mathcal{H}_{ext}$, and one can write simply $\mathcal{H}_{ext}=\mathcal{H}_{orb}(\mathbf{r},\mathbf{p})+\mathbf{\Omega}_{ext}\cdot\mathbf{s}$. The above variables will eventually be interpreted as quantum operators.

It will be useful below to consider alternative forms for $\mathbf{\Omega}_{ext}$ and $\mathbf{\Omega}_{rad}$ to elucidate various aspects of the polarization process. Specifically, dropping the subscripts ext, etc.,

$$\mathbf{\Omega}=-\frac{e}{mc}\left[\left[\frac{1}{\gamma+1}+a\right]\boldsymbol{\beta}\times(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}) - \frac{1+a}{\gamma}\mathbf{B}_v-\frac{1+a}{\gamma^2}\mathbf{B}_{tr}\right] \ , \qquad (6a)$$

$$\mathbf{\Omega}\cdot\mathbf{s}=-\frac{ge}{2mc\gamma}\mathbf{s}\cdot\mathbf{B}'+\boldsymbol{\omega}_T\cdot\mathbf{s}\equiv-\frac{1}{\gamma}\boldsymbol{\mu}\cdot\mathbf{B}'+\boldsymbol{\omega}_T\cdot\mathbf{s} \ . \qquad (6b)$$

The subscripts "v" and "tr" denote components parallel and transverse to $\boldsymbol{\beta}$, respectively; $\mathbf{B}'$ is the magnetic field in the particle rest frame, $\boldsymbol{\mu}\equiv ge\mathbf{s}/2mc$ is the magnetic moment, and

$$\boldsymbol{\omega}_T\equiv-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{dt}=-\frac{e}{mc}\frac{\gamma}{\gamma+1}\boldsymbol{\beta}\times(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}) \qquad (7)$$

is the Thomas precession vector. With an obvious notation, I shall write $\boldsymbol{\omega}_{T\text{ext}}$ and $\boldsymbol{\omega}_{T\text{rad}}$.

Let us now consider the solutions of the unperturbed equations of motion, i.e., the orbital and spin trajectories. The orbital motion in particle accelerators has been much studied elsewhere;[9] I discuss it at the end of this section. My principal interest is in the spin trajectories. These are obtained by solving the Thomas-Bargmann-Michel-Telegdi (BMT) equation[10,11] $d\mathbf{s}/dt=\mathbf{\Omega}_{ext}\times\mathbf{s}$. To begin with, I treat $\mathbf{s}$ as a classical vector obeying Poisson brackets, not as an operator. To diagonalize $\mathcal{H}_{ext}$, I seek to find a quantization axis the use of which yields stationary spin eigenstates in the quantum theory. Since $\mathbf{\Omega}_{ext}$ depends on the orbital trajectory, so does the spin motion, hence such an axis will not, in general, be the same on every orbital trajectory. For this reason, Derbenev and Kondratenko[2] introduced a variable vector $\hat{\mathbf{n}}(\mathbf{r},\mathbf{p})$ as the spin quantization axis. The vector $\hat{\mathbf{n}}$ associated with a given orbital trajectory is defined to be the explicitly time-independent solution of the Thomas-BMT equation on that trajectory;[12] states quantized along $\hat{\mathbf{n}}$ are stationary states of the Hamiltonian $\mathcal{H}_{ext}$. To see this, note that the Heisenberg equation of motion for the operator $\mathbf{s}\cdot\hat{\mathbf{n}}$ is

$$\frac{d}{dt}(\mathbf{s}\cdot\hat{\mathbf{n}})=\{\mathbf{s}\cdot\hat{\mathbf{n}},\mathcal{H}_{ext}\}+\frac{\partial}{\partial t}(\mathbf{s}\cdot\hat{\mathbf{n}}) \ , \qquad (8a)$$

or

$$\frac{d}{dt}(\mathbf{s}\cdot\hat{\mathbf{n}})=\frac{1}{i\hbar}[\mathbf{s}\cdot\hat{\mathbf{n}},\mathcal{H}_{ext}]+\frac{\partial}{\partial t}(\mathbf{s}\cdot\hat{\mathbf{n}}) \ , \qquad (8b)$$

where in Eq. (8a) the spin is treated classically and in Eq. (8b) it is treated quantum mechanically. Here $\{,\}$ denotes a Poisson bracket and $[,]$ denotes a commutator. By the properties of $\mathbf{s}$ and $\hat{\mathbf{n}}$, $\partial(\mathbf{s}\cdot\hat{\mathbf{n}})/\partial t=0$, and also, classically,

$$\frac{d}{dt}(\mathbf{s}\cdot\hat{\mathbf{n}})=(\mathbf{\Omega}_{ext}\times\mathbf{s})\cdot\hat{\mathbf{n}}+\mathbf{s}\cdot(\mathbf{\Omega}_{ext}\times\mathbf{s})=0 \ . \qquad (9)$$

Hence $\{\mathbf{s}\cdot\hat{\mathbf{n}},\mathcal{H}_{ext}\}$ vanishes, as required. Note that to diagonalize $\mathcal{H}_{ext}$, it is not sufficient that $\mathbf{s}\cdot\hat{\mathbf{n}}$ be a constant of the motion, it is also necessary that $\partial(\mathbf{s}\cdot\hat{\mathbf{n}})/\partial t=0$. For a given orbital trajectory, there are three linearly independent solutions of the Thomas-BMT equation, but not all are explicitly time independent. Yokoya has shown that

$\hat{\mathbf{n}}$ is unique, hence it is completely specified by the orbital trajectory.[12] Note that the above statements apply to *any* Hamiltonian; it is not necessary to assume that we are dealing with a particle accelerator.

In the quantum theory, with $\mathbf{\Omega}_{\rm ext}$ interpreted as an operator and $\mathbf{s}_{\rm op}$ denoting the electron-spin operator, the above criteria are satisfied only to the leading order in $\hbar$, in general. Let $\langle \mathbf{\Omega}_{\rm ext} \rangle$ denote the expectation value of $\mathbf{\Omega}_{\rm ext}$ over the orbital state of the electron; the classical analog of the operator $\mathbf{\Omega}_{\rm ext}$. Then

$$\frac{d}{dt}(\mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}})=(\mathbf{\Omega}_{\rm ext}\times\mathbf{s}_{\rm op})\cdot\hat{\mathbf{n}}+\mathbf{s}_{\rm op}\cdot(\langle \mathbf{\Omega}_{\rm ext}\rangle\times\hat{\mathbf{n}}) \ , \qquad (10)$$

which vanishes only to the leading order in $\hbar$. Consequently, so does the commutator of $\mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}$ with $\mathcal{H}_{\rm ext}$, now treated as an operator.[13]

In the special case when $\mathbf{\Omega}_{\rm ext}$ is constant, the solution for $\hat{\mathbf{n}}$ is trivial and familiar: it is just $\hat{\mathbf{n}}=\mathbf{\Omega}_{\rm ext}/|\mathbf{\Omega}_{\rm ext}|$, as can be verified by substitution into the above equations. This explains why the direction of the external field, which is usually constant in most models, is normally used as the spin quantization axis, but in general $\hat{\mathbf{n}}$ does not coincide with $\mathbf{\Omega}_{\rm ext}/|\mathbf{\Omega}_{\rm ext}|$. If the orbit is periodic every Nth turn around the ring, i.e., $\mathbf{r}(\theta+2\pi N)=\mathbf{r}(\theta)$ and $\mathbf{p}(\theta+2\pi N)=\mathbf{p}(\theta)$, where θ is the azimuth ($\theta\equiv 2\pi x/C$, where x is the arc length and C is the circumference of accelerator), then $\hat{\mathbf{n}}(\theta+2\pi N)=\hat{\mathbf{n}}(\theta)$. This is because $\hat{\mathbf{n}}=\hat{\mathbf{n}}(\mathbf{r},\mathbf{p})$ only, by definition. If the orbit is aperiodic, then so is $\hat{\mathbf{n}}$. Thus, for most orbits, $\hat{\mathbf{n}}$ is not periodic. See also Ref. 12. An algorithm for calculating $\hat{\mathbf{n}}$ is given in the companion work to this paper,[6] but is not needed here.

Let us now return to the orbital trajectories. The external fields in a storage ring consist mainly of static magnetic fields, which vary in both magnitude and direction. There are also accelerating electric fields in rf cavities to make up for the energy lost in synchrotron radiation. The orbital trajectories consist of oscillations around a central trajectory, called the equilibrium closed orbit. The equilibrium closed orbit is periodic around the ring, but in general the oscillations are not. Because the energy of an electron varies around the energy associated with the equilibrium closed orbit, and the path length of a trajectory around the machine depends on energy, the orbital oscillations have a longitudinal component. Modern storage rings have mechanisms for focusing both longitudinal and transverse oscillations. This constitutes the "phase space" of unperturbed orbital trajectories. It is six dimensional, hence the oscillations can be parameterized by three pairs of action-angle variables $\{I_\lambda,\psi_\lambda,\lambda=1,2,3\}$.

The fact that the energy of an electron is not constant, but oscillates around a central value, means that the system is not strictly conservative. This means that when we write $\hat{\mathbf{n}}=\hat{\mathbf{n}}(\mathbf{r},\mathbf{p})$ only, we must recognize that $\mathbf{r}$ and $\mathbf{p}$ contain energy and longitudinal oscillations, as well as transverse oscillations, and motion along the equilibrium closed orbit. The matter is largely formal. Details are given in Ref. 12.

IV. ORBITAL AND SPIN KINETICS

The effect of radiation (photon emission) is to cause a particle to make transitions between unperturbed trajectories. The concomitant energy loss is compensated in rf cavities. The net effect on the ensemble is to create an equilibrium statistical distribution of particles in the phase space of unperturbed orbital and spin trajectories. This distribution is constant in time.[14] The phases of the oscillations ψ_λ are uniformly distributed in the interval $[0,2\pi)$. The average values of the actions, say $\langle I_\lambda \rangle$, are known as the emittances, and they give a measure of the "beam size." The calculation of these emittances, at least in the approximation of linear orbital dynamics, is by now standard.[9]

The equilibrium in the orbital phase space is established principally by the emission of ordinary synchrotron radiation and energy gain in the rf cavities: this is essentially a classical spin-independent process,[15] with fluctuations around the classical synchrotron radiation spectrum due to the discrete nature of the emitted quanta. The time scale of this process is known as the radiation damping time, and is in the range of milliseconds to seconds in existing storage rings. The net effect on the orbital coordinates and momenta is to "shuffle" the particles around in the orbital phase space.

