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Shifted $1/N$ expansion for the Hulthén potential

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The shifted $1/N$ expansion method is applied to the Hulthén potential. Numerical results for the energy values for $n=1$–5 states are presented. For $l=0$ states, the agreement between our results and the exact analytical solution is excellent.

Large-N expansion approximations have recently received much attention[1,2] because they offer the possibility of analytically handling interactions which need not be small. In the context of the nonrelativistic Schrödinger equation, it is interesting to note that for spherically symmetric potentials N and l always appear together in the form $k \equiv N+2l$, N being the number of spatial dimensions and l the eigenvalue of the N-dimensional orbital angular momentum. For the power-law potential $V(r)=Ar^{\nu}$, systematic and analytic procedures for finding the energy eigenvalue have been developed to successive orders in the parameter $1/k$. At the end of the calculation N is set equal to the physical value 3. Since $1/k \simeq \frac{1}{3}$ is not a very small expansion parameter, to obtain accurate results with perturbation theory one must calculate many orders, each order getting progressively much more complicated.

To improve series convergence, Sukhatme, Imbo, and Pagnamenta[3,4] have recently constructed a modified version of the $1/N$ expansion for the Schrödinger equation. This method, called the shifted $1/N$ expansion, uses the quantity $1/\bar{k}$ as an expansion parameter, where $\bar{k}=k-a=N+2l-a$, and a is a suitable shift and a negative number. In a subsequent paper,[5] they applied the method to the Yukawa potential $V(r)=-Ae^{-\beta r}/r$. In comparison with the "exact" numerical results, the shifted $1/N$ expansion scheme indeed surpasses all other currently available approximation procedures "for its simplicity, accuracy, and wide range of applicability."

In this paper we study the Hulthén potential via the shifted $1/N$ expansion technique. The Hulthén potential is exactly solvable only for the $l=0$ states, thereby providing an excellent check for the method; such checks are important because they give us confidence in the results of $l \neq 0$ states. Atomic units will be used unless otherwise stated.

The Hulthén potential[6]

$$V(r)=-\frac{\delta e^{-\delta r}}{1-e^{-\delta r}} \tag{1}$$

is of considerable interest and importance in atomic physics. In Eq. (1) δ, a constant, is called the screening parameter. To the best of our knowledge, the exact analytic energy eigenvalues are available only for $l=0$ states[6–8] and are given by

$$E_{n0}^{\text{exact}}=-\frac{\delta^2}{2}\left[\frac{2/\delta-n^2}{2n}\right]^2, \tag{2}$$

with

$$n^2<\frac{2}{\delta}, \tag{3}$$

which means that the number of bound states is limited. For very small screening parameter δ, the Hulthén potential reduces to the Coulomb potential $1/r$, and the corresponding energy eigenvalues, Eq. (2), reduce to

$$E_n^{\text{Coulomb}}=-\frac{1}{2n^2}, \tag{4}$$

which is the Coulomb energy level, as expected.

We refer the reader to Refs. 4 and 5 for the detailed derivation of the formulas needed in this work. (Our notation follows that of these two references which will be referred to as IPS1 (Ref. 4) and IPS2,[5] respectively; IPS1 Eq. (4) means Eq. (4) of IPS1.)

The energy eigenvalues are given by IPS2 Eq. (3) as an expression in powers of $1/\bar{k}$,

