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Factorizations of two vector operators for the Coulomb problem

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We consider the factorization of two vector constants of the motion $\mathbf{C}^{\pm}$, which were recently introduced for the Coulomb problem [O. L. de Lange and R. E. Raab, Phys. Rev. A **34**, 1650 (1986)]. The operators $\mathbf{C}^{\pm}$ are quantum-mechanical analogs of the classical conserved vectors $(1\mp i\hat{\mathbf{L}}\times)\mathbf{A}_c$ where $\mathbf{L}$ is the orbital angular momentum vector and $\mathbf{A}_c$ is the Laplace-Runge-Lenz vector. It is shown that $\mathbf{C}^{\pm}$ can be factored in two different ways to yield operators which, apart from their dependence on the constants of the motion $\mathbf{L}^2$ and H, are linear in either $\mathbf{p}$ or $\mathbf{r}$. In this way we obtain 16 abstract operators. The properties of these operators are investigated and the following observations are made: (i) Twelve are ladder operators for the quantum numbers l, and l and m, in the eigenkets $|\,lm\,\rangle$ of $\mathbf{L}^2$ and L_z. In linearized, differential form six of these operators are ladder operators for the spherical harmonics in the coordinate representation, while the other six are the corresponding operators in the momentum representation. (ii) Two operators factorize an operator related to the Hamiltonian. In linearized, differential form they are the two ladder operators for the radial part of the coordinate-space wave functions which were discovered by Schrödinger. (iii) Two operators yield a factorization which is related to Hylleraas's equation. In linearized, differential form they are ladder operators for the radial part of the momentum-space wave function.

I. INTRODUCTION

Recently we discussed two vector operators which commute with the Hamiltonian

$$H=(2M)^{-1}\mathbf{p}^2-\hbar^2(Ma)^{-1}r^{-1} \tag{1}$$

for the Coulomb problem. [$a=\hbar^2(Mk)^{-1}$ is the Bohr radius if $k=e^2(4\pi\epsilon_0)^{-1}$.] These are the operators[1]

$$\mathbf{C}^{\pm}=\mathbf{A}+i\mathbf{B}^{\pm}\ , \tag{2}$$

where

$$\mathbf{A}=\tfrac{1}{2}\hbar^{-2}a(\mathbf{L}\times\mathbf{p}-\mathbf{p}\times\mathbf{L})+r^{-1}\mathbf{r} \tag{3}$$

is the Pauli-Lenz operator,[2]

$$\mathbf{B}^{\pm}=\mathbf{F}g^{\pm}(\mathbf{L}^2)\ , \tag{4}$$

and

$$\mathbf{F}=\mathbf{A}\times\mathbf{L} = r^{-1}\mathbf{r}\times\mathbf{L}+\hbar^{-2}a\mathbf{p}\mathbf{L}^2\ . \tag{5}$$

$\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is the orbital angular momentum operator. [The operators $\mathbf{B}^{\pm}$, unlike $\mathbf{A}$, are not Hermitian. Although an Hermitian form can be derived,[1] the expressions (4) are sufficient for our purposes.]

It can be shown that if the operators $g^{\pm}$ are solutions to the equation

$$\mathbf{L}^2g^2+\hbar g-1=0\ , \tag{6}$$

that is, if

$$g^{\pm}=2\hbar^{-1}(1\pm 2S)^{-1}\ , \tag{7}$$

where

$$S=(\hbar^{-2}\mathbf{L}^2+\tfrac{1}{4})^{1/2}\ , \tag{8}$$

then $\mathbf{C}^{\pm}$ are ladder operators for the quantum numbers l, and l and m, in the eigenkets $|\,nlm\,\rangle$ for the Coulomb problem.[1] (n labels the energy.) Specifically, $C_z^{\pm}$, $C_+^{\pm}=C_x^{\pm}+iC_y^{\pm}$, and $C_-^{\pm}=C_x^{\pm}-iC_y^{\pm}$ transform $|\,nlm\,\rangle$ into $|\,n,l\pm1,m\,\rangle$, $|\,n,l\pm1,m+1\,\rangle$, and $|\,n,l\pm1,m-1\,\rangle$, respectively. Thus the superscript (subscript) + or − on $\mathbf{C}$ indicates the effect on $l\,(m)$. The operators $\mathbf{C}^{\pm}$ are quantum-mechanical analogs of the two classical conserved vectors $\mathbf{A}_c\pm i\mathbf{B}_c$, where $\mathbf{A}_c=(Mk)^{-1}\mathbf{L}\times\mathbf{p}+r^{-1}\mathbf{r}$ is the Laplace-Runge-Lenz vector[3] and $\mathbf{B}_c=\mathbf{A}_c\times\hat{\mathbf{L}}$ (k is the constant in $\mathbf{f}=-kr^{-3}\mathbf{r}$.) Apart from their dependence on a constant of the motion ($\mathbf{L}^2$) the operators $\mathbf{C}^{\pm}$ are quadratic functions of the momentum operator $\mathbf{p}$ and are also nonlinear functions of the position operator $\mathbf{r}$.