It will prove convenient in this section to use the azimuth θ, instead of time, as the independent variable ($\theta\equiv 2\pi x/C$). The equilibrium polarization vector is then given by

$$\mathbf{P}_{\rm eq}(\theta)=\langle\langle \mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}\rangle\hat{\mathbf{n}}\rangle_{I,\psi} \ . \qquad (11)$$

The inner angular brackets denote an average over spin projections in a unit "phase-space volume element" centered at $\{I_\lambda,\psi_\lambda\}$ and the outer ones an average over $\{I_\lambda,\psi_\lambda\}$ at azimuth θ.[16]

Let us now consider the details of the interaction with the radiation. We have seen that when a photon is emitted the electron will, due to its energy loss, make a transition to a different orbital trajectory. The new orbit is tangent to the old orbit, but with a new γ. Let the quantization axes of the initial and final trajectories, at the azimuth θ where the photon is emitted, be $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_f$, respectively. Then, if we consider a spin whose projection is initially along $\hat{\mathbf{n}}_i$, it may flip (to point along $-\hat{\mathbf{n}}_f$), or not (to $\hat{\mathbf{n}}_f$), and the relevant matrix elements are $\langle -\hat{\mathbf{n}}_f,\gamma\,|\,\mathcal{H}_{\rm int}\,|\,\hat{\mathbf{n}}_i,0\rangle$ and $\langle \hat{\mathbf{n}}_f,\gamma\,|\,\mathcal{H}_{\rm int}\,|\,\hat{\mathbf{n}}_i,0\rangle$, respectively, where I have indicated that the final state contains a photon but the initial state does not. I shall often omit explicit mention of the photon where this does not lead to confusion. Since ordinary synchrotron radiation is almost spin independent, the vast majority of transitions are nonflip: the time scale of polarization buildup is typically tens to hundreds of minutes, as compared to approximately 10 msec for the radiation damping time. This means that the equilibrium value of $\langle \mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}\rangle$ in a unit phase-space volume element centered on $\{I_\lambda,\psi_\lambda\}$ at θ is in fact independent of I_λ, ψ_λ, and θ and is given by the average $\langle \mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}\rangle_{I,\psi,\theta}$: the relevant time scales are such that the nonflip processes make it uniform throughout the orbital phase space as they shuffle the orbital coordinates and momenta; the spins simply follow without being flipped.[17]

The above remarks show that, to a good approximation, the expression for the equilibrium polarization vector factorizes:

$$\mathbf{P}_{\rm eq} \simeq \langle \mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}\rangle_{I,\psi,\theta}\langle\hat{\mathbf{n}}\rangle_{I,\psi}$$
$$\equiv \langle \mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}\rangle_{I,\psi,\theta}\langle\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_{\rm eq}\rangle_{I,\psi}\hat{\mathbf{n}}_{\rm eq}(\theta) , \quad (12)$$

where $\hat{\mathbf{n}}_{\rm eq}(\theta)$ is a unit vector in the direction of $\langle\hat{\mathbf{n}}\rangle_{I,\psi}$. The quantity $\langle \mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}\rangle_{I,\psi,\theta}$ is determined by spin-flip interactions, while $\langle\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_{\rm eq}\rangle_{I,\psi}$ is determined solely by the orbital ensemble average. We may call $\langle \mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}\rangle_{I,\psi,\theta}\langle\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_{\rm eq}\rangle_{I,\psi}$ the equilibrium degree of polarization and $\hat{\mathbf{n}}_{\rm eq}$ the direction of the equilibrium polarization. In practice, the variation of $\hat{\mathbf{n}}$ with respect to $\{I_\lambda,\psi_\lambda\}$ is small except near so-called "spin resonances," and so $\langle\hat{\mathbf{n}},\hat{\mathbf{n}}_{\rm eq}\rangle_{I,\psi}\simeq 1$, and may be neglected.[18] We may then write $P_{\rm eq}\simeq\langle \mathbf{s}_{\rm op}\cdot\hat{\mathbf{n}}\rangle_{I,\psi,\theta}$ and $\hat{\mathbf{P}}_{\rm eq}=\hat{\mathbf{n}}_{\rm eq}$. This approximation is also made by Derbenev and Kondratenko.[19]

Returning to the spin-flip matrix elements, it is the second term in $\mathcal{H}_{\rm int}$, viz, $\mathbf{\Omega}_{\rm rad}\cdot\mathbf{s}_{\rm op}$, that is generally regarded as the source of spin-flip synchrotron radiation. Notice, however, that because $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_f$ are not necessarily parallel, even the spin-independent term $e(\Phi_{\rm rad}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm rad})$ can couple $|\hat{\mathbf{n}}_i\rangle$ and $|-\hat{\mathbf{n}}_f\rangle$, and this term is comparable in practice to that from $\mathbf{\Omega}_{\rm rad}\cdot\mathbf{s}_{\rm op}$. This is a mechanism not hitherto recognized *per se.* If the photon energy is $\hbar\omega$, and the electron energy is E, and we define $\Delta\gamma=-\hbar\omega/mc^2$, then it is convenient to write

$$\hat{\mathbf{n}}_f\simeq\hat{\mathbf{n}}_i+\Delta\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}=\hat{\mathbf{n}}_i-\frac{\hbar\omega}{E}\left[\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right] , \quad (13)$$

where the derivative $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ is a measure of the extent to which $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_f$ are not parallel [I have omitted the subscript i on $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$].[20] At a spin resonance $|\gamma(\partial\hat{\mathbf{n}}\,|\,\partial\gamma)|\rightarrow\infty$ (the present theory is then inapplicable), but in a nonresonant situation $|\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)|=O(1)$. This represents the "new" spin-flip mechanism that leads to terms in $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ in the Derbenev-Kondratenko formula. In the subsequent sections I shall study these spin-flip matrix elements in detail.

V. EQUILIBRIUM DEGREE OF POLARIZATION

To obtain $p_\uparrow$ and $p_\downarrow$, the spin-flip probabilities per unit time introduced in Sec. II, we must now integrate the relevant matrix elements. In the model used, spin-flip transitions are caused solely by photon emissions, not by the unperturbed motion, and the emission of distinct photons is uncorrelated, hence $p_\uparrow$ and $p_\downarrow$ are proportional to the number of photons which flip spin in the appropriate direction emitted per unit time; this, in turn, is related to the corresponding power spectra via

$$p_{\uparrow,\downarrow}\propto\left\langle\int\frac{d\omega}{\hbar\omega}\frac{d\mathcal{P}_{\uparrow,\downarrow}}{d\omega}\right\rangle , \quad (14)$$

where ω is the frequency of the photon and the angular brackets denote an average over the orbital actions and angles and accelerator azimuth. To obtain the power spectra, I follow Schwinger[4] and write

$$\frac{d\mathcal{P}}{d\omega}=\frac{\omega^2}{4\pi^2c}\,{\rm Re}\left\{\int\left[\frac{1}{c^2}\mathbf{j}(\mathbf{r}(t),t)\cdot\mathbf{j}^*(\mathbf{r}(t'),t')-\rho(\mathbf{r}(t),t)\rho^*(\mathbf{r}(t'),t')\right]\exp\left[-i\omega\left[t'-t-\frac{\hat{\mathbf{k}}}{c}\cdot[\mathbf{r}(t')-\mathbf{r}(t)]\right]\right]dt'd\Omega_{\hat{\mathbf{k}}}\right\} , \quad (15)$$

where $\hat{\mathbf{k}}$ is the direction of photon propagation. The polarization states of the photon have already been summed over in the derivation of the above formula. I get the charge density ρ and the current density $\mathbf{j}$ from the spin-flip matrix elements of $\mathcal{H}_{\rm int}$. To do so, I substitute for $\mathbf{E}_{\rm rad}$ and $\mathbf{B}_{\rm rad}$ in terms of $\Phi_{\rm rad}$ and $\mathbf{A}_{\rm rad}$ in $\mathcal{H}_{\rm int}$ by writing

$$\Phi_{\rm em},\ \mathbf{A}_{\rm em}\propto(e^{i\omega(\hat{\mathbf{k}}\cdot\mathbf{r}/c-t)})^*=e^{i\omega(t-\hat{\mathbf{k}}\cdot\mathbf{r}/c)} , \quad (16)$$

where the subscript "em" means "emission," whence

$$\mathbf{E}_{\rm em}=-\frac{i\omega}{c}(\mathbf{A}_{\rm em}-\hat{\mathbf{k}}\Phi_{\rm em}),\quad \mathbf{B}_{\rm em}=-\frac{i\omega}{c}\hat{\mathbf{k}}\times\mathbf{A}_{\rm em} . \quad (17)$$

I then read off the charge and current density operators which couple to $\Phi_{\rm em}$ and $\mathbf{A}_{\rm em}$, say $\rho_{\rm em}$ and $\mathbf{j}_{\rm em}$, via

$$\mathcal{H}_{\rm int}=\frac{1}{c}j_\mu A^\mu$$
$$=\rho_{\rm em}\Phi_{\rm em}-\frac{1}{c}\mathbf{j}_{\rm em}\cdot\mathbf{A}_{\rm em}+\text{(absorption terms)} . \quad (18)$$

The quantities ρ and $\mathbf{j}$ that appear in Eq. (15) are the appropriate matrix elements of $\rho_{\rm em}$ and $\mathbf{j}_{\rm em}$, respectively. Let us calculate the power spectrum for spin flip from $\hat{\mathbf{n}}_i$ to $-\hat{\mathbf{n}}_f$, so that $\rho=\langle -\hat{\mathbf{n}}_f\,|\,\rho_{\rm em}\,|\,\hat{\mathbf{n}}_i\rangle$, etc. For brevity, it is convenient to drop the subscript i in $\hat{\mathbf{n}}_i$, and to introduce two vectors $\hat{\boldsymbol{\eta}}_1$ and $\hat{\boldsymbol{\eta}}_2$, and $\boldsymbol{\eta}\equiv\hat{\boldsymbol{\eta}}_1+i\hat{\boldsymbol{\eta}}_2$, which are solutions of the Thomas-BMT equation such that $\{\hat{\mathbf{n}},\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2\}$ form a right-handed orthonormal triad. We see that $\boldsymbol{\eta}=(2/\hbar)\langle -\hat{\mathbf{n}}\,|\,\mathbf{s}_{\rm op}\,|\,\hat{\mathbf{n}}\rangle$, which also conforms to the Derbenev-Kondratenko normalization,[21] which will be discussed in Sec. VI. Then, to the leading order in $\hbar$, [define $\mathbf{D}\equiv(\hbar\omega/E)\gamma(\partial\hat{\mathbf{n}}\,|\,\partial\gamma)$]

$$\begin{aligned}\langle -\hat{\mathbf{n}}_f,\gamma \mid \mathcal{H}_{\rm int} \mid \hat{\mathbf{n}}_i,0\rangle &\simeq \langle -\hat{\mathbf{n}}+\mathbf{D},\gamma \mid \mathcal{H}_{\rm int} \mid \hat{\mathbf{n}},0\rangle \\ &\simeq \langle -\hat{\mathbf{n}},\gamma \mid e^{-i(\hat{\mathbf{n}}\times\mathbf{D})\cdot\mathbf{s}_{\rm op}/\hbar}\mathcal{H}_{\rm int} \mid \hat{\mathbf{n}},0\rangle \\ &\simeq \left\langle -\hat{\mathbf{n}},\gamma \left| \left[1-\frac{i}{\hbar}(\hat{\mathbf{n}}\times\mathbf{D})\cdot\mathbf{s}_{\rm op}\right]\mathcal{H}_{\rm int} \right| \hat{\mathbf{n}},0\right\rangle \\ &\simeq \left\langle -\hat{\mathbf{n}},\gamma \left| \left[-\frac{ie}{\hbar}(\hat{\mathbf{n}}\times\mathbf{D})\cdot\mathbf{s}_{\rm op}(\Phi_{\rm em}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm em})+\boldsymbol{\Omega}_{\rm em}\cdot\mathbf{s}_{\rm op}\right] \right| \hat{\mathbf{n}},0\right\rangle \\ &\simeq -\frac{ie\hbar\omega}{2\gamma mc^2}\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\cdot\boldsymbol{\eta}(\Phi_{\rm em}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm em})+\frac{\hbar}{2}\boldsymbol{\Omega}_{\rm em}\cdot\boldsymbol{\eta}\ ,\end{aligned} \tag{19}$$

where $\boldsymbol{\Omega}_{\rm em}$ is the part of $\boldsymbol{\Omega}_{\rm rad}$ containing emission terms. In the last line, $\Phi_{\rm em}$, $\mathbf{A}_{\rm em}$, and $\boldsymbol{\Omega}_{\rm em}$ are to be interpreted as c numbers, not operators. We see that, by treating the equations of motion of the various operators at the classical level, the leading terms in the matrix element are of $O(\hbar)$, and the power spectrum is therefore of $O(\hbar^2)$. This is to be compared with $O(1)$ for ordinary synchrotron radiation: we see that spin-flip photon emission is a highly suppressed process, which explains the great difference in time scales of the spin flip and nonflip processes. Using Eqs. (17), (18), and (19),

$$\rho\simeq\frac{ie\hbar\omega}{2mc^2}\left[-\frac{1}{\gamma}\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\cdot\boldsymbol{\eta} + \left[a+\frac{1}{\gamma+1}\right](\boldsymbol{\eta}\times\boldsymbol{\beta})\cdot\hat{\mathbf{k}}\right]\ , \tag{20a}$$

$$\frac{1}{c}\mathbf{j}\simeq\frac{ie\hbar\omega}{2mc^2}\left[-\frac{1}{\gamma}\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\cdot\boldsymbol{\eta}\boldsymbol{\beta}-\left[a+\frac{1}{\gamma}\right]\boldsymbol{\eta}\times\hat{\mathbf{k}} + \frac{a\gamma}{\gamma+1}\boldsymbol{\eta}\cdot\boldsymbol{\beta}\boldsymbol{\beta}\times\hat{\mathbf{k}}+\left[a+\frac{1}{\gamma+1}\right]\boldsymbol{\eta}\times\boldsymbol{\beta}\right]\ . \tag{20b}$$