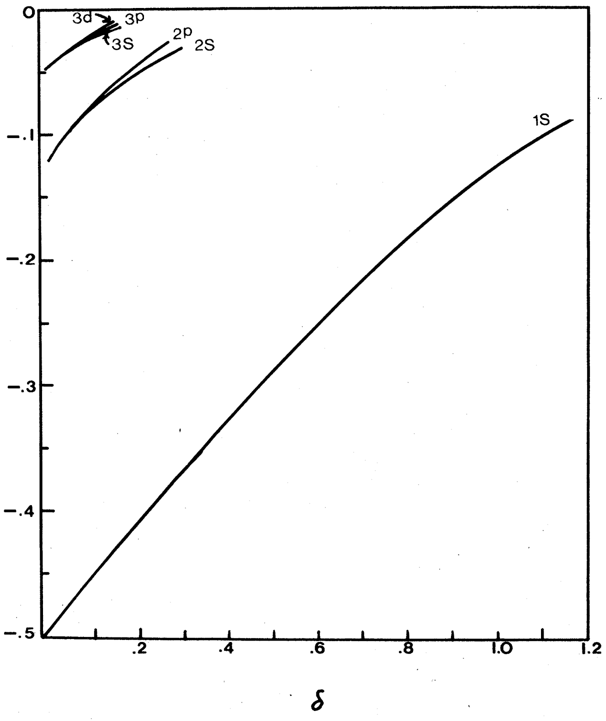

FIG. 1. Energy eigenvalues for $n=1$–3 Hulthén states as a function of δ. As $\delta \to 0$, curves for different l states (same n) approach each other owing to the accidental symmetry of the Coulomb potential.

TABLE I. Energy eigenvalues vs δ for $n=1$–5 Hulthén states. $E^{(0)}=(\bar{K}/r_0)^2[\frac{1}{8}+r_0^2V(r_0)/Q]$ is the leading contribution and the energy $E^{(2)}=(\bar{K}/r_0)^2\{[\frac{1}{8}+r_0^2V(r_0)/Q]+\Gamma^{(1)}/\bar{K}^2+\Gamma^{(2)}/\bar{K}^3\}$ includes two corrections in the $1/\bar{k}$ expansion. Note that for small δ our results are typically accurate to one part in 10^{14}–10^{9}.

States	δ	$-E^{(0)}$	$-E^{(2)}$	$-E_{\text{exact}}$ [Eq. (2)]
(1) 1s ($n_r=l=0$)	0.002	0.499 000 5	0.499 000 5	0.499 000 5
	0.01	0.495 012 5	0.495 012 5	0.495 012 5
	0.05	0.475 312 5	0.475 312 5	0.475 312 5
	0.2	0.404 998 1	0.404 996 2	0.405 000 0
	0.5	0.281 196 9	0.281 247 2	0.281 250 0
	1.00	0.123 488 7	0.124 816 0	0.125 000 0
	1.2	0.076 588 9	0.079 562 2	0.080 000 0
(2) 2s ($n_r=1, l=0$)	0.002	0.124 002 3	0.124 002 0	0.124 002 0
	0.01	0.120 058 3	0.120 050 0	0.120 050 0
	0.05	0.101 455 9	0.101 250 3	0.101 250 0
	0.1	0.080 794 0	0.080 004 6	0.080 000 0
	0.2	0.047 697 4	0.045 085 6	0.045 000 0
(3) 2p ($n_r=0, l=1$)	0.002	0.124 001 7	0.124 001 7	
	0.01	0.120 041 7	0.120 041 7	
	0.02	0.115 166 7	0.115 166 7	
	0.05	0.101 042 3	0.101 042 4	
	0.1	0.079 176 5	0.079 179 4	
	0.2	0.041 827 9	0.041 885 7	
(4) 3s ($n_r=2, l=0$)	0.002	0.054 561 1	0.054 560 1	0.054 560 1
	0.01	0.050 693 0	0.050 668 1	0.050 668 1
	0.02	0.046 104 6	0.046 005 7	0.046 005 6
	0.05	0.033 955 6	0.033 374 6	0.033 368 1
	0.1	0.018 713 0	0.016 927 4	0.016 805 6
(5) 3p ($n_r=1, l=1$)	0.002	0.054 560 06	0.054 559 72	
	0.01	0.050 668 05	0.050 659 73	
	0.02	0.046 005 42	0.045 972 36	
	0.05	0.033 362 26	0.033 165 18	
	0.1	0.016 699 58	0.016 067 72	
(6) 3D ($n_r=0, l=2$)	0.002	0.054 559 1	0.054 559 1	
	0.01	0.050 643 1	0.050 643 1	
	0.02	0.045 905 8	0.045 905 8	
	0.05	0.032 752 1	0.032 753 2	
	0.1	0.014 460 9	0.014 484 2	
(7) 4s ($n_r=3, l=0$)	0.002	0.030 260 0	0.030 258 0	0.030 258 0
	0.01	0.026 499 6	0.026 450 1	0.026 450 0
	0.02	0.022 243 9	0.022 051 2	0.022 050 0
	0.05	0.012 264 5	0.011 303 5	0.011 250 0
(8) 4p ($n_r=2, l=1$)	0.002	0.030 258 7	0.030 257 7	
	0.01	0.026 466 5	0.026 441 7	
	0.02	0.022 114 4	0.022 017 4	
	0.05	0.011 574 5	0.011 072 5	
(9) 4D ($n_r=1, l=2$)	0.002	0.030 257 3	0.030 257 0	
	0.01	0.026 433 4	0.026 425 1	
	0.02	0.021 983 6	0.021 951 0	
	0.05	0.010 842 0	0.010 669 0	
(10) 4F ($n_r=0, l=3$)	0.002	0.030 256 0	0.030 256 0	
	0.01	0.026 400 1	0.026 400 1	
	0.02	0.021 851 4	0.021 851 5	
	0.05	0.010 057 5	0.010 062 0	