The purpose of this paper is twofold. Firstly, we show that by suitably factorizing $\mathbf{C}^{\pm}$ we obtain operators which, apart from depending on constants of the motion such as $\mathbf{L}^2$ and H, are linear functions of either $\mathbf{p}$ or $\mathbf{r}$. Thus we can derive linear operators by replacing constants of the motion with their eigenvalues. Secondly, we study the properties of the operators derived by factorization.

The factorizations are presented in Secs. II and III; they yield a total of 16 operators. The properties of these abstract operators are investigated in Sec. IV. Twelve of them are shown to be ladder operators for the quantum numbers l, and l and m, of the eigenkets $|\,lm\,\rangle$ of $\mathbf{L}^2$ and L_z. After linearization (that is, replacing constants of the motion with their eigenvalues), six of these operators are linear in $\mathbf{p}$, the other six are linear in $\mathbf{r}$. The wave-mechanical versions of the former are ladder operators for the spherical harmonics in the coordinate representation, and of the latter are the corresponding operators in the momentum representation. Four of these differential operators are related to those derived by Infeld and Hull[4] in their study of factorization of the Sturm-Liouville equation.

Two of the remaining four operators are linear in $\mathbf{p}$, apart from their dependence on $\mathbf{L}^2$. They factorize the operator $2MH+\hbar^2a^{-2}(S\pm\frac{1}{2})^{-2}$ [Eq. (44)]. Linearization yields operators which factorize the radial Hamiltonian [Eqs. (48) and (49)] and which are ladder operators for the quantum number l in the eigenkets of this Hamiltonian. The corresponding wave-mechanical forms are the ladder operators for the radial part of the coordinate-space wave functions which were discovered by Schrödinger.[5]

Finally, there are two operators which, apart from their dependence on $\mathbf{L}^2$ and H, are linear in $\mathbf{r}$. They factorize the operator $\hbar^{-2}\mathbf{r}^2(\mathbf{p}^2-2MH)^2+(2MH)(2S\pm1)^2$ [Eq. (52)]. Linearization yields operators which factorize the radial Hylleraas operator [Eqs. (60) and (61)] and which are ladder operators for the quantum number l in the eigenkets of this radial operator [Eq. (63)]. The corresponding wave-mechanical forms are ladder operators for the radial part of the momentum-space wave functions. As such they provide a simple method for calculating these functions.

II. FACTORIZATIONS YIELDING OPERATORS LINEAR IN p

We state the result of the first of the two factorizations of $\mathbf{C}^{\pm}$ and then outline the proof;

$$\mathbf{C}^{\pm}=\hbar^{-2}a\mathbf{U}^{\pm}R^{\pm}\ , \tag{9}$$

from which it follows that

$$C_z^{\pm}=\hbar^{-2}aU_z^{\pm}R^{\pm}\ , \tag{10}$$

$$C_+^{\pm}=\hbar^{-2}aU_+^{\pm}R^{\pm}\ , \tag{11}$$

$$C_-^{\pm}=\hbar^{-2}aU_-^{\pm}R^{\pm}\ . \tag{12}$$

Here

$$\mathbf{U}^{\pm}=\pm ir^{-1}\mathbf{r}\times\mathbf{L}+\hbar r^{-1}\mathbf{r}(S\pm\tfrac{1}{2})\ , \tag{13}$$

$$R^{\pm}=\pm ir^{-1}\mathbf{r}\cdot\mathbf{p}-\hbar(S\mp\tfrac{1}{2})r^{-1}+\hbar a^{-1}(S\pm\tfrac{1}{2})^{-1}\ , \tag{14}$$

and S is given by Eq. (8).