To calculate the power spectrum Eq. (15), I integrate over solid angles $\Omega_{\hat{k}}$ (direction of photon emission) first, followed by the integral over t'. The approximations made are the usual ones in the field,[4] and the integrals encountered are the same (they are listed in Appendix B). For locally circular motion, the radius of curvature of the trajectory at azimuth θ is $c\beta^2/|\dot{\boldsymbol{\beta}}|\simeq c\,|\dot{\boldsymbol{\beta}}|^{-1}$. Let us also write, for brevity, $\omega_0=|\dot{\boldsymbol{\beta}}|/\beta$, $\omega_c=3\gamma^3\omega_0/2$, $\xi=\omega/\omega_c$, and denote the direction of the local magnetic field $\mathbf{B}_{\rm ext}$ by $\hat{\mathbf{b}}$. I assume, as is standard, that $\boldsymbol{\beta}$ is perpendicular to $\hat{\mathbf{b}}$ and $|\boldsymbol{\beta}|$ is constant, whence $\hat{\mathbf{b}}=\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}/\beta\,|\dot{\boldsymbol{\beta}}|$. For brevity, I define

$$\begin{aligned}&\mathbf{r}'=\mathbf{r}(t'),\quad \boldsymbol{\beta}'=\boldsymbol{\beta}(t')\ ,\\ &\mathbf{j}'=\mathbf{j}^*(\mathbf{r}',t')\ ,\quad \rho'=\rho^*(\mathbf{r}',t')\ ,\\ &j^\mu=(c\rho,\mathbf{j}),\quad j'^\mu=(c\rho',\mathbf{j}')\ ,\\ &\frac{1}{c^2}j_\mu j'^\mu=\rho\rho'-\frac{1}{c^2}\mathbf{j}\cdot\mathbf{j}',\\ &\mathbf{s}'=\mathbf{s}^*(\mathbf{r}',t'),\quad \tau=t'-t\ ,\\ &\zeta=\frac{\omega}{c}(\mathbf{r}'-\mathbf{r})\equiv\zeta\hat{\boldsymbol{\zeta}},\quad \chi=\hat{\mathbf{k}}\cdot\hat{\boldsymbol{\zeta}}\ .\end{aligned} \tag{21}$$

Then Eq. (15) can be rewritten as

$$\frac{d\mathcal{P}}{d\omega}=-\frac{\omega^2}{4\pi^2c}\int\left[\operatorname{Re}\left[\frac{1}{c^2}j_\mu j'^\mu\right]\cos(\omega\tau-\chi\zeta) + \operatorname{Im}\left[\frac{1}{c^2}j_\mu j'^\mu\right]\sin(\omega\tau-\chi\zeta)\right]d\tau\,d\Omega_{\hat{k}}. \tag{22}$$

Because I have summed over the photon polarizations, the integrand is independent of azimuth around $\hat{\mathbf{k}}$, so I take the polar axis of the integral over $\Omega_{\hat{k}}$ along $\hat{\mathbf{k}}$, in which case $d\Omega_{\hat{k}}=2\pi d\chi$. Then, from the form of $j_\mu j'^\mu$, I find three classes of angular integrals, viz.,

$$\int f(\tau,\chi)d\chi,\quad \int \hat{\mathbf{k}}_i f(\tau,\chi)d\chi,\quad \int \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j f(\tau,\chi)d\chi\ , \tag{23}$$

which I shall label scalar, vector, and tensor integrals, respectively. The subscripts i and j refer to the components of $\hat{\mathbf{k}}$. The quantity $f(\tau,\chi)$ denotes a function invariant under rotation around $\hat{\mathbf{k}}$. It is readily verified that

$$\int \hat{\mathbf{k}}_i f(\tau,\chi)d\chi=\hat{\xi}_i\int \chi f(\tau,\chi)d\chi\ , \tag{24}$$

$$\int \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j f(\tau,\chi)d\chi=A\delta_{ij}+B\hat{\xi}_i\hat{\xi}_j\quad \begin{cases}A=\frac{1}{2}\left[\int d\chi\, f(\tau,\chi)-\int d\chi\,\chi^2 f(\tau,\chi)\right]\\ B=\frac{1}{2}\left[3\int d\chi\,\chi^2 f(\tau,\chi)-\int d\chi\, f(\tau,\chi)\right]\ .\end{cases} \tag{25}$$

These results are exact; they do not depend on the angular distribution of the radiated power, because they assume only that the integrand is azimuthally symmetric about $\hat{\mathbf{k}}$. To proceed further, I need the detailed form of $f(\tau,\chi)$. At this point it becomes easier to expand $j_\mu j'^\mu$ in powers of a. I shall justify this step below. In that case, we find $f(\tau,\chi)$ is of the form

$$f(\tau,X)=\begin{cases}(\omega_0\tau)^q\cos(\omega\tau-X\xi) & q=0,2,\ldots\\ (\omega_0\tau)^q\sin(\omega\tau-X\xi) & q=1,3,\ldots\ .\end{cases} \tag{26}$$

For the leading term, which is independent of a, we need only $q=0,1,2$ to the degree of approximation of interest.

The above expressions for ρ and $\mathbf{j}$ were used to derive expressions for the spin-flip power spectra, thence the equilibrium degree of polarization.[3] No assumptions were made about the angular distribution of the radiated power: this makes the algebra lengthy, and not very illuminating; I shall present a simpler derivation below, after the salient features of the spin-flip matrix elements have been elucidated, thereby simplifying the expressions for ρ and $\mathbf{j}$. To anticipate these simplifications, the final results are quoted here. The expression for the power spectrum is, to first order in $a=(g-2)/2$,

$$\begin{aligned}\frac{d\mathcal{P}_\downarrow}{d\xi}=\frac{27\sqrt{3}}{32\pi}\frac{e^2\hbar^2\gamma^8\omega_0^4}{m^2c^5}\xi^3\Bigg\{&\left[K_{2/3}(\xi)+\frac{a}{2}\left[9K_{2/3}(\xi)-\int_\xi^\infty K_{1/3}(\xi')d\xi'\right]\right]\\ &+(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2\left[\int_\xi^\infty K_{1/3}(\xi')d\xi'-K_{2/3}(\xi)+\frac{a}{2}\left[3\int_\xi^\infty K_{1/3}(\xi')d\xi'-11K_{2/3}(\xi)\right]\right]\\ &+\hat{\mathbf{n}}\cdot\hat{\mathbf{b}}\left[K_{1/3}(\xi)+a\left[4K_{1/3}(\xi)+\frac{4}{3\xi}\int_\xi^\infty K_{1/3}(\xi')d\xi'\right]\right]\\ &+\frac{1}{2}\left|\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right|^2\left[2K_{2/3}(\xi)-\int_\xi^\infty K_{1/3}(\xi')d\xi'\right]-(1+a)\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{b}}K_{1/3}(\xi)\Bigg\}\ ,\end{aligned} \tag{27}$$

where $\hat{\mathbf{v}}$ is a unit vector in the direction of the particle velocity $\hat{\mathbf{b}}\equiv\mathbf{v}\times\dot{\mathbf{v}}/|\mathbf{v}\times\dot{\mathbf{v}}|$, and the $K(\xi)$ are modified Bessel functions. The power spectrum $d\mathcal{P}_\uparrow/d\xi$, for spin flip from $-\hat{\mathbf{n}}_i$ to $\hat{\mathbf{n}}_f$, is obtained by replacing $\hat{\mathbf{n}}$ by $-\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ by $-\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. From Sec. II and using Eq. (14), the equilibrium degree of polarization P_{eq} is obtained via

$$P_{\text{eq}}=\frac{\left\langle\int\frac{d\xi}{\xi}\left[\frac{d\mathcal{P}_\uparrow}{d\xi}-\frac{d\mathcal{P}_\downarrow}{d\xi}\right]\right\rangle}{\left\langle\int\frac{d\xi}{\xi}\left[\frac{d\mathcal{P}_\uparrow}{d\xi}+\frac{d\mathcal{P}_\downarrow}{d\xi}\right]\right\rangle}\ . \tag{28}$$

The angular brackets denote an average over the orbital actions and angles and accelerator azimuth. Performing the integral over ξ, I find

$$P_{\text{eq}}=-\frac{8}{5\sqrt{3}}\frac{\left\langle|\dot{\mathbf{v}}|^3\hat{\mathbf{b}}\cdot\left[(1+\frac{14}{3}a)\hat{\mathbf{n}}-(1+a)\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\right\rangle}{\left\langle|\dot{\mathbf{v}}|^3\left[1+\frac{37}{9}a-(\frac{2}{9}+\frac{13}{3}a)(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2+\frac{11}{18}\left|\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right|^2\right]\right\rangle}\ , \tag{29}$$

and

$$\tau_{\text{pol}}^{-1}=\frac{5\sqrt{3}}{8}\frac{e^2\hbar\gamma^5}{m^2c^8}\left\langle|\dot{\mathbf{v}}|^3\left[1+\frac{37}{9}a-(\frac{2}{9}+\frac{13}{3}a)(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2+\frac{11}{18}\left|\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right|^2\right]\right\rangle\ . \tag{30}$$

Derbenev and Kondratenko obtained the above results for $a=0$. Since $a\ll1$ for electrons, the a terms in Eqs. (29) and (30) make very little difference. This may appear surprising, since at the high energies of interest to us, $a\gg\gamma^{-1}$, hence it dominates the coefficients of the spin-dependent terms in $\mathcal{H}_{\text{ext}}$ and $\mathcal{H}_{\text{int}}$, yet it has a strong effect only in $\mathcal{H}_{\text{ext}}$, but not in $\mathcal{H}_{\text{int}}$. It is therefore clear that subtle cancellations are taking place between the various terms in $\mathcal{H}_{\text{int}}$, which shall now be elucidated. Note, from Sec. III, that $\mathcal{H}_{\text{int}}$ can be written in the form

$$\mathcal{H}_{\text{int}}=e(\Phi-\boldsymbol{\beta}\cdot\mathbf{A})-\frac{e}{mc}\mathbf{s}_{\text{op}}\cdot\left[\left[\frac{1}{\gamma+1}+a\right]\boldsymbol{\beta}\times(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B})-\frac{1+a}{\gamma}\mathbf{B}_v-\frac{1+a}{\gamma^2}\mathbf{B}_{\text{tr}}\right]\ , \tag{31}$$

where radiation field operators are to be understood throughout. The subscripts v and tr denote components parallel and transverse to $\boldsymbol{\beta}$, respectively, i.e., $\mathbf{B}_v\equiv\boldsymbol{\beta}\cdot\mathbf{B}\boldsymbol{\beta}/\beta^2$ and $\mathbf{B}_{\text{tr}}\equiv\mathbf{B}-\mathbf{B}_v$. For a plane wave $\{\hat{\mathbf{k}},\mathbf{E}_{\text{em}},\mathbf{B}_{\text{em}}\}$ form a right-handed orthogonal triad of vectors, whence

$$\hat{\mathbf{k}}\times\mathbf{E}_{\text{em}}=\mathbf{B}_{\text{em}},\quad \hat{\mathbf{k}}\times\mathbf{B}_{\text{em}}=-\mathbf{E}_{\text{em}}\ . \tag{32}$$