TABLE I. (*Continued.*)

States	δ	$-E^{(0)}$	$-E^{(2)}$	$-E_{\text{exact}}$ [Eq. (2)]
(11) $5s$ ($n_r=4, l=0$)	0.001	0.019 504 0	0.019 503 1	0.019 503 1
	0.005	0.017 598 9	0.017 578 1	0.017 578 1
	0.01	0.015 394 3	0.015 312 8	0.015 312 5
	0.02	0.011 558 3	0.011 255 4	0.011 250 0
	0.025	0.009 913 2	0.094 669 4	0.094 531 3
(12) $5p$ ($n_r=3, l=1$)	0.002	0.019 014 2	0.019 012 2	
	0.01	0.015 353 4	0.015 304 3	
	0.02	0.011 404 1	0.011 219 7	
	0.025	0.009 682 6	0.009 408 7	
(13) $5D$ ($n_r=2, l=2$)	0.002	0.019 012 5	0.019 011 5	
	0.01	0.015 312 3	0.015 287 7	
	0.02	0.011 246 8	0.011 153 3	
	0.025	0.009 445 0	0.009 305 0	
(14) $5F$ ($n_r=1, l=3$)	0.002	0.019 010 8	0.019 010 5	
	0.01	0.015 271 0	0.015 262 8	
	0.02	0.011 086 2	0.011 054 7	
	0.025	0.009 199 7	0.009 152 3	
(15) $5G$ ($n_r=0, l=4$)	0.002	0.019 009 2	0.019 009 2	
	0.01	0.015 229 5	0.015 229 5	
	0.02	0.010 922 1	0.010 922 4	
	0.025	0.008 945 9	0.008 946 5	

$$E_{n_r,l}=\left[\frac{\bar{K}}{r_0}\right]^2\left[\left[\frac{1}{8}+r_0^2V(r_0)/Q\right]+\frac{\Gamma^{(1)}}{\bar{K}^2}+\frac{\Gamma^{(2)}}{\bar{K}^3}+O\left[\frac{1}{\bar{K}^4}\right]\right], \qquad (5)$$

where

$$a=2-2(2n_r+1)\omega\ , \qquad (6)$$

$$\omega=\tilde{\omega}/2\ , \qquad (7)$$

$$\tilde{\omega}=[3+r_0V''(r_0)/V'(r_0)]^{1/2}\ , \qquad (8)$$

$$Q=4r_0^3V'(r_0)\ , \qquad (9)$$

and the quantity r_0 satisfies the equation

$$N+2l-2+(2n_r+1)[3+r_0V''(r_0)/V'(r_0)]^{1/2}=[4r_0^3V'(r_0)]^{1/2}\ . \qquad (10)$$