To prove Eq. (9) we first write out Eq. (13)

$$\mathbf{U}^{\pm}=\pm ir^{-1}\mathbf{r}(\mathbf{r}\cdot\mathbf{p})\mp ir\mathbf{p}+\hbar r^{-1}\mathbf{r}(S\pm\tfrac{1}{2})\ .$$

Now S is a function of $\mathbf{L}^2$ and therefore commutes with all the operators in $R^{\pm}$ in Eq. (14). Multiplying out the product $\mathbf{U}^{\pm}R^{\pm}$, then using the commutators

$$[\mathbf{r}\cdot\mathbf{p},r^{-1}]=i\hbar r^{-1} \tag{15}$$

and

$$[\mathbf{p},r^{-1}]=i\hbar r^{-3}\mathbf{r}\ , \tag{16}$$

and collecting terms we find

$$\begin{aligned}\mathbf{U}^{\pm}R^{\pm}=\{&-r^{-2}\mathbf{r}[(\mathbf{r}\cdot\mathbf{p})^2-i\hbar\mathbf{r}\cdot\mathbf{p}+\mathbf{L}^2]+\mathbf{p}(\mathbf{r}\cdot\mathbf{p})\\&+\hbar^{-2}a^{-1}r^{-1}\mathbf{r}\}\\&+i\hbar^2a^{-1}[r^{-1}\mathbf{r}(\mathbf{r}\cdot\mathbf{p})-r\mathbf{p}\\&+\hbar^{-2}a\mathbf{p}\mathbf{L}^2][\pm\hbar(S\pm\tfrac{1}{2})]^{-1}\ .\end{aligned} \tag{17}$$

Use of the identity

$$\mathbf{L}^2=\mathbf{r}^2\mathbf{p}^2-(\mathbf{r}\cdot\mathbf{p})^2+i\hbar\mathbf{r}\cdot\mathbf{p} \tag{18}$$

and Eq. (3) shows that the term in curly brackets in Eq. (17) is equal to $\hbar^2a^{-1}\mathbf{A}$. Comparison with Eqs. (4) and (5) shows that the second term in Eq. (17) is equal to $i\hbar^2a^{-1}\mathbf{B}^{\pm}$. This proves Eq. (9).

A noteworthy feature of the abstract operators (13) and (14) is that apart from their dependence on $\mathbf{L}^2$ they are linear in p. It is natural to inquire whether $\mathbf{C}^{\pm}$ can also be factored to yield operators which, apart from any dependence on constants of the motion, are linear in $\mathbf{r}$. This is considered below.

III. FACTORIZATIONS YIELDING OPERATORS LINEAR IN r

Again we state the results and then outline the proof;

$$\mathbf{C}^{\pm}=i\tfrac{1}{2}\hbar^{-1}a\mathbf{V}^{\pm}P^{\pm}g^{\pm}\ , \tag{19}$$

from which it follows that

$$C_z^{\pm}=i\tfrac{1}{2}\hbar^{-1}aV_z^{\pm}P^{\pm}g^{\pm}\ , \tag{20}$$

$$C_+^{\pm}=i\tfrac{1}{2}\hbar^{-1}aV_+^{\pm}P^{\pm}g^{\pm}\ , \tag{21}$$

$$C_-^{\pm}=i\tfrac{1}{2}\hbar^{-1}aV_-^{\pm}P^{\pm}g^{\pm}\ . \tag{22}$$

Here

$$\mathbf{V}^{\pm}=\pm ip^{-1}\mathbf{p}\times\mathbf{L}+\hbar p^{-1}\mathbf{p}(S\pm\tfrac{1}{2})\ , \tag{23}$$

$$\begin{aligned}P^{\pm}=&\mp i\hbar^{-1}p^{-1}\mathbf{r}\cdot\mathbf{p}(\mathbf{p}^2-2MH)+p^{-1}(\mathbf{p}^2+2MH)(S\pm\tfrac{1}{2})\\&\mp2p^{-1}(\mathbf{p}^2-2MH)\ ,\end{aligned} \tag{24}$$

$g^{\pm}$ are given by Eq. (7) and $p=(\mathbf{p}^2)^{1/2}$.