Since the photon is emitted almost parallel to $\boldsymbol{\beta}$, the angle between $\boldsymbol{\beta}$ and $\hat{\mathbf{k}}$ being of $O(\gamma^{-1})$, and $|\boldsymbol{\beta}|\simeq1$,

$$\boldsymbol{\beta}\times(\mathbf{E}_{\rm em}+\boldsymbol{\beta}\times\mathbf{B}_{\rm em})=O(\gamma^{-2}),\quad \boldsymbol{\beta}\cdot\mathbf{B}_{\rm em}=O(\gamma^{-1}),\quad \boldsymbol{\beta}\cdot\mathbf{E}_{\rm em}=O(\gamma^{-1})\ . \tag{33}$$

Hence

$$\frac{1}{\gamma+1}\boldsymbol{\beta}\times(\mathbf{E}_{\rm em}+\boldsymbol{\beta}\times(\mathbf{B}_{\rm em})=O(\gamma^{-3})\ , \tag{34a}$$

$$\frac{1}{\gamma}\mathbf{B}_v=O(\gamma^{-2})\ , \tag{34b}$$

$$\frac{1}{\gamma^2}\mathbf{B}_{\rm tr}=\frac{1}{\gamma^2}(\mathbf{B}-\mathbf{B}_v)=O(\gamma^{-2})\ , \tag{34c}$$

and so the leading contribution of the a-independent explicitly spin-dependent terms in $\mathcal{H}_{\rm int}$ is $-(e/mc)\mathbf{s}_{\rm op}\cdot(\mathbf{B}_v/\gamma+\mathbf{B}_{\rm tr}/\gamma^2)$, which is of $O(\gamma^{-2})$. As for the terms in a, they are of $O(a\gamma^{-2})$, firstly because

$$a\left[\boldsymbol{\beta}\times(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B})-\frac{1}{\gamma}\mathbf{B}_v-\frac{1}{\gamma^2}\mathbf{B}_{\rm tr}\right]=O(a\gamma^{-2})\ , \tag{35}$$

which is of $O(a)$ relative to the a-independent terms. Furthermore, the time evolution of $\boldsymbol{\eta}\propto\langle -\hat{\mathbf{n}}\,|\,\mathbf{s}_{\rm op}\,|\,\hat{\mathbf{n}}\rangle$ is $d\boldsymbol{\eta}/dt=\boldsymbol{\Omega}_{\rm ext}\times\boldsymbol{\eta}$, where $\boldsymbol{\Omega}_{\rm ext}=(1+\gamma a)\omega_0\hat{\mathbf{b}}$, in a locally uniform magnetic field, which leads to terms proportional to $(a\gamma\omega_0\tau)^q$, $q=0,1,2,\ldots$ in the integral Eq. (15). It is shown in Ref. 4 that the dimensionless variable of integration is $\gamma\omega_0\tau$, hence such integrands contribute terms of $O(a^q)$, $q=1,2,\ldots$ relative to the a-independent result. It remains to determine the magnitude of the terms in $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. To do so, recall that

$$\mathbf{E}_{\rm em}=-\frac{i\omega}{c}(\mathbf{A}_{\rm em}-\hat{\mathbf{k}}\Phi_{\rm em})\ , \tag{36}$$

whence

$$\boldsymbol{\beta}\cdot\mathbf{E}_{\rm em}=\frac{i\omega}{c}(\boldsymbol{\beta}\cdot\hat{\mathbf{k}}\Phi_{\rm em}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm em})\simeq\frac{i\omega}{c}(\Phi_{\rm em}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm em})\ , \tag{37}$$

and so

$$-\frac{ie\hbar\omega}{2\gamma mc^2}\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\cdot\boldsymbol{\eta}(\Phi_{\rm em}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm em})\simeq-\frac{e\hbar}{2\gamma mc}\boldsymbol{\beta}\cdot\mathbf{E}_{\rm em}\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\cdot\boldsymbol{\eta}=O(\gamma^{-2})\ , \tag{38}$$

using Eq. (33). This term is thus of the same magnitude as the other a-independent terms in the matrix element.

The weak explicit dependence of the equilibrium degree of polarization on the value of $a=(g-2)/2$ is therefore fundamentally because the photon is emitted almost parallel to $\boldsymbol{\beta}$ and $|\boldsymbol{\beta}|\simeq 1$. Clearly, cancellations such as the above do *not* occur between the terms in the external fields $\mathbf{E}_{\rm ext}$ and $\mathbf{B}_{\rm ext}$ in $\mathcal{H}_{\rm ext}$, hence the fact that $a\gg\gamma^{-1}$ *does* have a pronounced effect on the unperturbed equation of spin motion, thence on $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, and thus a strong implicit effect on the equilibrium degree of polarization.

I now present the mathematical details of the integration of the spin-flip matrix elements. The calculation is displayed for $a=0$ only, the result for $a\neq 0$ merely requiring more labor. The simplified matrix element is

$$\langle -\hat{\mathbf{n}}_f,\gamma\,|\,\mathcal{H}_{\rm int}\,|\,\hat{\mathbf{n}}_i,0\rangle\simeq-\frac{e\hbar}{2\gamma mc}\left[\frac{i\omega}{c}\boldsymbol{\eta}\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right](\Phi_{\rm em}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm em})+\boldsymbol{\eta}\cdot\boldsymbol{\beta}\boldsymbol{\beta}\cdot\mathbf{B}_{\rm em}+\frac{1}{\gamma}\boldsymbol{\eta}\cdot\mathbf{B}_{\rm em}\right]\ , \tag{39}$$

whence

$$\rho_s=-\frac{ie\hbar\omega}{2\gamma mc^2}\boldsymbol{\eta}\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\ , \tag{40a}$$

$$\frac{1}{c}\mathbf{j}_s=-\frac{ie\hbar\omega}{2\gamma mc^2}\left[\frac{1}{\gamma}\boldsymbol{\eta}\times\hat{\mathbf{k}}+\boldsymbol{\eta}\cdot\boldsymbol{\beta}\boldsymbol{\beta}\times\hat{\mathbf{k}}+\boldsymbol{\eta}\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\boldsymbol{\beta}\right]\ , \tag{40b}$$

where the subscript "s" stands for "simplified." The notation otherwise follows that used in Eq. (20). Then

$$\frac{1}{c^2}\mathbf{j}_s\cdot\mathbf{j}_s'-\rho_s\rho_s'=\frac{e^2\hbar^2\omega^2}{4m^2c^4}\left\{\frac{1}{\gamma^4}(\boldsymbol{\eta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\eta}'\times\hat{\mathbf{k}})+\frac{1}{\gamma^2}\boldsymbol{\eta}\cdot\boldsymbol{\beta}\boldsymbol{\eta}'\cdot\boldsymbol{\beta}'(\boldsymbol{\beta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\beta}'\times\hat{\mathbf{k}})\right.$$
$$+\frac{1}{\gamma^3}[(\boldsymbol{\eta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\beta}'\times\hat{\mathbf{k}})\boldsymbol{\eta}'\cdot\boldsymbol{\beta}+(\boldsymbol{\beta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\eta}'\times\hat{\mathbf{k}})\boldsymbol{\eta}\cdot\boldsymbol{\beta}]$$
$$+\frac{1}{\gamma^2}\boldsymbol{\eta}\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\boldsymbol{\eta}'\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right](\boldsymbol{\beta}\cdot\boldsymbol{\beta}'-1)$$
$$+\frac{1}{\gamma^3}\left[\boldsymbol{\eta}\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right](\boldsymbol{\eta}'\times\hat{\mathbf{k}})\cdot\boldsymbol{\beta}+\boldsymbol{\eta}'\cdot(\hat{\mathbf{n}}\times\hat{\mathbf{k}})\cdot\boldsymbol{\beta}'\right]$$
$$\left.+\frac{1}{\gamma^2}\left[\boldsymbol{\eta}\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\boldsymbol{\eta}'\cdot\boldsymbol{\beta}'(\boldsymbol{\beta}'\times\hat{\mathbf{k}})\cdot\boldsymbol{\beta}+\boldsymbol{\eta}'\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\boldsymbol{\eta}\cdot\boldsymbol{\beta}(\boldsymbol{\beta}\times\hat{\mathbf{k}})\cdot\boldsymbol{\beta}'\right]\right\}. \quad (41)$$

To evaluate the above expression, I follow Jackson[8] and introduce a right-handed basis $\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$, where $\hat{\mathbf{x}}=\hat{\boldsymbol{\beta}}$, $\hat{\mathbf{y}}=\dot{\boldsymbol{\beta}}/|\dot{\boldsymbol{\beta}}|$, and $\hat{\mathbf{z}}=\hat{\mathbf{b}}$ at time t ($\tau=0$). Expressions for the above vectors and dot products, as functions of τ, are given in Appendix A. The following simplifications can now be made in Eq. (41).

(i) In the term $(\boldsymbol{\eta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\eta}'\times\hat{\mathbf{k}})/\gamma^4$, the overall coefficient is already γ^{-4}, so it is permissible to set $\hat{\mathbf{k}}\simeq\boldsymbol{\beta}$. Also, since each power of $\omega_0\tau$ costs a factor γ^{-1}, the variation of $\boldsymbol{\eta}'$ with τ can be neglected.[22] Hence this term simplifies to

$$(\boldsymbol{\eta}\times\boldsymbol{\beta})\cdot(\boldsymbol{\eta}^*\times\boldsymbol{\beta})/\gamma^4\simeq(|\boldsymbol{\eta}|^2-|\eta_x|^2)/\gamma^4=[1+(\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2]/\gamma^4 .$$

(ii) In the term $\boldsymbol{\eta}\cdot\boldsymbol{\beta}\boldsymbol{\eta}'\cdot\boldsymbol{\beta}'(\boldsymbol{\beta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\beta}'\times\hat{\mathbf{k}})/\gamma^2$, it is not permissible to put $\hat{\mathbf{k}}\simeq\boldsymbol{\beta}$, since $|\boldsymbol{\beta}\times\hat{\mathbf{k}}|=O(\gamma^{-1})$ and I wish to keep terms up to $O(\gamma^{-4})$. However, note that $\boldsymbol{\eta}\cdot\boldsymbol{\beta}=\eta_x\beta$ and, for $a=0$, $\boldsymbol{\eta}'\cdot\boldsymbol{\beta}'=\eta_x^*\beta$ (see Appendix A), which are both constant, and so we need to integrate only $(\boldsymbol{\beta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\beta}'\times\hat{\mathbf{k}})$. Since this is simply the integrand of the classical synchrotron radiation power spectrum,[5] the contribution of this term to $d\mathcal{P}/d\xi$, to $O(\gamma^{-4})$, is just the classical synchrotron radiation power spectrum multiplied by $|\eta_x|^2/\gamma^2=[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2]/\gamma^2$.