Equations (6)—(10) are simply IPS2 Eqs. (4) and (5) except that we use n_r, the radial quantum number, in place of n in IPS1 and IPS2. The explicit expressions for the corrections to the leading order [the terms inside the square brackets in Eq. (5)], $\Gamma^{(i)}$ ($i=1,2$), can readily be written down from IPS1 Eqs. (13) and (14), and (B6)—(B11). Notice that the choice of the shift a, given in Eq. (6), gives exact energy eigenvalues from the leading term of the $1/\bar{k}$ expansion for Coulomb and harmonic-oscillator potentials.[3]

We have applied Eqs. (5)—(10) and the corresponding expressions for $\Gamma^{(1)}$ and $\Gamma^{(2)}$ in IPS1 and IPS2 to the case where $V(r)$ is the Hulthén potential [Eq. (1)] with $N=3$. In Table I we have listed numerically the energy eigenvalues for a wide range of n_r, l, and δ. For $l=0$ states, our numerical results obtained by means of the shifted $1/N$ expansion and the exact analytical results [Eq. (2)] match to at least three places. Thus the reliability of the method and our numerical codes are established. A more qualitative picture is provided by Figs. 1 and 2. In partic-

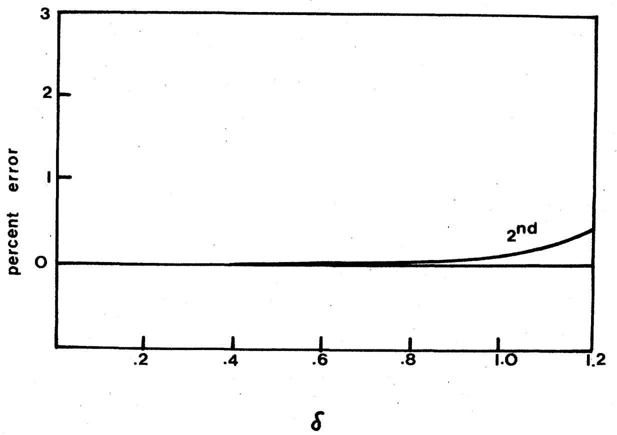

FIG. 2. Percent error in the shifted $1/N$ expansion for the ground-state energy eigenvalues vs δ. The curve marked 2nd corresponds to the leading term and two corrections.

ular, we have plotted the percent error of our 1*s* results as a function of δ in Fig. 2. Note that in the region $\delta < 1.2$, the first three terms in the $1/\bar{k}$ expansion converge nicely, whereas in the region $\delta > 1.2$ (also see Table I) successive contributions from the perturbation series are substantial and the expansion is not valid. The reason for the δ dependence is that the value of the shift a is fixed such as to reproduce the exact analytic results for the Coulomb potential (corresponding to the Hulthén potential with $\delta \to 0$). Hence our results are especially accurate for small δ. On the other hand, as δ increases and approaches a critical value δ_c, the energy eigenvalues approach zero, a situation corresponding to the case of no bound states, and the expansion technique breaks down. Furthermore, we do not expect that the shifted $1/N$ expansion works well for very high n_r since $\Gamma^{(i)}$ contains terms proportional to n_r^3. Anyhow, the results presented in Table I are quite accurate for both $l = 0$ and $l \neq 0$ states.

In conclusion, we would like to point out that although the shifted $1/N$ expansion scheme works quite well for the Hulthén potential, extensive applications to other potentials are needed to test the credibility of the method and therefore would be of great value. We are currently carrying out such an investigation.

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