The proof of these results is more cumbersome than that in Sec. II. It is useful to note that S, being a function of L^2, commutes with all the operators in $P^\pm$ in Eq. (24). To prove Eq. (19) consider first the product of $\mathbf{V}^\pm$ and the second term on the right-hand side of Eq. (24). Writing $\mathbf{p}^2+2MH=2\mathbf{p}^2-2\hbar^2a^{-1}r^{-1}$, multiplying out, and using Eqs. (2)—(5) yields

$$i\hbar^{-1}a\mathbf{V}^\pm p^{-1}(\mathbf{p}^2+2MH)(S\pm\tfrac{1}{2}) =2(\mathbf{C}^\pm-r^{-1}\mathbf{r}\mp i\mathbf{V}^\pm p^{-1}r^{-1})(g^\pm)^{-1}-2ir^{-1}\mathbf{r}\times\mathbf{L}\ . \tag{25}$$

Solving Eq. (25) for $\mathbf{C}^\pm$ and using Eqs. (24) and (7) we find

$$\mathbf{C}^\pm=i\tfrac{1}{2}\hbar^{-1}a\mathbf{V}^\pm P^\pm g^\pm+\mathbf{\Delta}^\pm g^\pm\ , \tag{26}$$

where

$$\mathbf{\Delta}^\pm=(r^{-1}\mathbf{r}\pm i\mathbf{V}^\pm p^{-1}r^{-1})(g^\pm)^{-1}+ir^{-1}\mathbf{r}\times\mathbf{L} \mp\tfrac{1}{2}\hbar^{-2}a\mathbf{V}^\pm p^{-1}(\mathbf{r}\cdot\mathbf{p}-2i\hbar)(\mathbf{p}^2-2MH)\ . \tag{27}$$

If in Eq. (27) we substitute

$$\mathbf{V}^\pm=\mp ip^{-1}\mathbf{p}(\mathbf{r}\cdot\mathbf{p})\pm ip\mathbf{r}+\hbar p^{-1}\mathbf{p}(S\mp\tfrac{5}{2})\ ,$$

$$\mathbf{r}\times\mathbf{L}=\mathbf{r}(\mathbf{r}\cdot\mathbf{p})-\mathbf{r}^2\mathbf{p}\ ,$$

$$\mathbf{p}^2-2MH=2\hbar^2a^{-1}r^{-1}\ ,$$

and use Eqs. (7), (15), (16), and (18) and the commutators

$$[r,\mathbf{p}]=i\hbar r^{-1}\mathbf{r}\ ,$$

$$[r,\mathbf{p}^2]=2i\hbar r^{-1}\mathbf{r}\cdot\mathbf{p}+2\hbar^2r^{-1}\ ,$$

some manipulation yields $\mathbf{\Delta}^\pm=0$. This proves Eq. (19). Apart from their dependence on the constants of the motion $\mathbf{L}^2$ and H, the abstract operators (23) and (24) are linear functions of $\mathbf{r}$.

IV. INTERPRETATION AND DISCUSSION

A. The operators $\mathbf{U}^\pm$ and $\mathbf{V}^\pm$

Consider first the operators $\mathbf{U}^\pm$ and $\mathbf{V}^\pm$ defined by Eqs. (13) and (23). It is straightforward to show that

$$[L_z,W_z^\pm]=0\ , \tag{28}$$

$$[L_z,W_+^\pm]=\hbar W_+^\pm\ , \tag{29}$$

$$[L_z,W_-^\pm]=-\hbar W_-^\pm\ , \tag{30}$$

$$[\mathbf{L}^2,\mathbf{W}^\pm]=\pm2\hbar^2\mathbf{W}^\pm(S\pm\tfrac{1}{2})\ , \tag{31}$$

where $W=U$ or V. If $|\,lm\,\rangle$ is an eigenket of the operators $\mathbf{L}^2$ and L_z, it follows from Eqs. (28)—(31) and the eigenvalue equation

$$S\,|\,lm\,\rangle=(l+\tfrac{1}{2})\,|\,lm\,\rangle \tag{32}$$

that

$$W_z^\pm\,|\,lm\,\rangle=\hbar\beta^\pm\,|\,l\pm1,m\,\rangle\ , \tag{33}$$

$$W_+^\pm\,|\,lm\,\rangle=\hbar\gamma^\pm\,|\,l\pm1,m+1\,\rangle\ , \tag{34}$$

$$W_-^\pm\,|\,lm\,\rangle=\hbar\delta^\pm\,|\,l\pm1,m-1\,\rangle\ , \tag{35}$$

where the factors β, γ, and δ commute with $\mathbf{L}^2$ and L_z. These factors can be evaluated, either by using the Wigner-Eckart theorem or from first principles. The results are