(iii) As for

$$[(\boldsymbol{\eta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\beta}'\times\hat{\mathbf{k}})\boldsymbol{\eta}'\cdot\boldsymbol{\beta}+(\boldsymbol{\beta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\eta}'\times\hat{\mathbf{k}})\boldsymbol{\eta}\cdot\boldsymbol{\beta}]/\gamma^3 ,$$

to $O(\gamma^{-4})$ I need only expand the τ dependence to $O(\omega_0\tau)$, and the angular dependence of the dot products to $O(\gamma^{-1})$. Then

$$\begin{aligned}\boldsymbol{\eta}\cdot\boldsymbol{\beta}(\boldsymbol{\beta}\times\hat{\mathbf{k}})\cdot(\boldsymbol{\eta}'\times\hat{\mathbf{k}})&\simeq\eta_x(\boldsymbol{\eta}'\cdot\boldsymbol{\beta}-\boldsymbol{\beta}\cdot\hat{\mathbf{k}}\boldsymbol{\eta}'\cdot\hat{\mathbf{k}})\simeq\eta_x(\boldsymbol{\eta}'\cdot\boldsymbol{\beta}-\boldsymbol{\eta}'\cdot\hat{\mathbf{k}})\\&\simeq\eta_x(\boldsymbol{\eta}'\cdot\boldsymbol{\beta}-\boldsymbol{\eta}'\cdot\hat{\boldsymbol{\xi}}\chi)\\&\simeq\eta_x(\eta_x^*-\omega_0\tau\eta_y^*-\eta_x^*+\tfrac{1}{2}\omega_0\tau\eta_y^*)=-\tfrac{1}{2}\omega_0\tau\eta_x\eta_y^*\end{aligned} \quad (42a)$$

and

$$\boldsymbol{\eta}'\cdot\boldsymbol{\beta}'(\boldsymbol{\beta}'\times\hat{\mathbf{k}})\cdot(\boldsymbol{\eta}\times\hat{\mathbf{k}})\simeq\tfrac{1}{2}\omega_0\tau\eta_y\eta_x^* . \quad (42b)$$

Hence, to $O(\gamma^{-4})$, this integrand is $\omega_0\tau(\eta_y\eta_x^*-\eta_x\eta_y^*)/2\gamma^3=i\hat{\mathbf{n}}\cdot\hat{\mathbf{z}}\omega_0\tau/\gamma^3$.

(iv) In the term

$$\boldsymbol{\eta}\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]\boldsymbol{\eta}'\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)](\boldsymbol{\beta}\cdot\boldsymbol{\beta}'-1)/\gamma^2 ,$$

since $\boldsymbol{\beta}\cdot\boldsymbol{\beta}'-1$ is the integrand of the classical synchrotron radiation power spectrum,[4] which is of $O(\gamma^{-2})$, and there is already an overall coefficient of γ^{-2}, I neglect the variation of $\boldsymbol{\eta}'$. From Appendix A,

$$\boldsymbol{\eta}\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]\boldsymbol{\eta}'\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]=|\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)|^2 .$$

The contribution of this term to $d\mathcal{P}/d\xi$ is therefore the classical synchrotron radiation power spectrum multiplied by $|\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)|^2/\gamma^2$.

(v) The term

$$\{\boldsymbol{\eta}\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)](\boldsymbol{\eta}'\times\hat{\mathbf{k}})\cdot\boldsymbol{\beta}+\boldsymbol{\eta}'\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)](\boldsymbol{\eta}\times\hat{\mathbf{k}})\cdot\boldsymbol{\beta}'\}/\gamma^3$$

yields only vector integrals, and so from Eq. (24) I replace $\hat{\mathbf{k}}$ by $\chi\hat{\boldsymbol{\xi}}$. Then, as in (iii), I approximate $\chi\simeq1$ and keep the τ

dependence only up to $O(\omega_0\tau)$. I obtain the products $(\boldsymbol{\eta}'\times\hat{\boldsymbol{\xi}})\cdot\boldsymbol{\beta}(\simeq-\eta_z^*\omega_0\tau/2)$ and $(\boldsymbol{\eta}\times\hat{\boldsymbol{\xi}})\cdot\boldsymbol{\beta}'(\simeq\eta_z\omega_0\tau/2)$, whence the integrand reduces to

$$-i\,\mathrm{Im}\{\eta_z^* s\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]\}\omega_0\tau/\gamma^3=-i\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)\cdot\hat{\mathbf{z}}\omega_0\tau/\gamma^3 \ .$$

(vi) As for

$$\{\boldsymbol{\eta}\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]\boldsymbol{\eta}'\cdot\boldsymbol{\beta}'(\boldsymbol{\beta}'\times\hat{\mathbf{k}})\cdot\boldsymbol{\beta}+\boldsymbol{\eta}'\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]\boldsymbol{\eta}\cdot\boldsymbol{\beta}(\boldsymbol{\beta}\times\hat{\mathbf{k}})\cdot\boldsymbol{\beta}'\}/\gamma^2 \ ,$$

I replace $\hat{\mathbf{k}}$ by $\chi\hat{\boldsymbol{\xi}}$, as in (v), whereupon I discover the integrals of these terms vanish because I obtain the products $(\boldsymbol{\beta}'\times\hat{\boldsymbol{\xi}})\cdot\boldsymbol{\beta}$ and $(\boldsymbol{\beta}\times\hat{\boldsymbol{\xi}})\cdot\boldsymbol{\beta}'$, which vanish since the vectors are coplanar.

This concludes the discussion of the individual terms in $\mathbf{j}_s\cdot\mathbf{j}_s'/c^2-\rho_s\rho_s'$. The total integrand is

$$\frac{1}{c^2}\mathbf{j}_s\cdot\mathbf{j}_s'-\rho_s\rho_s'=\frac{e^2\hbar^2\omega^2}{4m^2c^4}\left[\frac{1}{\gamma^4}[1+(\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2]-\frac{1}{\gamma^2}[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2]\left[\frac{1}{\gamma^2}+\frac{1}{2}\omega_0^2\tau^2\right]\right.$$
$$\left.+\frac{i}{\gamma^3}\hat{\mathbf{n}}\cdot\hat{\mathbf{z}}\omega_0\tau-\frac{1}{\gamma^2}\left|\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right|^2\left[\frac{1}{\gamma^2}+\frac{1}{2}\omega_0^2\tau^2\right]-\frac{i}{\gamma^3}\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{z}}\omega_0\tau\right] , \tag{43}$$

and the relevant integrals, to the degree of accuracy required, are

$$\mathrm{Re}\left[\int(1,i\omega_0\tau,\omega_0^2\tau^2)e^{-i(\omega\tau-\chi\xi)}d\tau\,d\chi\right]\simeq\frac{2}{\sqrt{3}\omega\beta}\int_\xi^\infty K_{1/3}(\xi')d\xi'$$
$$\simeq\frac{4}{\sqrt{3}\omega\beta\gamma}K_{1/3}(\xi)$$
$$\simeq-\frac{8}{\sqrt{3}\omega\beta\gamma^2}K_{2/3}(\xi) , \tag{44}$$

with an obvious notation, and so we arrive at the result

$$\frac{d\mathcal{P}_\downarrow}{d\omega}\simeq\frac{e^2\hbar^2\omega^3}{4\sqrt{3}\pi m^2c^5\gamma^4}\left[[1+(\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2]\int_\xi^\infty K_{1/3}(\xi')d\xi'+2\hat{\mathbf{z}}\cdot\left[\hat{\mathbf{n}}-\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]K_{1/3}(\xi)\right.$$
$$+[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2]\left[2K_{2/3}(\xi)-\int_\xi^\infty K_{1/3}(\xi')d\xi'\right]$$
$$\left.+\left|\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right|^2\left[2K_{2/3}(\xi)-\int_\xi^\infty K_{1/3}(\xi')d\xi'\right]\right] , \tag{45}$$

or

$$\frac{d\mathcal{P}_\downarrow}{d\xi}=\frac{27\sqrt{3}}{32\pi}\frac{e^2\hbar^2\gamma^8\omega_0^4}{m^2c^5}\xi^3\left[K_{2/3}(\xi)+(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2\left[\int_\xi^\infty K_{1/3}(\xi')d\xi'-K_{2/3}(\xi)\right]\right.$$
$$\left.+\frac{1}{2}\left|\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right|^2\left[2K_{2/3}(\xi)-\int_\xi^\infty K_{1/3}(\xi')d\xi'\right]+\hat{\mathbf{b}}\cdot\left[\hat{\mathbf{n}}-\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]K_{1/3}(\xi)\right] , \tag{46}$$

which leads to the Derbenev-Kondratenko formula, or, with $a\neq0$ included to first order, to Eq. (29).

VI. DERBENEV-KONDRATENKO APPROACH

In this Section I prove the equivalence of my approach with that of Derbenev and Kondratenko,[2] because they do not follow the above procedure. I begin by deriving their interaction Hamiltonian, following their approach. We have seen in Sec. III that $\mathbf{s}\cdot\hat{\mathbf{n}}$ is a constant of the unperturbed motion. In the presence of radiation, Derbenev and Kondratenko write, for the instantaneous rate of change, and to the leading order in spin,[23]

$$\frac{d}{dt}(\mathbf{s}\cdot\hat{\mathbf{n}})=\{\mathbf{s}\cdot\hat{\mathbf{n}},\mathcal{H}_{\rm int}\}$$
$$\simeq(\boldsymbol{\Omega}_{\rm rad}\times\mathbf{s})\cdot\hat{\mathbf{n}}+\mathbf{s}\cdot\{\hat{\mathbf{n}},e(\Phi_{\rm rad}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm rad})\} , \tag{47}$$

where $\{,\}$ denotes a Poisson bracket. At this point an important difference in our formulations exhibits itself: I associate $\hat{\mathbf{n}}$ (and $\{I_\lambda,\psi_\lambda\}$) with a *trajectory*, and use a phase-space average, whereas they attach $\hat{\mathbf{n}}$ to a *particle*, and calculate the time average of $\mathbf{s}\cdot\hat{\mathbf{n}}$ as the particle moves through phase space.[24] By the ergodic hypothesis, the final result is the same, and Derbenev and Kondratenko present their final result as a phase-space average. This is how the spin-independent part of $\mathcal{H}_{\rm int}$ [thence $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$]

enters their calculation. To quote them "With the spin-orbit coupling taken into account, the radiation can thus act on the polarization not only directly, but also via the trajectory, perturbing the quantization axis."[2] Now

$$\{\hat{\mathbf{n}},\mathcal{H}_{\rm int}\}=\frac{\partial\hat{\mathbf{n}}}{\partial\mathbf{r}_\alpha}\frac{\partial\mathcal{H}_{\rm int}}{\partial\mathbf{p}_\alpha}-\frac{\partial\hat{\mathbf{n}}}{\partial\mathbf{p}_\alpha}\frac{\partial\mathcal{H}_{\rm int}}{\partial\mathbf{r}_\alpha}\ . \tag{48}$$

Because of the assumption of "point emission," they neglect $\partial\hat{\mathbf{n}}/\partial\mathbf{r}$, and to the desired order in spin they also neglect $\boldsymbol{\Omega}_{\rm rad}\cdot\mathbf{s}$ in $\mathcal{H}_{\rm int}$, whence

$$\begin{aligned}\{\hat{\mathbf{n}},\mathcal{H}_{\rm int}\}&\simeq\{\hat{\mathbf{n}},e(\Phi_{\rm rad}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm rad})\}\\&=-e\frac{\partial\hat{\mathbf{n}}}{\partial\mathbf{p}_\alpha}\frac{\partial}{\partial\mathbf{r}_\alpha}(\Phi_{\rm rad}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm rad})\\&\simeq e(\mathbf{E}_{\rm rad}+\boldsymbol{\beta}\times\mathbf{B}_{\rm rad})\cdot\frac{\partial}{\partial\mathbf{p}}\hat{\mathbf{n}}\equiv\mathbf{f}\cdot\frac{\partial}{\partial\mathbf{p}}\hat{\mathbf{n}}\ ,\end{aligned} \tag{49}$$

where $\mathbf{f}=e(\mathbf{E}_{\rm rad}+\boldsymbol{\beta}\times\mathbf{B}_{\rm rad})$. They now introduce

$$\boldsymbol{\omega}_{\rm DK}=\boldsymbol{\Omega}_{\rm rad}-\hat{\mathbf{n}}\times\left[\mathbf{f}\cdot\frac{\partial}{\partial\mathbf{p}}\right]\hat{\mathbf{n}}\ . \tag{50}$$

They call this $\boldsymbol{\omega}$, but I use ω to denote the photon frequency. In terms of $\boldsymbol{\omega}_{\rm DK}$, one can construct an effective interaction Hamiltonian $\boldsymbol{\omega}_{\rm DK}\cdot\mathbf{s}$, called V_ψ by Dervenev and Kondratenko, whence

$$\begin{aligned}\frac{d}{dt}(\mathbf{s}\cdot\hat{\mathbf{n}})&\simeq\{\mathbf{s}\cdot\hat{\mathbf{n}},\boldsymbol{\omega}_{\rm DK}\cdot\mathbf{s}\}\\&=\hat{\mathbf{n}}_\alpha\boldsymbol{\omega}_{{\rm DK}\beta}\{\mathbf{s}_\alpha,\mathbf{s}_\beta\}\\&=(\boldsymbol{\omega}_{\rm DK}\times\mathbf{s})\cdot\hat{\mathbf{n}}\\&=(\boldsymbol{\Omega}_{\rm rad}\times\mathbf{s})\cdot\hat{\mathbf{n}}+\mathbf{s}_\beta\left[\mathbf{f}_\alpha\frac{\partial}{\partial\mathbf{p}_\alpha}\right]\hat{\mathbf{n}}_\beta\ ,\end{aligned} \tag{51}$$

in agreement with Eqs. (47) and (49). For the case of interest, viz., "point" photon emission, almost parallel to $\boldsymbol{\beta}$, by an electron following a classical orbital trajectory, evaluation to the leading ordering in $\hbar$ yields

$$\mathbf{f}\cdot\frac{\partial}{\partial\mathbf{p}}\hat{\mathbf{n}}\simeq\frac{\mathbf{f}\cdot\boldsymbol{\beta}}{mc}\frac{\partial}{\partial\gamma}\hat{\mathbf{n}}=\frac{e}{mc}\boldsymbol{\beta}\cdot\mathbf{E}_{\rm rad}\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\ , \tag{52}$$

whence

$$\boldsymbol{\omega}_{\rm DK}\simeq\boldsymbol{\Omega}_{\rm rad}-\frac{e}{\gamma mc}\boldsymbol{\beta}\cdot\mathbf{E}_{\rm rad}\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\ . \tag{53}$$

Neglecting the terms in a, this simplifies to

$$\boldsymbol{\omega}_{\rm DK}\simeq-\frac{e}{\gamma mc}\left[\mathbf{B}_v+\frac{1}{\gamma}\mathbf{B}_{\rm tr}+\boldsymbol{\beta}\cdot\mathbf{E}\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\ , \tag{54}$$

which is the final form quoted by Derbenev and Kondratenko. In the quantum (spin-$\frac{1}{2}$) theory, $\mathbf{s}\cdot\hat{\mathbf{n}}$ can have, instantaneously, one of only two values, and the radiation causes transitions between them; recall that $\mathbf{s}\cdot\hat{\mathbf{n}}$ is otherwise constant. The equilibrium degree of polarization is the time-averaged value of $\mathbf{s}\cdot\hat{\mathbf{n}}$. The relevant matrix elements are $\langle\mp\hat{\mathbf{n}},\gamma\,|\,\boldsymbol{\omega}_{\rm DK}\cdot\mathbf{s}_{\rm op}\,|\,\pm\hat{\mathbf{n}},0\rangle$, i.e., $|\pm\hat{\mathbf{n}}\rangle$ to $|\mp\hat{\mathbf{n}}\rangle$, since the difference between the initial and final quantization axes has been absorbed into the effective interaction Hamiltonian. We can compare this result with mine; using Eqs. (19) and (39) we see that

$$\begin{aligned}\langle-\hat{\mathbf{n}},\gamma\,|\,\boldsymbol{\omega}_{\rm DK}\cdot\mathbf{s}_{\rm op}\,|\,\hat{\mathbf{n}},0\rangle&=\left\langle-\hat{\mathbf{n}},\gamma\left|\left[\boldsymbol{\Omega}_{\rm rad}\cdot\mathbf{s}_{\rm op}-\frac{e}{\gamma mc}\boldsymbol{\beta}\cdot\mathbf{E}_{\rm rad}\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\right|\hat{\mathbf{n}},0\right\rangle\\&\simeq\frac{\hbar}{2}\boldsymbol{\Omega}_{\rm em}\cdot\boldsymbol{\eta}-\frac{ie\hbar\omega}{2\gamma mc^2}\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]\cdot\boldsymbol{\eta}(\Phi_{\rm em}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm em})\\&\simeq\langle-\hat{\mathbf{n}}+\mathbf{D},\gamma\,|\,\mathcal{H}_{\rm int}\,|\,\hat{\mathbf{n}},0\rangle\ .\end{aligned} \tag{55}$$

I introduce $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ and the various simplifications after determining the matrix elements; they do it before. They next introduce two functions, α_+ and α_-, given mathematically by[21]

$$\alpha_\pm=\frac{\hbar^2}{4}\left\langle\int_{-\infty}^{\infty}d\tau\langle 0\,|\,[(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta})_{t+\tau/2},(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta}^*)_{t-\tau/2}]_\pm\,|\,0\rangle\right\rangle\ , \tag{56}$$

and obtain the equilibrium degree of polarization via $P_{\rm eq}=\alpha_-/\alpha_+$. The large angular brackets denote an ensemble average over the electron distribution and accelerator azimuth, while the subscript "±" in $[,]_\pm$ denotes an anticommutator or commutator, respectively, and "Here $\langle 0|\ldots|0\rangle$ denotes averaging over the state of the photon vacuum,"[2] this statement will be explained in more detail below. They state, and I shall show below, that for spin $\frac{1}{2}$, $\alpha_\pm=p_\uparrow\pm p_\downarrow$. I do this by noting that the probability of emitting a photon, accompanied by spin-flip from "up" to "down" in the time interval $(-T/2,T/2)$ is[5]

$$\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{2\pi}{\hbar\omega}\left|\int_{-T/2}^{T/2}dt'\langle-\hat{\mathbf{n}},\gamma\,|\,\boldsymbol{\omega}_{\rm DK}(t')\cdot\mathbf{s}_{\rm op}(t')\,|\,\hat{\mathbf{n}},0\rangle\right|^2\ , \tag{57}$$

and the corresponding probability per unit time $p_\downarrow(\mathbf{r},\mathbf{p},t)$ is

$$p_\downarrow(\mathbf{r},\mathbf{p},t)\approx \int d\tau \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{2\pi}{\hbar\omega}\langle \hat{\mathbf{n}},0\,|\,(\boldsymbol{\omega}_{\rm DK}\cdot\mathbf{s}_{\rm op})_{t+\tau/2}\,|\,-\hat{\mathbf{n}},\gamma\rangle\langle -\hat{\mathbf{n}},\gamma\,|\,(\boldsymbol{\omega}_{\rm DK}\cdot\mathbf{s}_{\rm op})_{t-\tau/2}\,|\,\hat{\mathbf{n}},0\rangle \;, \tag{58}$$

in the limit $T\to\infty$. Then $p_\downarrow$ is the average of $p_\downarrow(\mathbf{r},\mathbf{p},t)$ over the electron distribution and the accelerator azimuth; $p_\downarrow=\langle p_\downarrow(\mathbf{r},\mathbf{p},t)\rangle$. In keeping with notation of the original authors, I have not complex conjugated $\boldsymbol{\omega}_{\rm DK}$, which is permissible because $\boldsymbol{\omega}_{\rm DK}\cdot\mathbf{s}_{\rm op}$ is Hermitian. However, this is only true if both photon emission *and* absorption terms are retained in $\boldsymbol{\omega}_{\rm DK}$, hence in this approach one cannot neglect photon absorption terms. Now, $\boldsymbol{\eta}$ and $\boldsymbol{\eta}^*$ are introduced by writing

$$p_\downarrow(\mathbf{r},\mathbf{p},t)=\frac{\hbar^2}{4}\int d\tau \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{2\pi}{\hbar\omega}\langle 0\,|\,(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta}^*)_{t+\tau/2}\,|\,\gamma\rangle\langle\gamma\,|\,(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta})_{t-\tau/2}\,|\,0\rangle \;. \tag{59}$$

Next, $\int d^3\mathbf{k}(2\pi)^{-2}(\hbar\omega)^{-1}\,|\,\gamma\rangle\langle\gamma\,|$ is replaced by the unit operator by using the identity $\sum\,|\,\gamma,\ldots,\gamma\rangle\langle\gamma,\ldots,\gamma\,|=1$, where the sum extends over all photon states. Hence

$$p_\downarrow(\mathbf{r},\mathbf{p},t)=\frac{\hbar^2}{4}\int_{-\infty}^{\infty} d\tau\langle 0\,|\,(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta}^*)_{t+\tau/2}(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta})_{t-\tau/2}\,|\,0\rangle \;, \tag{60}$$

with a similar expression for $p_\uparrow$. We see now the meaning of the phrase "averaging over the state of the photon vacuum" alluded to above. Thus

$$p_\uparrow(\mathbf{r},\mathbf{p},t)\pm p_\downarrow(\mathbf{r},\mathbf{p},t)=\frac{\hbar^2}{4}\int_{-\infty}^{\infty} d\tau\langle 0\,|\,[(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta})_{t+\tau/2}(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta}^*)_{t-\tau/2}\pm(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta}^*)_{t+\tau/2}(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta})_{t-\tau/2}]\,|\,0\rangle \;. \tag{61}$$

Finally, because the range of integration is symmetric about $\tau=0$, one can replace τ by $-\tau$ in the second term, whence

$$p_\uparrow(\mathbf{r},\mathbf{p},t)\pm p_\downarrow(\mathbf{r},\mathbf{p},t)=\frac{\hbar^2}{4}\int_{-\infty}^{\infty} d\tau\langle 0\,|\,[(\boldsymbol{\omega}_{DK}\cdot\boldsymbol{\eta})_{t+\tau/2},(\boldsymbol{\omega}_{\rm DK}\cdot\boldsymbol{\eta}^*)_{t-\tau/2}]_\pm\,|\,0\rangle \;. \tag{62}$$

Ensemble averaging yields the expressions for $\alpha_\pm$. As has been explained in Sec. II, the equilibrium degree of polarization is given by $(p_\uparrow-p_\downarrow)/(p_\uparrow+p_\downarrow)$, or, as we now see, by α_-/α_+.

VII. CONCLUSIONS

A number of concluding remarks are in order. To begin, let us recall the argument for the origin of polarization in electron storage rings. When an electron emits a photon, its spin sometimes flips. This can be either because of the direct interaction of the spin operator with the photon field in the interaction Hamiltonian, or because the initial and final spin quantization axes are not parallel. Polarization develops because the transition probabilities are not equal for flips in opposite directions.

The calculations in Sec. V elucidate the effect of the anomalous magnetic moment on the polarization and justify the way in which the equilibrium polarization has thus far been calculated in practice—the Derbenev-Kondratenko expression is used, *but* $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ are calculated with $g\neq 2$. An important goal of these calculations is the analysis of the phenomenon of "spin resonances." Near a resonance, the polarization almost vanishes, and so the accurate determination of resonances is essential to the design of an electron accelerator in which it is desired to achieve a high degree of polarization. A clear understanding of the processes which lead to the equilibrium polarization, and of the roles they play, is thus desirable.[6]

Semiclassical quantum electrodynamics, statistical mechanics, and classical mechanics have been used to derive the equilibrium polarization in a high-energy electron storage ring. First, the Hamiltonian was specified and divided into appropriate unperturbed and interaction parts. The unperturbed Hamiltonian was then diagonalized, to first order in $\hbar$, and the subtleties encountered in so doing were elucidated, such as the need for a trajectory-dependent spin quantization axis. Furthermore, many workers in the field have had difficulty in understanding the meaning of the quantity $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. Perhaps the reason for this is the failure to realize that $\hat{\mathbf{n}}$ is not merely a fixed vector, but a vector field—a function of the orbital trajectory on which it is defined. From the above analysis, it is clear that $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ is also a vector field. The effects of perturbations on the unperturbed motion were then included. The relevant processes were described, their time scales, and their effect on the electron beam. This enabled us to determine the matrix elements and statistical averages needed to calculate the equilibrium state of the beam. The Derbenev-Kondratenko formula was the result, and, because of the relative simplicity of Schwinger's formalism, it was also possible to extend the formula to elucidate the role of the anomalous magnetic moment. The approach used by Derbenev and Kondratenko[2] was explained in Sec. VI, and was shown to be equivalent to that used in this paper. In summary, the above work helps to elucidate several aspects of the polarization process not evident from the work of the original authors.