$$|\,\beta^\pm\,|^2=(l-m+\tfrac{1}{2}\pm\tfrac{1}{2})(l+m+\tfrac{1}{2}\pm\tfrac{1}{2})a_l^\pm\ , \tag{36}$$

$$|\,\gamma^\pm\,|^2=(l\pm m+\tfrac{1}{2}\pm\tfrac{1}{2})(l\pm m+\tfrac{1}{2}\pm\tfrac{3}{2})a_l^\pm\ , \tag{37}$$

$$|\,\delta^\pm\,|^2=(l\mp m+\tfrac{1}{2}\pm\tfrac{1}{2})(l\mp m+\tfrac{1}{2}\pm\tfrac{3}{2})a_l^\pm\ , \tag{38}$$

where $a_l^\pm=(2l+1)(2l+1\pm2)^{-1}$. Thus we find that the twelve operators $W_z^\pm$, $W_+^\pm$, and $W_-^\pm$ ($W=U$ or V) are ladder operators for the quantum numbers l, and l and m, of the eigenkets $|\,lm\,\rangle$.

If we replace the operator S by its eigenvalues $l+\frac{1}{2}$ in Eq. (13), we obtain the ladder operators derived by Corben and Schwinger.[6] It is useful to express these linear operators in spherical polar forms using the canonical conjugates p_θ,θ and p_ϕ,ϕ. The latter obey the fundamental commutation rules that a member of one pair commutes with each member of the other pair and also that $[p_\theta,\Theta(\theta)]=-i\hbar\partial\Theta/\partial\theta$ and $[p_\phi,\Phi(\phi)]=-i\hbar\partial\Phi/\partial\phi$, where Θ and Φ are arbitrary functions of θ and ϕ, respectively. It can be shown that[7] in terms of p_θ and p_ϕ

$$\mathbf{L}=(r\sin\theta)^{-1}[-\mathbf{e}_\theta p_\theta+\mathbf{e}_\phi(p_\theta+i\tfrac{1}{2}\hbar\cot\theta)] \tag{39}$$

and

$$L_z=p_\phi\ . \tag{40}$$

Using these expressions and $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, and $z=r\cos\theta$ in Eq. (13) with S replaced by $l+\frac{1}{2}$ we find the spherical polar forms

$$U_z^\pm=\pm i(\sin\theta)p_\theta+\hbar(l+\tfrac{1}{2})\cos\theta\ , \tag{41}$$

$$U_+^\pm=\mp e^{i\phi}[i(\cos\theta)p_\theta\mp\hbar(l+\tfrac{1}{2})\sin\theta -(p_\phi+\tfrac{1}{2}\hbar)(\sin\theta)^{-1}]\ , \tag{42}$$

$$U_-^\pm=\pm e^{-i\phi}[-i(\cos\theta)p_\theta\pm\hbar(l+\tfrac{1}{2})\sin\theta -(p_\phi-\tfrac{1}{2}\hbar)(\sin\theta)^{-1}]\ . \tag{43}$$

From Eqs. (41)—(43) we can write down wave-mechanical operators by substituting[7] $p_\theta=-i\hbar(\frac{1}{2}\cot\theta+\partial/\partial\theta)$ and $p_\phi=-i\hbar\partial/\partial\phi$. The resulting differential operators are ladder operators for the quantum numbers l and m of spherical harmonics in the coordinate representation. The differential forms of $U_z^\pm$ which follow from Eq. (41) are closely related to the operators derived by Infeld and Hull[4] in their study of factorization of the Sturm-Liouville equation.

Similar results apply to the operators linear in $\mathbf{r}$, Eq. (23). The linearized, differential forms of $V_z^\pm$, $V_+^\pm$, and $V_-^\pm$ are the same as the corresponding operators which follow from Eqs. (41)—(43), except that now θ and ϕ are the angular coordinates of $\mathbf{p}$ rather than $\mathbf{r}$. These differential operators are ladder operators for the quantum numbers l and m of spherical harmonics in the momentum representation.