ACKNOWLEDGMENTS

I am especially grateful to K. Yokoya for much help and encouragement in carrying out this research, and for checking early versions of the manuscript. I would also like to thank R. G. Glasser for first suggesting that I use a statistical mechanical approach to calculate the polarization, M. T. Bernius and R. Talman for their comments on the final draft of the manuscript, and also M. E. Fisher, K. Gottfried, and the staff of the Laboratory of Nuclear

Studies, Cornell University. This work was supported in part by the National Science Foundation, under Grant Nos. PHY84-12263 and PHY84-1531.

APPENDIX A: USEFUL MATHEMATICAL EXPRESSIONS

I begin with the "spin-flip" vector" $\boldsymbol{\eta}$. In this appendix $\boldsymbol{\eta}$ denotes $\boldsymbol{\eta}(\tau=0)$. Following Jackson,[8] I write $\hat{\mathbf{n}}=(\sin\theta_{\hat{n}}\cos\phi_{\hat{n}}, \sin\theta_{\hat{n}}\sin\phi_{\hat{n}}, \cos\theta_{\hat{n}})$ in the basis $\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$. Then, quoting his results in my notation,

$$\begin{aligned}\boldsymbol{\eta}\cdot\hat{\mathbf{z}}&=-\sin\theta_{\hat{n}}\ ,\\ \boldsymbol{\eta}\cdot(\hat{\mathbf{x}}+i\hat{\mathbf{y}})&=-e^{i\phi_{\hat{n}}}(1-\cos\theta_{\hat{n}})\ ,\\ \boldsymbol{\eta}\cdot(\hat{\mathbf{x}}-i\hat{\mathbf{y}})&=e^{-i\phi_{\hat{n}}}(1+\cos\theta_{\hat{n}})\ .\end{aligned} \tag{A1}$$

It is easy to check that $|\boldsymbol{\eta}|^2=2$ and $|\eta_x|^2\equiv|\boldsymbol{\eta}\cdot\hat{\mathbf{x}}|^2=[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2]$. I also need the value of $\boldsymbol{\eta}\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]$ at $\tau=0$, which is

$$\begin{aligned}\boldsymbol{\eta}\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]=-i\Bigg[(\sin\theta_{\hat{n}})\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{z}}-(\cos\theta_{\hat{n}})\left[(\cos\phi_{\hat{n}})\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{x}}+(\sin\phi_{\hat{n}})\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{y}}\right]\\ +i\left[(\sin\phi_{\hat{n}})\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{x}}-(\cos\phi_{\hat{n}})\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{y}}\right]\Bigg]\ ,\end{aligned} \tag{A2}$$

which can be simplified, using the constraint $\hat{\mathbf{n}}\cdot[\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]=0$, to

$$\boldsymbol{\eta}\cdot\left[\hat{\mathbf{n}}\times\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right]=-i\left[\frac{1}{(\sin\theta_{\hat{n}})}\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{z}}+i\left[(\sin\phi_{\hat{n}})\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{x}}-(\cos\phi_{\hat{n}})\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\cdot\hat{\mathbf{y}}\right]\right]\ . \tag{A3}$$

Note that $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)\cdot\hat{\mathbf{z}}$ vanishes at $\theta_{\hat{n}}=0,\pi$, hence the term in $(\sin\theta_{\hat{n}})^{-1}$ is not divergent. It is easily verified that $|\boldsymbol{\eta}\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]|^2=|\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)|^2$ and

$$\mathrm{Im}\{\eta_z^*\boldsymbol{\eta}\cdot[\hat{\mathbf{n}}\times\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)]\}=\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)\cdot\hat{\mathbf{z}}\ .$$

Furthermore, following Derbenev and Kondratenko[2] and Jackson,[8] I neglect the variation of $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ along the electron trajectory over the duration of the photon emission, in keeping with the assumption of "point" photon emission.

To evaluate the integrand in Sec. V, it is useful to prepare a list of relevant vectors as functions of τ. Primed vectors are evaluated at time τ, unprimed ones at $\tau=0$. Then

$$\begin{aligned}&\mathbf{r}\equiv0,\quad \mathbf{r}'\simeq\omega_0\tau\hat{\mathbf{x}}+\frac{\omega_0^2\tau^2}{2}\hat{\mathbf{y}}\ ,\\ &\boldsymbol{\beta}\equiv\beta\hat{\mathbf{x}},\quad \boldsymbol{\beta}'\simeq\beta\left[\left[1-\frac{\omega_0^2\tau^2}{2}\right]\hat{\mathbf{x}}+\omega_0\tau\hat{\mathbf{y}}\right]\ ,\\ &\boldsymbol{\eta}\equiv\eta_x\hat{\mathbf{x}}+\eta_y\hat{\mathbf{y}}+\eta_z\hat{\mathbf{z}},\quad \hat{\boldsymbol{\xi}}\simeq\left[1-\frac{\omega_0^2\tau^2}{8}\right]\hat{\mathbf{x}}+\frac{\omega_0\tau}{2}\hat{\mathbf{y}}\ ,\\ &\boldsymbol{\eta}'\simeq\boldsymbol{\eta}^*+\tau\boldsymbol{\Omega}_{\mathrm{ext}}\times\boldsymbol{\eta}^*+\frac{\tau^2}{2}\boldsymbol{\Omega}_{\mathrm{ext}}\times(\boldsymbol{\Omega}_{\mathrm{ext}}\times\boldsymbol{\eta}^*)\\ &\quad=\eta_z^*\hat{\mathbf{z}}-(\eta_y^*\hat{\mathbf{x}}-\eta_x^*\hat{\mathbf{y}})\Omega\tau+(\eta_x^*\hat{\mathbf{x}}+\eta_y^*\hat{\mathbf{y}})\left[1-\frac{\Omega^2\tau^2}{2}\right]\ ,\end{aligned} \tag{A4}$$

where η_x, η_y, and η_z all refer to $\tau=0$. To obtain the expression for $\boldsymbol{\eta}'$, the Thomas-BMT equation $d\boldsymbol{\eta}'/dt=\boldsymbol{\Omega}_{\mathrm{ext}}\times\boldsymbol{\eta}'$ is solved using $\boldsymbol{\Omega}_{\mathrm{ext}}=\Omega\hat{\mathbf{z}}$, assumed constant over the duration of the photon emission, in which case $\Omega=(1+\gamma a)\omega_0$. It is also helpful to prepare a list of relevant dot products of the above vectors (putting $a=0$ in all expressions involving $\boldsymbol{\eta}'$):

$$\begin{aligned}&\boldsymbol{\beta}\cdot\boldsymbol{\beta}'\simeq\beta^2\left[1-\frac{\omega_0^2\tau^2}{2}\right]\ ,\\ &\boldsymbol{\eta}\cdot\boldsymbol{\beta}=\eta_x\beta,\quad \boldsymbol{\eta}'\cdot\boldsymbol{\beta}'\simeq\eta_x^*\beta\ ,\\ &\boldsymbol{\eta}\cdot\boldsymbol{\beta}'\simeq\left[\eta_x\left[1-\frac{\omega_0^2\tau^2}{2}\right]+\eta_y\omega_0\tau\right],\quad \boldsymbol{\eta}'\cdot\boldsymbol{\beta}\simeq\beta\left[\eta_x\left[1-\frac{\omega_0^2\tau^2}{2}\right]-\eta_y^*\omega_0\tau\right]\ ,\\ &\boldsymbol{\eta}\cdot\hat{\boldsymbol{\xi}}\simeq\eta_x\left[1-\frac{\omega_0^2\tau^2}{8}\right]+\eta_y\frac{\omega_0\tau}{2},\quad \boldsymbol{\eta}'\cdot\hat{\boldsymbol{\xi}}\simeq\eta_x^*\left[1-\frac{\omega_0^2\tau^2}{8}\right]-\eta_y^*\frac{\omega_0\tau}{2}\ ,\\ &(\boldsymbol{\eta}\times\boldsymbol{\beta})\cdot\hat{\boldsymbol{\xi}}\simeq\eta_z\beta\frac{\omega_0\tau}{2},\quad (\boldsymbol{\eta}'\times\boldsymbol{\beta}')\cdot\hat{\boldsymbol{\xi}}\simeq-\eta_z^*\beta\frac{\omega_0\tau}{2}\ .\end{aligned} \tag{A5}$$

APPENDIX B: EVALUATION OF THE INTEGRALS

Consider the angular integrals first. There are six of them, viz.,

$$\int_{-1}^{1} d\chi \cos(\omega\tau-\chi\zeta)=2\cos(\omega\tau)\frac{\sin\zeta}{\zeta} ,$$

$$\int_{-1}^{1} d\chi \sin(\omega\tau-\chi\zeta)=2\sin(\omega\tau)\frac{\sin\zeta}{\zeta} ,$$

$$\int_{-1}^{1} d\chi\, \chi \cos(\omega\tau-\chi\zeta)=2\sin(\omega\tau)\left[\frac{\sin\zeta}{\zeta^2}-\frac{\cos\zeta}{\zeta}\right] ,$$

$$\int_{-1}^{1} d\chi\, \chi \sin(\omega\tau-\chi\zeta)=-2\cos(\omega\tau)\left[\frac{\sin\zeta}{\zeta^2}-\frac{\cos\zeta}{\zeta}\right] , \tag{B1}$$

$$\int_{-1}^{1} d\chi\, \chi^2 \cos(\omega\tau-\chi\zeta) =2\cos(\omega\tau)\left[\frac{\sin\zeta}{\zeta}+2\frac{\cos\zeta}{\zeta^2}-2\frac{\sin\zeta}{\zeta^3}\right] ,$$

$$\int_{-1}^{1} d\chi\, \chi^2 \sin(\omega\tau-\chi\zeta) =2\sin(\omega\tau)\left[\frac{\sin\zeta}{\zeta}+2\frac{\cos\zeta}{\zeta^2}-2\frac{\sin\zeta}{\zeta^3}\right] .$$

I am now left with the integral over $\tau\equiv t'-t$. I follow the usual approach, which is to expand ζ for small $|\tau|$, up to $|\tau|^3$. The rapid oscillation of $\sin\zeta$ and $\cos\zeta$ kills off contributions from large values of $|\tau|$, so the limits of integration can be taken to be $-\infty<\tau<\infty$. From the main body of the text, the integrands, say $f(\tau,\chi)$, are all of the form $(\omega_0\tau)^q\cos(\omega\tau-\chi\zeta)$, $q=0,2,\ldots$ or $(\omega_0\tau)^q\sin(\omega\tau-\chi\zeta)$, $q=1,3,\ldots$, and, to the degree of approximation of interest, only the values $q=0,1,2$ matter. It will be helpful to classify the distinct integrals that appear, and to prepare a list of results.

We see that only three distinct integrals appear in the simplified derivation in Sec. V, and they are the same as those that occur in the theory of classical synchrotron radiation, viz.,

$$\mathrm{Re}\left[\int (1,i\omega_0\tau,\omega_0^2\tau^2)e^{-i(\omega\tau-\chi\zeta)}d\tau\, d\chi\right] , \tag{B2}$$

or, consulting the list of angular integrals,

$$2\int_{-\infty}^{\infty}\cos(\omega\tau)\frac{\sin\zeta}{\zeta}d\tau,\quad 2\int_{-\infty}^{\infty}\omega_0\tau\sin(\omega\tau)\frac{\sin\zeta}{\zeta}d\tau,\quad 2\int_{-\infty}^{\infty}(\omega_0\tau)^2\cos(\omega\tau)\frac{\sin\zeta}{\zeta}d\tau . \tag{B3}$$

We therefore consult Ref. 4 to evaluate them. Consider the first of the three integrals above. Expanding ζ for small $|\tau|$, $\zeta\simeq\omega\beta\,|\tau-\omega_0^2\tau^3/24|$, whence

$$\begin{aligned}
2\int_{-\infty}^{\infty} d\tau\cos(\omega\tau)\frac{\sin\zeta}{\zeta} &\simeq \frac{4}{\omega\beta}\int_0^{\infty} d\tau\cos(\omega\tau)\frac{\sin\zeta}{\tau-\omega_0^2\tau^3/24}\\
&\simeq \frac{4}{\omega\beta}\int_0^{\infty}\frac{d\tau}{\tau}\left[1+\frac{\omega_0^2\tau^2}{24}\right]\cos(\omega\tau)\sin\left[\omega\beta\tau\left[1-\frac{\omega_0^2\tau^2}{24}\right]\right]\\
&\simeq \frac{2}{\omega\beta}\int_0^{\infty}\frac{d\tau}{\tau}\left[\sin(2\omega\tau)-\sin\left[\omega(1-\beta)\tau+\omega\frac{\omega_0^2\tau^3}{24}\right]\right]\\
&\quad+\frac{\omega_0^2}{12\omega\beta}\int_0^{\infty}d\tau\,\tau\left[\sin(2\omega\tau)-\sin\left[\omega(1-\beta)\tau+\omega\frac{\omega_0^2\tau^3}{24}\right]\right] .
\end{aligned} \tag{B4}$$

It is shown in Ref. 4 that the above integrals are modified Bessel functions, specifically

$$\int_0^{\infty}\frac{d\tau}{\tau}\left[\sin(2\omega\tau)-\sin\left[\omega(1-\beta)\tau+\omega\frac{\omega_0^2\tau^3}{24}\right]\right]=\frac{1}{\sqrt{3}}\int_{\xi}^{\infty}K_{1/3}(\xi')d\xi' , \tag{B5}$$

where $\xi=\omega/\omega_c$ and $\omega_c=3\omega_0\gamma^3/2$ (see Sec. V), and

$$\int_0^{\infty}d\tau\,\tau\left[\sin(2\omega\tau)-\sin\left[\omega(1-\beta)\tau+\omega\frac{\omega_0^2\tau^3}{24}\right]\right]=-\frac{4}{\sqrt{3}\omega_0^2\gamma^2}K_{2/3}(\xi) , \tag{B6}$$

whence the sum total is

$$\mathrm{Re}\left[\int e^{-i(\omega\tau-\chi\zeta)}d\tau\, d\chi\right]=\frac{2}{\sqrt{3}\omega\beta}\int_{\xi}^{\infty}K_{1/3}(\xi')d\xi'-\frac{1}{3\sqrt{3}\omega\beta\gamma^2}K_{2/3}(\xi) . \tag{B7}$$

It can similarly be shown that

$$2\int_{-\infty}^{\infty} d\tau\,\omega_0\tau\sin(\omega\tau)\frac{\sin\zeta}{\zeta} \simeq \frac{2\omega_0}{\omega\beta}\int_{-\infty}^{\infty} d\tau \frac{\tau}{|\tau|}\left[1+\frac{\omega_0^2\tau^2}{24}\right]\sin(\omega\tau)\sin\zeta$$

$$\simeq \frac{4\omega_0}{\omega\beta}\int_0^{\infty} d\tau\left[1+\frac{\omega_0^2\tau^2}{24}\right]\sin(\omega\tau)\sin\zeta$$

$$\simeq \frac{2\omega_0}{\omega\beta}\int_0^{\infty} d\tau\left[\cos\left[\omega(1-\beta)\tau+\omega\frac{\omega_0^2\tau^3}{24}\right]-\cos(2\omega\tau)\right] = \frac{4}{\sqrt{3}\omega\beta\gamma}K_{1/3}(\xi)\ , \quad \text{(B8)}$$

and

$$2\int_{-\infty}^{\infty} d\tau(\omega_0\tau)^2\cos(\omega\tau)\frac{\sin\zeta}{\zeta} \simeq \frac{4}{\omega\beta}\int_0^{\infty}\frac{d\tau}{\tau}(\omega_0\tau)^2\left[1+\frac{\omega_0^2\tau^2}{24}\right]\cos(\omega\tau)\sin\zeta \simeq \frac{4\omega_0^2}{\omega\beta}\int_0^{\infty} d\tau\,\tau\cos(\omega\tau)\sin\zeta$$

$$= -\frac{8}{\sqrt{3}\omega\beta\gamma^2}K_{2/3}(\xi)\ . \quad \text{(B9)}$$

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[2]Ya. S. Derbenev and A. M. Kondratenko, Zh. Eksp. Teor. Fiz. **64**, 1918 (1973) [Sov. Phys.—JETP **37**, 968 (1973)].

[3]S. R. Mane, Phys. Rev. Lett. **57**, 78 (1986).

[4]J. Schwinger, Phys. Rev. **75**, 1912 (1949).

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[6]S. R. Mane, following paper, Phys. Rev. A **36**, 120 (1987). An algorithm to evaluate the formula for the equilibrium polarization, and numerical results, is presented.

[7]If the beam is initially polarized, the time evolution of the polarization vector is more complicated. The component orthogonal to $\hat{\mathbf{P}}_{eq}$ decays to zero, while the component parallel to $\hat{\mathbf{P}}_{eq}$ approaches the value P_{eq}. The time scales of these processes are in general not equal. In this paper I consider only the equilibrium polarization.

[8]The use of the above Hamiltonian for semiclassical work may be justified in various ways. Derbenev and Kondratenko state, in Ref. 2, that it follows from the Dirac Hamiltonian by a Foldy-Wouthuysen transformation to first order in $\hbar$. In J. D. Jackson, Rev. Mod. Phys. **48**, 417 (1976), it is stated that it may be obtained "via a Pauli reduction of the momentum-space matrix element of the Dirac-Pauli current (with $\sigma_{\mu\nu}$ coupling for the anomalous magentic moment) in the soft-photon limit."

[9]M. Sands, Stanford Linear Accelerator Center Report No. SLAC-121, 1970 (unpublished).

[10]L. Thomas, Philos. Mag. **3**, 1 (1927); V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Lett. **2**, 435 (1959). See also J. D. Jackson, *Classical Electrodynamics,* 2nd ed. (Wiley, New York, 1975).

[11]B. W. Montague (private communication) points out that this equation should only be called the Thomas-BMT equation when $\mathbf{\Omega}_{ext}$ is appropriately specified in terms of laboratory fields. It is otherwise a first-order precession equation, the most general in three dimensions. Many of the points made in this paper concerning spin motion in fact only use this latter property; an explicit knowledge of $\mathbf{\Omega}_{ext}$ in terms of accelerator fields is unnecessary.

[12]K. Yokoya, DESY Report No. 86-057, 1986 (unpublished). This report corrects some errors in the original Derbenev-Kondratenko argument in Ref. 2.

[13]At higher orders in $\hbar$ the spin affects the orbital motion, hence one cannot "attach" a spin trajectory to an orbital trajectory. The matter is beyond the scope of this paper.

[14]I am considering here only the simplest case: there has been much interest, with reference to the orbital dynamics, in "limit cycles," etc. Such sophistication is beyond the scope of this paper.

[15]The relevant terms in $\mathcal{H}_{int}$ are $e(\Phi_{rad}-\boldsymbol{\beta}\cdot\mathbf{A}_{rad})$ which do not involve the spin operator. Even so, we shall see below that they *can* lead to spin flip, but their main coupling is naturally nonflip. With regard to the matrix elements, ordinary synchrotron radiation is of $O(1)$, whereas the spin-dependent matrix elements are of $O(\hbar)$.

[16]In general, had we not diagonalized the unperturbed Hamiltonian, the unperturbed motion would also cause transitions between up and down spin states. The calculation of the equilibrium polarization would then be more difficult.

[17]Yokoya (private communication) states the matter thus: "In the Derbenev-Kondratenko (DK) formula $P_{eq}\propto\langle\alpha_-\rangle/\langle\alpha_+\rangle$, (the) average is taken in the numerator and denominator separately. (We shall define and study α_+ and α_- in Sec. VI. Suffice it for now to say $\langle\alpha_\pm\rangle=p_\uparrow\pm p_\downarrow$.) $\alpha_\pm$ are functions of (the) orbit action variable I. It is obviously wrong to think that P_{eq} is a function of I and is given by $P_{eq}(I)\propto\alpha_-(I)/\alpha_+(I)$. If this (were) true, I could happily carry out (a) phase-space average within the perturbation theory. On the contrary, as you know $\langle\alpha_\pm\rangle$ gives the rate of spin flip in the beam *as a whole.* There are polarization-gaining and polarization-losing areas in the phase space . . . but still P_{eq} is independent of I. If we measure P_{eq} as a function of I (possible in principle, not a polarization of one single particle), it is uniform (assumed so in the DK formula). Why? The reason is simply $\tau_{pol}\gg\tau_{orbit\ damp}$. (The) orbit action variable I changes considerably within τ_{damp} but the spin cannot catch up. The polarization (strictly $\langle\langle s_{op}\cdot\hat{\mathbf{n}}\rangle\rangle$) is shuffled well by the orbit excitation damping."

[18]At a "spin resonance" $\hat{\mathbf{n}}$ is ill defined (this may be taken as a definition of a spin resonance) and so near one $\hat{\mathbf{n}}$ is obviously a badly behaved function of $\{I_\lambda,\Psi_\lambda\}$. In that case the variation of $\hat{\mathbf{n}}$ as a function of $\{I_\lambda,\psi_\lambda\}$ may be large. In a nonresonant

situation, however, a simple numerical estimate indicates that the direction of $\hat{\mathbf{n}}$ varies by approximately a milliradian (or even less), hence $|\langle \hat{\mathbf{n}} \rangle_{I,\psi}| \simeq 1$.

[19]Derbenev and Kondratenko further approximate $\hat{\mathbf{n}}_{eq}$ by $\hat{\mathbf{n}}_0$, the value of $\hat{\mathbf{n}}$ on the equilibrium closed orbit, but this is not necessary. In a nonresonant situation, the difference between $\hat{\mathbf{n}}_{eq}$ and $\hat{\mathbf{n}}_0$ is not large.

[20]I am assuming point photon emission. The "length of formation of the radiation" (Ref. 2) is typically a few centimeters in modern machines, and $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ do not change appreciably over such an interval.

[21]Derbenev and Kondratenko introduce two vectors $\hat{\boldsymbol{\eta}}_1$ and $\hat{\boldsymbol{\eta}}_2$, defined as solutions of the Thomas-BMT equation, such that $\{\hat{\mathbf{n}},\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2\}$ is a right-handed orthonormal triad, and then they define $\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}_1+i\hat{\boldsymbol{\eta}}_2$. My definition coincides with theirs. In both cases the definition of $\boldsymbol{\eta}$ is arbitrary up to an overall phase factor, which has no physical significance. Because they treat the spin classically, Derbenev and Kondratenko in fact write $\alpha_{\pm}=(\frac{1}{4})\langle \ldots \rangle$, whereas I have an explicit factor of $\hbar^2$ in Eq. (56) below.

[22]From Ref. 4, the dimensionless variable of integration is $\gamma\omega_0\tau$, so $\omega_0\tau$ itself costs a factor of γ^{-1}.

[23]In other words, this is *before* smoothing (averaging) over the effects of photon emissions—this is not the equation for the time evolution of the degree of polarization, which is a deterministic equation.

[24]As in Sec. IV, the equilibrium polarization is taken to be the average of $\mathbf{s}\cdot\hat{\mathbf{n}}$, and the factor $\langle \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_{eq} \rangle$ is neglected.