B. The operators $R^{\pm}$

It is straightforward to show that

$$(R^{\pm})^{\dagger}R^{\pm}=2MH+\hbar^2a^{-2}(S\pm\tfrac{1}{2})^{-2}\ , \tag{44}$$

where $R^{\pm}$ are given by Eq. (14), H by Eq. (1), S by Eq. (8) and $\dagger$ denotes the adjoint operator. Thus the operators $R^{\pm}$ factorize the right-hand side of Eq. (44). One can linearize $R^{\pm}$ by replacing S with its eigenvalues $l+\frac{1}{2}$ in Eq. (14). This yields the abstract operators

$$R_l^{\pm}=\pm ip_r-\hbar(l+\tfrac{1}{2}\pm\tfrac{1}{2})r^{-1}+\hbar a^{-1}(l+\tfrac{1}{2}\pm\tfrac{1}{2})^{-1}\ , \tag{45}$$

where the operator

$$p_r=r^{-1}\mathbf{r}\cdot\mathbf{p}-i\hbar r^{-1} \tag{46}$$

is the canonical conjugate of r. From Eq. (45) one can show that the adjoint operators are given by

$$(R_l^{\pm})^{\dagger}=R_{l\pm 1}^{\mp}\ . \tag{47}$$

Replacing S with $l+\frac{1}{2}$ in Eq. (44) and using Eq. (47) we find

$$R_{l\pm 1}^{\mp}R_l^{\pm}=2MH_l+\hbar^2a^{-2}(l+\tfrac{1}{2}\pm\tfrac{1}{2})^{-2}\ . \tag{48}$$

Here

$$H_l=(2M)^{-1}[p_r^2+\hbar^2l(l+1)r^{-2}-2\hbar^2a^{-1}r^{-1}] \tag{49}$$

is the radial Hamiltonian obtained from Eq. (1) by using the identity $\mathbf{L}^2=\mathbf{r}^2\mathbf{p}^2-\mathbf{r}^2p_r^2$ to eliminate $\mathbf{p}^2$ in favor of p_r^2 and then replacing $\mathbf{L}^2$ with its eigenvalues. The factorization (48) has been discussed by Newmarch and Golding.[8] If $|\,nl\,\rangle$ denotes a bound-state eigenket of H_l,

$$H_l\,|\,nl\,\rangle=-\hbar^2(2Ma^2n^2)^{-1}\,|\,nl\,\rangle\ , \tag{50}$$

it follows from Eqs. (47), (48), and (50) that

$$R_l^{\pm}\,|\,nl\,\rangle=\hbar a^{-1}[(l+\tfrac{1}{2}\pm\tfrac{1}{2})^{-2}-n^{-2}]^{1/2}\,|\,n,l\pm 1\,\rangle\ . \tag{51}$$

[In Eq. (51) an overall phase factor has been set equal to $+1$.]

One can obtain wave-mechanical operators by substituting

$$p_r=-i\hbar r^{-1}-i\hbar\partial/\partial r$$

in Eq. (45). The resulting differential operators were first presented long ago by Schrödinger;[5] the solutions to the wave-mechanical form of Eq. (51) are the familiar radial coordinate-space wave functions for the bound states of the Coulomb problem.[9]

C. The operators $P^{\pm}$

The properties of the operators $P^{\pm}$ can be analyzed in a similar manner to those of $R^{\pm}$. Starting from Eq. (24) one can show that

$$(P^{\pm})^{\dagger}P^{\pm}=\hbar^{-2}\mathbf{r}^2(\mathbf{p}^2-2MH)^2+(2MH)(2S\pm 1)^2\ , \tag{52}$$

which is the analog of the factorization in Eq. (44).

From Eq. (24) one can obtain operators $P_E^{\pm}$ by replacing H with its eigenvalue E:

$$P_E^{\pm}=\mp i\hbar^{-1}p^{-1}\mathbf{r}\cdot\mathbf{p}(\mathbf{p}^2-2ME)+p^{-1}(\mathbf{p}^2+2ME)(S\pm\tfrac{1}{2})\mp 2p^{-1}(\mathbf{p}^2-2ME)\ . \tag{53}$$

Then it can be shown that

$$[(P_E^{\pm})^{\dagger}\mp 2p]P_E^{\pm}=\hbar^{-2}\Lambda_E+(2ME)(2S\pm 1)^2\ , \tag{54}$$

where

$$\Lambda_E=\mathbf{r}^2(\mathbf{p}^2-2ME)^2-2i\hbar\mathbf{p}\cdot\mathbf{r}(\mathbf{p}^2-2ME)+4\hbar^2(\mathbf{p}^2-2ME)\ . \tag{55}$$

The action of the operator Λ_E on a ket $|\,nlm\,\rangle$ is the same as that of $\mathbf{r}^2(\mathbf{p}^2-2MH)^2$, as is evident from expanding the latter and using

$$\mathbf{r}^2[H,\mathbf{p}^2]=2i\hbar^3(Ma)^{-1}(\mathbf{p}\cdot\mathbf{r}r^{-1}+2i\hbar r^{-1})\ .$$

Thus, noting Eq. (1), it follows that

$$\Lambda_E\,|\,nlm\,\rangle=4\hbar^4a^{-2}\,|\,nlm\,\rangle\ , \tag{56}$$

an equation which was first deduced by Hylleraas.[10]

We can obtain operators $P_{El}^{\pm}$ which are linear in $\mathbf{r}$ by replacing both H and S with their eigenvalues in Eq. (24),

$$P_{El}^{\pm}=\mp i\hbar^{-1}r_p(\mathbf{p}^2-2ME)+(l+\tfrac{1}{2}\pm\tfrac{1}{2})p^{-1}(\mathbf{p}^2+2ME)\ , \tag{57}$$

where the operator

$$r_p=p^{-1}\mathbf{p}\cdot\mathbf{r}+i\hbar p^{-1} \tag{58}$$

is the canonical conjugate of p. From Eq. (57) one can show that the adjoint operators are given by

$$(P_{El}^{\pm})^{\dagger}=P_{E,l\pm 1}^{\mp}\pm 2p\ , \tag{59}$$

which differs from the corresponding result for the operators $R_l^{\pm}$ in Eq. (47). The operators (57) provide the factorization

$$P_{E,l\pm 1}^{\mp}P_{El}^{\pm}=\hbar^{-2}\Lambda_{El}+(2l+1\pm 1)^2(2ME)\ , \tag{60}$$

where

$$\Lambda_{El}=[r_p^2+p^{-2}\hbar^2l(l+1)](\mathbf{p}^2-2ME)^2-2i\hbar\mathbf{p}\cdot\mathbf{r}(\mathbf{p}^2-2ME)+4\hbar^2(\mathbf{p}^2-2ME)\ . \tag{61}$$

Equations (60) and (61) follow directly from Eqs. (54) and (55) if we use the identity $\mathbf{L}^2=\mathbf{p}^2\mathbf{r}-\mathbf{p}^2r_p^2$ to eliminate $\mathbf{r}^2$ in favor of r_p^2 in Eq. (55), then replace $\mathbf{L}^2$ and S by their eigenvalues and use Eq. (59). It is clear that the manner in which the radial operator Λ_{El} [Eq. (61)] is obtained from the operator Λ_E [Eq. (55)] is analogous to the way in which the radial Hamiltonian H_l [Eq. (49)] follows from the Hamiltonian [Eq. (1)].

Let $|\,nl\,\rangle$ denote a bound-state eigenket of the operator Λ_{El} with $E=-\hbar^2(2Ma^2n^2)^{-1}$,

$$\Lambda_{El}\,|\,nl\,\rangle=4\hbar^4a^{-2}\,|\,nl\,\rangle\ . \tag{62}$$

[For notational convenience we do not distinguish between the eigenkets of Eqs. (50) and (62).] Then it follows from Eqs. (60) and (62) that

TABLE I. Summary of operators and their matrix elements.

	Operator	Matrix elements[a]
1.	$\mathbf{W}^{\pm}=\pm i\mathbf{n}\times\mathbf{L}+\hbar\mathbf{n}(S\pm\frac{1}{2})$ (If $\mathbf{n}=r^{-1}\mathbf{r}$, $\mathbf{W}=\mathbf{U}$ [Eq. (13)]; if $\mathbf{n}=p^{-1}\mathbf{p}$, $\mathbf{W}=\mathbf{V}$ [Eq. (23)].)	$\langle l'm' \mid W_z^{\pm} \mid lm\rangle=\hbar[(l-m+\frac{1}{2}\pm\frac{1}{2})(l+m+\frac{1}{2}\pm\frac{1}{2})a_l^{\pm}]^{1/2}\delta_{l',l\pm1}\delta_{m'm}$ $\langle l'm' \mid W_+^{\pm} \mid lm\rangle=\hbar[(l\pm m+\frac{1}{2}\pm\frac{1}{2})(l\pm m+\frac{1}{2}\pm\frac{3}{2})a_l^{\pm}]^{1/2}\delta_{l',l\pm1}\delta_{m',m+1}$ $\langle l'm' \mid W_-^{\pm} \mid lm\rangle=\hbar[(l\mp m+\frac{1}{2}\pm\frac{1}{2})(l\mp m+\frac{1}{2}\pm\frac{3}{2})a_l^{\pm}]^{1/2}\delta_{l',l\pm1}\delta_{m',m-1}$ $a_l^{\pm}=(2l+1)(2l+1\pm2)^{-1}$
2.	$R_l^{\pm}=\pm ip_r-\hbar(l+\frac{1}{2}\pm\frac{1}{2})r^{-1}+\hbar a^{-1}(l+\frac{1}{2}\pm\frac{1}{2})^{-1}$ [Eq. (45)]	$\langle nl' \mid R_l^{\pm} \mid nl\rangle=\hbar a^{-1}[(l+\frac{1}{2}\pm\frac{1}{2})^{-2}-n^{-2}]^{1/2}\delta_{l',l\pm1}$
3.	$P_{El}^{\pm}=\mp i\hbar^{-1}r_p(\mathbf{p}^2-2ME)+(l+\frac{1}{2}\pm\frac{1}{2})p^{-1}(\mathbf{p}^2+2ME)$ [Eq. (57)]	$\langle nl' \mid P_{El}^{\pm} \mid nl\rangle=2\hbar a^{-1}[1-(l+\frac{1}{2}\pm\frac{1}{2})^2n^{-2}]^{1/2}\delta_{l',l\pm1}$

[a] (i) Phase factors have been omitted. (ii) In the matrix elements for $R_l^{\pm}$, $\mid nl\rangle$ denotes a bound-state eigenket of H_l [Eq. (50)]. In the matrix elements for $P_{El}^{\pm}$, $\mid nl\rangle$ denotes a bound-state eigenket of Λ_{El} [Eq. (62)].

$$P_{El}^{\pm} \mid nl\rangle=\alpha_{nl}^{\pm} \mid n,l\pm1\rangle \ . \qquad (63)$$

The coefficients in Eq. (63) can be evaluated by a straightforward calculation. The results are

$$\alpha_{nl}^{\pm}=2\hbar a^{-1}[1-(l+\tfrac{1}{2}\pm\tfrac{1}{2})^2n^{-2}]^{1/2} \ , \qquad (64)$$

where an overall phase factor has been set equal to + 1.

The wave-mechanical versions of the abstract operators $P_{El}^{\pm}$ are found by putting $r_p=i\hbar p^{-1}+i\hbar\partial/\partial p$ in Eq. (57),

$$P_{El}^{\pm}=\pm(\mathbf{p}^2-2ME)\frac{\partial}{\partial p}+(l+\tfrac{1}{2}\pm\tfrac{7}{2})p+(l+\tfrac{1}{2}\mp\tfrac{1}{2})(2ME)p^{-1} \ . \qquad (65)$$

If Eqs. (64) and (65) are substituted in Eq. (63), the resulting first-order differential equations can be readily solved. The normalized wave functions obtained in this manner are

$$\mid nl\rangle=[\tfrac{1}{2}(\hbar^{-1}an)^3n(n-l-1)!/(n+l)!]^{1/2}\times(1-z)^2(1-z^2)^{l/2}T_{n-l-1}^{l+1/2}(z) \ , \qquad (66)$$

where

$$z=(n^2\mathbf{p}^2-\hbar^2a^{-2})(n^2\mathbf{p}^2+\hbar^2a^{-2})^{-1}$$

and $T_\alpha^\beta(z)$ denotes a Gegenbauer polynomial.[11] These are just the radial momentum-space wave functions for the bound states of the Coulomb problem. They were first calculated by Fourier-transforming the coordinate-space wave functions.[12] Momentum-space wave functions have also been derived by solving Hylleraas's equation (56) in spherical polar coordinates[10] and in toroidal coordinates,[13] and by solving an integral equation for the Coulomb problem.[14]

We conclude by giving a summary of the 16 operators derived in this paper, together with their matrix elements, in Table I.

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